ON THE POSSIBILITY OF AN $\alpha^2\omega$-TYPE DYNAMO IN A THIN LAYER INSIDE THE SUN

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ABSTRACT

If the solar dynamo operates in a thin layer of $10^4$ km thickness at the interface between the convection zone and the radiative core, using the facts that the dynamo should have a period of 22 years and a half-wavelength of 40° in the $\theta$-direction, it is possible to impose restrictions on the values which various dynamo parameters are allowed to have. We point out that the dynamo should be of $\alpha^2\omega$ nature, and we present kinematical calculations for free dynamo waves and for dynamos in thin rectangular slabs with appropriate boundary conditions. An $\alpha^2\omega$ dynamo is expected to produce a significant poloidal field which does not leak to the solar surface. We find that the turbulent diffusivity $\eta$ and $\alpha$-coefficient are restricted to values within about a factor of 10, the median values being $\eta \sim 10^{10}$ cm$^2$ s$^{-1}$ and $\alpha \sim 10$ cm s$^{-1}$. On the basis of mixing length theory, we point out that such values imply a reasonable turbulent velocity of the order 30 m s$^{-1}$, but rather small turbulent length scales like 300 km.

Subject headings: hydrodynamics — Sun: interior — Sun: rotation — turbulence

1. INTRODUCTION

The solar magnetic fields were traditionally assumed to be generated by an $\alpha\omega$-type dynamo operating in the convection zone (Moffatt 1978; Parker 1979; Stix 1981). It has, however, been recognized in the last few years that there are two main difficulties with a dynamo operating in the solar convection zone: (i) magnetic buoyancy would remove any magnetic flux from the convection zone quickly, cutting down the efficiency of any dynamo action (Parker 1975; Moreno-Insertis 1983), and (ii) self-consistent dynamical models of convection zone dynamos are unable to reproduce the surface rotation pattern and the butterfly diagram simultaneously (Gilman 1983; Glatzmaier 1985). In response to these difficulties, it has been suggested that the dynamo operates in a thin layer at the bottom of the convection zone or underneath it rather than operating in the main bulk of the convection zone. The overshoot region at the interface between the radiative core and the convection zone is supposed to be a suitable location for the operation of the solar dynamo, since the subadiabatic gradient there can suppress the magnetic buoyancy (Spiegel and Weiss 1980; van Ballegooijen 1982). Alternative effects such as thermal shadows (Parker 1987) and the drag due to the meridional flow (van Ballegooijen and Choudhuri 1988) have also been proposed in order to suppress magnetic buoyancy in a thin layer at the bottom of the convection zone. The differential rotation and the $\alpha$-coefficient driving the dynamo action there are also expected to have correct signs, at least in the lower latitudes, so as to make the dynamo waves go in the equatorward direction, in conformity with the butterfly diagram (Gilman, Morrow, and DeLuca 1989). So it seems that one can get around the two above-mentioned difficulties if the dynamo operates in a thin layer at the bottom of the convection zone.

There are, however, some new difficulties if the dynamo operates in such a thin layer rather than in the main bulk of the convection zone. Calculations on the buoyant rise of the magnetic flux produced at the bottom of the convection zone show that the flux is deflected by the Coriolis force to move parallel to the rotation axis and emerge at rather high latitudes poleward of where sunspots are seen to appear (Choudhuri and Gilman 1987; Choudhuri 1989). More recently, however, Choudhuri and D'Silva (1990) have studied the interactions between the rising flux and the surrounding turbulence in the convection zone. They conclude that if the flux rises in the form of flux tubes of sufficiently small radius (a few hundred kilometers or so), then the tube can exchange enough angular momentum with the surroundings to suppress the growth of the Coriolis force and can emerge out radially. So, at least if the flux rises in the form of sufficiently thin tubes, then there is hope that the flux produced at low latitudes may still emerge at low latitudes on the solar surface. Another difficulty pointed out by Parker (1987) is that a very thin layer may not be able to contain sufficient magnetic flux to account for all the flux that emerges on the solar surface. This is an issue which remains rather ill-understood at the present time and deserves further attention. Another important question is whether the values of turbulent diffusion, differential rotation, $\alpha$-coefficient, etc., that are expected in the thin layer at the bottom of the convection zone can give rise to a dynamo with the right characteristics (such as a period of 22 yr and a half-wavelength of about 40° in the $\theta$-direction). This is one of central questions addressed in this paper.

Even when the dynamo was supposed to operate in the convection zone, most of the kinematical models tended to give a period on the lower side, and it was necessary to push various parameters to the utmost limit of the allowed values in order to get a period of 22 yr (Choudhuri 1984). This problem of obtaining the right period with an $\alpha\omega$-dynamo becomes even more aggravated if we want to make the dynamo operate in a thin layer. It has been noted by Gilman, Morrow, and DeLuca (1989) that considering an $\alpha^2\omega$-type dynamo rather than an $\alpha\omega$ dynamo relieves this difficulty to a large extent. One important property of a pure $\alpha\omega$ dynamo is that it produces a predominantly toroidal field with a much smaller poloidal field. Since the poloidal field on the surface of the Sun is known to be rather small, an $\alpha\omega$ dynamo was regarded as a natural candidate for generating the solar magnetic fields. However, if the
dynamo operates underneath the convection zone, then even if it produced a large poloidal field, it is not clear what the subsequent manifestations of that field will be. Since the electrically conducting plasmas of the convection zone would be on top of the dynamo region, the poloidal field produced in the dynamo region deep down at the bottom of the convection zone would not be able to leak directly to the solar surface. It may just give rise to some twist in the flux tubes that break away from the dynamo region and may not produce any other observable effects at the solar surface. So it is conceivable that a poloidal field as large in magnitude as a few tens of percent of the toroidal field may be present in the dynamo region. This allows one to abandon the rather restrictive \( \omega_0 \) limit of the dynamo and consider \( \omega^2 \omega \) dynamos, which are expected to produce significant poloidal fields. In the calculations in this paper, we have used the \( \omega^2 \omega \) equations in which the \( \omega \)-term is retained in the generation of the toroidal field in addition to the \( \omega \)-term. Though \( \omega^2 \omega \) dynamos and \( \omega^2 \) dynamos have been studied by many authors (see Moffatt 1978; Krause and Radler 1980), the properties of an \( \omega^2 \omega \) dynamo remain largely unexplored.

It is difficult to estimate the thickness of the layer in which the dynamo action takes place. The overshoot region is supposed to have a thickness equal to a few tenths of a pressure scale height (van Ballegooijen 1982; Schmitt, Rosner, and Bohn 1984), and \( 10^4 \) km is probably a good guess for the thickness of the dynamo region (DeLuca and Gilman 1986). So we can suppose that an \( \omega^2 \omega \) dynamo operates in a layer of \( 10^4 \) km and demand that it gives rise to a period of 22 years and a half-wavelength of about 40° in the \( \theta \)-direction. We now ask the following questions. What constraints do we have to impose on the properties of the dynamo so that these demands are met? What range of values is allowed for quantities like the \( \alpha \)-coefficient, the differential rotation, and the turbulent diffusion? Are those values reasonable? We are going to carry on a kinematical calculation and show that the various quantities entering the dynamo equation have to be restricted within certain regions of the parameter space if the dynamo has to have the desired characteristics. Perhaps the limitations of a kinematical approach also should be kept in mind. Kinematical calculations do not tell us whether it is dynamically possible for various quantities to have the simultaneous values that we propose. In fact, self-consistent dynamical calculations failed to reproduce the combinations of various dynamo parameters used in the kinematical modelings of convection zone dynamos to explain the butterfly diagram (Gilman 1983). In spite of this caveat, a kinematical approach can still be a powerful tool for an initial exploration of dynamo properties if we are judicious and keep the limitations in mind. Something halfway between a purely kinematical approach and a full dynamical approach for a thin layer dynamo has been attempted in a series of papers by DeLuca and Gilman (1986, 1988, 1989). Though the motions induced by the backreactions of the magnetic field are calculated explicitly, the \( \alpha \)-coefficient is taken to be a given parameter. Even these halfway calculations are quite computer-intensive, and to the best of our knowledge, nobody has yet published full dynamical calculations for a thin layer dynamo. The main advantage of the kinematical approach is that one can proceed analytically up to a certain point, and it is a lot less painful to explore the parameter space and study how the behavior of the dynamo changes with the change of different parameters. Also, the solutions obtained by kinematical methods can be used as checks for numerical codes.

As we have pointed out, the kinematical approach merely isolates the regions of parameter space which give rise to a dynamo with the desired characteristics, but it does not shed any light on whether the values of various parameters in those regions make sense on the basis of dynamical considerations. Though full dynamical calculations are beyond the scope of this paper, one can use crude mixing length arguments to check whether it is at all likely that the different dynamo parameters like the \( \alpha \)-coefficient and the turbulent diffusivity have values that we would wish them to have. We eventually find from mixing length arguments that, in order to have the right kind of dynamo behavior, the turbulence in the overshoot region has to have reasonable velocities like \( 30 \) m s\(^{-1}\), but rather small length scales of the order of only a few hundred kilometers. This is much smaller than all the relevant scale heights, and we do not know at all what is expected at those length scales. The scanty knowledge that we have of convective turbulence in compressible fluids is mostly based on numerical simulations, and such simulations often suggest convective cells extending over several scale heights rather than being much smaller than the scale height (see Gilman 1986). It should, however, be borne in mind that all three-dimensional simulations use rather coarse grids and hence cannot resolve any phenomena taking place at small length scales. We should also remember that the overshoot region is a place where the motions of penetrating plumes are braked by subadiabatic gradient, and it is conceivable that this braking action may lead to turbulence at small length scales. We merely point out in this paper that a solar dynamo operating in a thin layer requires small-scale turbulence.

The basic equations are introduced in the next section along with a discussion of the boundary conditions. Before getting into the calculations for the dynamo action in a layer, we look at plane dynamo waves in free space and present some illustrative results in § III. The dispersion relation for the dynamo waves in a layer is then derived in § IV. In § V, we solve this dispersion relation and discuss the properties of the magnetic fields which are produced by the dynamo acting in a thin layer. The final section discusses the implications of the results of our calculations.

II. BASIC EQUATIONS AND BOUNDARY CONDITIONS

Since the thickness of the dynamo region is small compared to its curvature and to its horizontal extents, one can hope to obtain reasonable results by solving the dynamo equations in a thin slab with a rectangular geometry. This is the approach taken by Parker (1971, 1979) in his early work on the solar dynamo, and following him, we consider a slab bounded by the planes \( x = -h, x = +h \), where \( x \) corresponds to the vertical direction, \( y \) corresponds to the azimuthal direction, and \( z \) corresponds to the latitudinal direction. We assume a constant shear \( G = d\nu_y/dx \) within the slab, and the \( \alpha \)-coefficient and the turbulent diffusion \( \eta \) are also taken to be constants. We shall be considering mainly dynamo waves propagating freely in the latitude (i.e., \( z \)-direction). The effects of sphericity and the effects of interfering dynamo waves from opposite hemispheres are not taken into account. The final results may change to some extent when these effects are considered. However, we are interested only in roughly locating the allowed region of the parameter space, and the simplifications adopted here should be good enough for our purposes.

Since \( y \) is an ignorable direction in which nothing varies (i.e.,
\[-d\partial y = 0\], the magnetic field can be written down quite generally:
\[
B = B_x \hat{y} + \nabla \times (A_y \hat{y}) ,
\]
so that \( B_x \) corresponds to the azimuthal field and \( A_y \) gives the poloidal field. From the dynamo equation
\[
\frac{\partial B}{\partial t} = \nabla \times (\omega \times B) + \nabla \times (\eta \nabla \times B) ,
\]
we can obtain the following pair of equations for \( B_x \) and \( A_y \) in the dynamo region (see Parker 1979, p. 619):
\[
\left( \frac{\partial}{\partial t} - \eta \nabla^2 \right) B_x = -G \frac{\partial A_y}{\partial z} - \alpha \nabla^2 A_y , \tag{3}
\]
\[
\left( \frac{\partial}{\partial t} - \eta \nabla^2 \right) A_y = \alpha B_x , \tag{4}
\]
For solutions of the form \( \exp [\iota (\omega t' - k_x x - k_z z)] \), equations (3) and (4) can be combined to give
\[
[\iota \omega + \eta (k_z^2 + k_x^2)]^2 = \iota \alpha G k_x + \alpha^2 (k_z^2 + k_x^2) . \tag{5}
\]
If \( \omega \) is real, then the solutions neither grow nor decay in time. The usual approach of kinematical calculations is to find conditions (i.e., values of \( \alpha, G, \eta \)) that make \( \omega \) real, with the hope that in reality the nonlinear feedback reactions of the magnetic fields would drive the parameters to such values as to make the dynamo stationary. Not only does \( \omega \) have to be real, its value has to correspond to a period of 22 years, which means
\[
\omega = 9.1 \times 10^{-9} \text{ s}^{-1} . \tag{6}
\]
We know further that \( k_x \) has to correspond to a half-wavelength of 40° in the \( \theta \)-direction at the overshoot layer, which can be taken to have a radius of 0.7 \( R_\odot \). This implies
\[
k_x = 9.2 \times 10^{-11} \text{ cm}^{-1} . \tag{7}
\]
Our aim now is to solve equations (3) and (4) within the layer extending from \( x = -h \) to \( x = h \) such that the \( t \) and \( z \) dependencies of the solutions are of the form \( \exp [\iota (\omega t - k_z z)] \), with \( \omega \) and \( k_z \) given by equations (6) and (7). Let us figure out the boundary conditions that have to be applied. The boundary conditions at the lower surface are easy to find. The radiative core is expected to have very low resistivity so that the poloidal field may not penetrate there. This means
\[
A_y = 0 \quad \text{at} \quad x = -h . \tag{8}
\]
To find the other boundary condition, we integrate equation (2) from \( -h - \epsilon \) to \( -h + \epsilon \) and use the fact that \( \alpha \) and \( \eta \) have to be close to zero at \( -h - \epsilon \) within the radiative region. Making \( \epsilon \) tend to 0 gives us
\[
\eta \frac{\partial B_x}{\partial x} = \alpha \frac{\partial A_y}{\partial x} \quad \text{at} \quad x = -h . \tag{9}
\]
Figuring out the boundary conditions at the upper boundary of the dynamo region is a much more tricky problem. If the upper boundary were an impervious boundary with high conductivity, then the same conditions as in equations (8) and (9) would be applicable there also. In fact, DeLuca and Gilman (1986) use the same conditions at the upper boundaries also in all their calculations. The convection zone above the upper boundary has a turbulent resistivity probably comparable to that in the dynamo region, and the upper boundary is also not impervious to the magnetic flux in the sense that flux passing through that boundary may be subject to the destabilizing effect of magnetic buoyancy and start rising. Hence, using the same boundary conditions at the upper boundary is at best an approximation. Since using symmetric boundary conditions makes our dispersion relations somewhat less formidable and since we do not know how to treat the upper boundary better, we use the same conditions
\[
A_y = 0 \quad \text{at} \quad x = h , \tag{10}
\]
\[
\eta \frac{\partial B_x}{\partial x} = \alpha \frac{\partial A_y}{\partial x} \quad \text{at} \quad x = h . \tag{11}
\]
However, while presenting the results of our calculations in \( \S \) V, we discuss other possible approximations for the upper boundary and point out how they modify the results. We notice that the pair of dynamo equations (3) and (4) are invariant under the transformation
\[
\alpha \rightarrow -\alpha , \quad G \rightarrow -G , \quad A_y \rightarrow -A_y .
\]
It is well known that we have to have \( \alpha G < 0 \) in the northern hemisphere and \( \alpha G > 0 \) in the southern hemisphere in order to make the dynamo waves propagate toward the equator. Throughout this paper, however, we shall be mainly interested in figuring out the appropriate magnitudes of different quantities, without paying much attention to the signs. Once we have the magnitudes and remember the symmetry of our equations under the above transformation, it is fairly straightforward to figure out what combinations of signs we may need in a particular circumstance.

III. PLANE DYNAMO WAVES IN FREE SPACE

The equation (5) is fourth order in \( k_z \). With \( \omega \) and \( k_z \) specified by equations (6) and (7), \( k_z \) has four roots which are generally complex and are functions of \( \alpha, G, \eta \). Since we have to satisfy four boundary conditions, it is necessary to superpose the four solutions corresponding to the four roots of \( k_z \) in order to construct a solution satisfying all the boundary conditions. This is done in the next section. Before plunging into the full calculations, let us take a look at plane dynamo waves in infinite space for which \( k_z \) can be taken as real. If \( k_x, k_z, \omega \) are all real, then we can separate the real and imaginary parts of equation (5) to give
\[
\eta^2 K^4 - \omega^2 = \alpha^2 K^2 , \tag{12}
\]
\[
2\omega \eta K^2 = \alpha G k_z , \tag{13}
\]
where
\[
K = (k_x^2 + k_z^2)^{1/2} , \tag{14}
\]
To have a rough idea of the behavior of a dynamo operating in a layer of thickness \( 2h \), we may take \( 2h \) to correspond to half a wavelength in the \( x \)-direction so that
\[
k_x = \frac{\pi}{2h} . \tag{15}
\]
With \( \omega \) and \( k_z \) given by equations (6) and (7), once we choose a particular thickness \( 2h \) and hence a corresponding \( k_z \) from equation (15), we can have only one free parameter in the pair of equations (12) and (13). If we take \( \alpha \) to be the free parameter, then \( \eta \) and \( G \) can be found for any given value of \( \alpha \) from these
two equations. In much of our discussion, \(a\) and \(G\) may actually mean \(|a|\) or \(|G|\). We have avoided using modulus signs to keep the notation less cumbersome.

We have shown in Figure 1a how the shear \(G\) varies with \(\alpha\) for the three thicknesses 5000 km, 10,000 km, and 20,000 km for the dynamo layer. This figure is very similar to Figure 2d of Gilman, Morrow, and DeLuca (1989). Though the procedure for solving the problem with the boundary conditions (8)-(11) is described in the next two sections, we present here the results for a layer of thickness 10,000 km for comparison. The dashed curve in Figure 1a shows how \(G\) has to vary with \(\alpha\) in the case when the boundary conditions are imposed. In Figure 1b, we plot how \(\eta\) varies with \(\alpha\) for the three given thicknesses. Again, the dashed curve corresponds to a dynamo layer of 10,000 km with boundary conditions imposed. The left-hand portions of Figures 1a and 1b correspond to the \(\alpha \rightarrow 0\) limit. The three curves in Figure 1a coalesce in this limit, whereas we see in Figure 1b that \(\eta\) remains constant for any given thickness. These can be understood from equations (12) and (13) which give in this limit:

\[
\eta \rightarrow \frac{\omega}{K^2}, \quad aG \rightarrow 2\alpha^2/k_z.
\]

The right-hand portions of the figures correspond to the large-\(\alpha\) limit, in which \(\eta\) increases linearly with \(\alpha\) and \(G\) remains constant. These also follow from equations (12) and (13) which, in the large-\(\alpha\) limit, yield:

\[
\eta \rightarrow \frac{\alpha}{K}, \quad G \rightarrow 2\omega K/k_z.
\]

We note that the actual results with the appropriate boundary conditions also qualitatively behave in the same way.

We know that the poloidal field is small compared to the toroidal field in the \(\alpha \rightarrow 0\) limit, whereas they are expected to be comparable in the large-\(\alpha\) limit. Since the poloidal field is of the order \(K\alpha\), we see from equation (4) that a rough ratio \(r\) of the magnitude of the poloidal field to that of the toroidal field is given by

\[
r \sim \left| \frac{aK}{\eta K^2 + i\omega} \right| \sim \frac{1}{\sqrt{(\eta K / \alpha)^2 + (\omega / \alpha K)^2}}. \tag{16}
\]

Figure 1c plots this ratio \(r\) as a function of \(\alpha\) for the three thicknesses we are considering. We note that \(r\) changes from values close to 0 to values nearly 1 as \(\alpha\) increases.

From crude mixing length arguments, one expects the diffusivity \(\eta\) in the convection zone to be around \(10^{12} \text{ cm}^2 \text{ s}^{-1}\). The \(\alpha\)-coefficient, which has the same dimension as velocity, is expected to be a fraction of convective velocities so that one would normally suppose \(10^3 \text{ cm} \text{ s}^{-1}\) to be a reasonable value for it. Looking at Figures 1a and 1b, it seems that by extending our plots towards the right-hand side, we should be able to reach the desired values of \(\alpha\) and \(\eta\). Figure 1c clearly shows that the poloidal field then will have to be comparable to the toroidal field, contrary to the statements found in Gilman, Morrow, and DeLuca (1989). There is also another additional reason for worry. We know that the dynamo equations are not exact equations, but rather of the nature of stochastic equations for mean fields obtained through an averaging procedure (see Moffatt 1978). We see from equation (13) that the frequency \(\omega\) is produced by the shear \(G\), and the dynamo would be nonper-
iodic if the shear were zero. In the large-$\alpha$ regime, the shear term in equation (3) would be much smaller than the other source term involving $\alpha$. A direct consequence of this follows from equation (12), where we see that $\omega^2$ arises out of the small imbalance between two much larger terms $\eta^2 K^4$ and $\alpha^2 K^2$, which are expected to have stochastic fluctuations. One wonders whether it would have been possible to get a fairly regular period if the term giving rise to the periodicity were much smaller compared to other large stochastic terms in the equations.

The shear term loses importance precisely where the curves in Figure 1a start moving away from the $\omega \alpha$ regime (where all the curves coalesce) and bend to become horizontal. If we want the shear term to remain comparable to the other terms in our equations, we cannot allow $\alpha$ to have values much larger than the value it has in the region where the curves in Figure 1a bend. It is to be noted that the curves in Figure 1b also bend for the same values of $\alpha$, and the ratio of the poloidal to the toroidal fields also jumps from 0 to 1 around the same region. If we look at the bends in the curves corresponding to free dynamo waves, we seem to be restricted to values of $\alpha$ and $\eta$ rather small compared to their expected values. However, the bends in the curves corresponding to the solutions with boundaries allow for considerably larger $\alpha$ and $\eta$. A discussion of the possible values of various dynamo parameters and their implications will be taken up in § VI. Here we note that if we want to push $\alpha$ and $\eta$ to their largest allowable values, then we have to be around the bends in the curves where the shear term just hovers around being comparable to the other terms. Within the framework of kinematical models, it may seem like an accident if the two terms on the right-hand side of equation (3) are comparable in magnitude. However, we should remember that both the shear and the $\alpha$-effect are ultimately supposed to be caused by the same prime mover—the turbulent motions driven by the convective instability (see Gilman 1986). So it is conceivable that there are some subtle dynamical reasons why the two terms in equation (3) may be comparable, making the solar dynamo a truly $\omega \alpha$ dynamo.

Recently, Feynman and Gabriel (1989) have analyzed the periodicities of various phenomena connected with the solar cycle and conclude that the periodicity of the solar cycle lies just on the brink of becoming chaotic. Drawing on the calculations of Weiss, Cattaneo, and Jones (1984), Feynman and Gabriel (1989) suggest that the nonlinearities of the system are responsible for the occasional displays of chaos. Though nonlinear equations are known to produce chaotic behavior, it is still necessary to understand why the periodicity is just on the brink of becoming chaotic. Feynman and Gabriel (1989) consider the possibility of some feedback mechanism that may drive the dynamo to the borderline of chaos. One wonders whether the periodicity being on the verge of chaos is related to the fact that the term giving rise to the periodicity in the dynamo equation just hovers around being comparable to the other stochastic terms in the equation, as we seem to conclude here from completely different considerations. We leave this as a provocative conjecture and hope that some of our colleagues possessing appropriate numerical codes may consider this question worth investigating.

IV. DERIVATION OF THE DISPERSION RELATION

We now proceed to solve equations (3) and (4) which can be combined to form equation (5), subject to the values of $\omega$ and $k_z$ given by equations (6) and (7), and subject to boundary conditions (8)-(11). We employ a method very similar to the method used in Choudhuri (1984). Dividing equation (5) by $\eta^2 K^2$, we get

$$ (1 + S^2 + i\Omega)^2 = iN + \beta(1 + S^2) , $$

where

$$ S = k_x/k_z , $$

$$ \Omega = \omega/\eta k_z^2 , $$

$$ N = \left( \frac{G \eta}{\alpha} \right) k_z^2 , $$

$$ \beta = \alpha^2 / \eta^2 k_z^2 . $$

Since equation (17) is fourth order in $S$, there should be four roots for $S$. We can write $S^2$ in terms of the other quantities in equation (17):

$$ S^2 = -1 - i\Omega + \frac{\beta}{2} \pm \sqrt{\left( \frac{\beta}{2} \right)^2 - 2\omega \left( \frac{\beta}{2} \right) + iN} . $$

Let $s_1, s_2 = -s_3$ be the roots of $S$ corresponding to the plus sign in equation (18), and let $s_3, s_4 = -s_5$ be the roots corresponding to the minus sign. We see that $S$ and $k_z$ are in general complex. Superposing the solutions corresponding to the four roots of $S$, we can write $B_y$ as

$$ B_y = e^{i(k_y-xa)} \sum_{j=1}^4 c_j e^{-is_j k_x} , $$

where the $c_j$'s are coefficients which are to be determined from boundary conditions. It then follows from equation (4) that $A_y$ has to be written in the following form:

$$ A_y = e^{i(k_y-xa)} \frac{1}{k_z} [ (\lambda - \mu)(c_1 e^{-is_1 k_x} + c_2 e^{is_1 k_x}) $$

$$ + (\lambda + \mu)(c_3 e^{-is_3 k_x} + c_4 e^{is_3 k_x})] , $$

where

$$ \lambda = \frac{\beta^{3/2}}{2\Omega(\beta/2) - iN} , $$

$$ \mu = \frac{\beta^{3/2}}{2\Omega(\beta/2) - iN} . $$

Using the boundary conditions (8) and (10), we obtain

$$ (\lambda - \mu)(c_1 e^{is_1 u} + c_2 e^{-is_1 u}) + (\lambda + \mu)(c_3 e^{is_3 u} + c_4 e^{-is_3 u}) = 0 , $$

and

$$ (\lambda - \mu)(c_1 e^{-is_1 u} + c_2 e^{is_1 u}) + (\lambda + \mu)(c_3 e^{-is_3 u} + c_4 e^{is_3 u}) = 0 , $$

where $u = k_z h$. Adding and subtracting these two equations gives

$$ (\lambda - \mu)[e^{is_1 u} + e^{-is_1 u}](c_1 + c_2) $$

$$ + (\lambda + \mu)[e^{is_3 u} + e^{-is_3 u}](c_3 + c_4) = 0 , $$

and

$$ (\lambda - \mu)[e^{is_1 u} - e^{-is_1 u}](c_1 - c_2) $$

$$ + (\lambda + \mu)[e^{is_3 u} - e^{-is_3 u}](c_3 - c_4) = 0 . $$

Similarly, boundary conditions (9) and (11) give

$$ c_1 s_1 e^{is_1 u} - c_2 s_1 e^{-is_1 u} [1 - \beta^{1/2}(\lambda - \mu)] $$

$$ + (c_3 s_3 e^{is_3 u} - c_4 e^{-is_3 u}) [1 - \beta^{1/2}(\lambda + \mu)] = 0 , $$

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Again, addition and subtraction of these two equations yields
\[ s_1(e^{i\omega u} + e^{-i\omega u})[1 - \beta^{1/2}(\lambda - \mu)](c_1 + c_2) \]
\[ + s_3(e^{i\omega y} + e^{-i\omega y})[1 - \beta^{1/2}(\lambda + \mu)](c_3 - c_4) = 0 , \quad (25) \]
and
\[ s_1(e^{i\omega y} - e^{-i\omega y})[1 - \beta^{1/2}(\mu - \lambda)](c_1 + c_2) \]
\[ + s_3(e^{i\omega y} - e^{-i\omega y})[1 - \beta^{1/2}(\mu + \lambda)](c_3 + c_4) = 0 . \quad (26) \]

Looking at equations (23)-(26), it is clear that the solutions can be of two types: (i) even modes for which \( c_3 = c_1, c_4 = -c_1; \) and (ii) odd modes for which \( c_2 = -c_1, c_4 = -c_3. \) We are interested here only in studying the lowest mode without any nodes within the dynamo layer. From the symmetry of the problem, we consider equations (23) and (26) only. The coefficients of \((e^{i\omega y} + e^{-i\omega y})\) in these equations constitute a \(2 \times 2\) matrix of which the determinant must be zero. This gives the final dispersion relation:
\[ s_2(\lambda - \mu)[1 - \beta^{1/2}(\lambda + \mu)](e^{i\omega y} + e^{-i\omega y}) \]
\[ = s_3(\lambda + \mu)[1 - \beta^{1/2}(\lambda - \mu)] \]
\[ \times (e^{i\omega y} + e^{-i\omega y})(e^{i\omega y} - e^{-i\omega y}) . \quad (27) \]

Since \( s_1, s_3, \lambda, \mu\) are functions of \(\beta, \Omega, N\) as expressed by equations (18), (21), and (22), the dispersion relation (27) essentially relates \(\beta, \Omega, N\) for any value of \(u\). The real and imaginary parts of equation (27) would give us two equations of the type
\[ f_1(\beta, \Omega, N, u = u_{\text{given}}) = 0 , \]
\[ f_2(\beta, \Omega, N, u = u_{\text{given}}) = 0 . \]

In addition to \(u\), if one other parameter, say \(\beta\), is specified, we can solve for \(\Omega\) and \(N\). Thus, for a particular value of \(u\), by varying the parameter \(\beta\), we can obtain sets of values of \(\{\beta, \Omega, N\}\) which satisfy our dispersion relation. We have obtained such solutions of equation (27) numerically and present them in the next section. Once we have a set of values \(\{\beta, \Omega, N\}\) for a given \(u\), we can obtain the values of various dynamo parameters which would give rise to the right sort of dynamo behavior with \(\omega\) and \(k_x\) given by equations (6) and (7):
\[ \eta = \frac{\alpha \Omega k_x^2}{\Omega} = \frac{1.1 \times 10^{12}}{\Omega} \text{ cm}^2 \text{ s}^{-1} , \quad (28) \]
\[ \alpha = \eta \sqrt{\frac{k_x}{\beta}} = \frac{1.0 \times 10^2}{\Omega} \sqrt{\beta} \text{ cm s}^{-1} , \quad (29) \]
\[ G = \frac{\alpha^2 N}{\Omega^2 k_x} = 9.0 \times 10^{-9} \frac{N}{\Omega \sqrt{\beta}} \text{ s}^{-1} . \quad (30) \]

V. NATURE OF THE GENERATED FIELDS

We can obtain the \(\pi\omega\)-dynamo limit by putting \(\beta = 0\) in our equations. For any value of \(u\) (i.e., thickness of the slab in units of horizontal wavelength) in this limit, we can then find values of \(\Omega\) and \(N\) which will sustain the steady dynamo. Table 1 lists values of \(\Omega\) and \(N\) against \(u\) for \(\pi\omega\) dynamos. In the asymptotic limit of \(u \to \infty\), the dispersion relation can be solved analytically by the method described in the Appendix of Choudhuri (1984). The analytical expressions are:
\[ N = 2 + \frac{n^2 \pi^2}{u^2} \left(1 - \frac{1.10}{u}\right) + O \left(\frac{1}{u^{1/4}}\right) , \quad (31) \]
\[ \Omega = 1 + \frac{n^2 \pi^2}{4u^2} \left(1 - \frac{1.54}{u}\right) + O \left(\frac{1}{u^{3/4}}\right) , \quad (32) \]
where \(n\) is an integer. The numbers within brackets in Table 1 correspond to values obtained from equations (31) and (32) by taking \(n = 1\). A slab of 10,000 km corresponds to \(u = 0.045\). It is seen from Table 1 that \(N\) has the very large value of \(5.5 \times 10^4\) for \(u = 0.045\). To understand the significance of this large number, we should remember that \(N\) is a sort of dynamo number using a horizontal length scale. For a dynamo in a thin layer, the actual length scale is set by the depth. So, if we use the depth to construct a dynamo number for \(u = 0.045\), we find
\[ N_d = \frac{2 \pi \rho h^3}{\eta^2} = N u^3 = 5.0 , \]
which is of the order of unity. In the \(\pi^2\) limit, the dynamo equation becomes
\[ \frac{\partial B}{\partial t} = \nabla \times (\pi B) + \eta \nabla^2 B . \quad (33) \]

It is easy to see that any magnetic field satisfying
\[ \nabla \times B = \frac{\alpha}{\eta} B , \quad (34) \]
would constitute a time-independent solution of equation (33). We note that equation (34) has exactly the same form as the force-free equation for magnetic-pressure-dominated plasmas. Hence, the mean field solutions of the \(\pi^2\)-dynamo equation would have the same mathematical structure as a force-free field. Such solutions have been discussed at length by DeLuca and Gilman (1986, 1988). The solution satisfying our boundary conditions is
\[ B_y = B_0 e^{-i(\pi x)/2h} \cos \frac{\pi x}{2h} , \quad A_y = \frac{\eta}{\alpha} B_y , \quad (35) \]
where
\[ \frac{\pi^2}{4h^2} = \frac{\alpha^2}{\eta^2} - k_z^2 . \quad (36) \]
TABLE 2A

| Results for $\alpha^2\omega$ Dynamos with $u = 0.5$ ($\beta_m = 10.9$) |
|---------------|---|---|
| $\beta$       | $\Omega$ | $N$ |
| 0.1           | 3.4     | 48. |
| 0.5           | 3.4     | 48. |
| 2.5           | 3.0     | 42. |
| 6.0           | 2.4     | 30. |
| 9.0           | 1.4     | 17. |

TABLE 2B

| Results for $\alpha^2\omega$ Dynamos with $u = 0.045$ ($\beta_m = 1.22 \times 10^3$) |
|---------------|---|---|
| $\beta$       | $\Omega$ | $N$ |
| 1             | 38.     | $5.5 \times 10^4$ |
| 100           | 35.     | $5.2 \times 10^4$ |
| 200           | 32.     | $4.7 \times 10^4$ |
| 600           | 27.     | $3.6 \times 10^4$ |

Dividing equation (36) by $k^2$,

$$\beta_m = 1 + \left( \frac{\pi}{2u} \right)^2,$$  \hspace{1cm} (37)

where $\beta_m$ is the limiting value of $\beta = \alpha^2/\eta^2k^2$ in the $\alpha^2$ limit. Since $\beta$ has the value 0 in the $\alpha^\omega$ limit and the value $\beta_m = 1 + (\pi/2u)^2$ in the $\alpha^2$ limit, we can cover the whole $\alpha^2\omega$ range by making $\beta$ go from 0 to $\beta_m = 1 + (\pi/2u)^2$.

In order to understand the special properties of a thin layer dynamo as opposed to a dynamo operating in a thick layer, we present solutions of the full dispersion relation for a thin layer of 10,000 km (i.e., $u = 0.045$) and also for a layer with $u = 0.5$ corresponding to $1.1 \times 10^5$ km, which is more than half the depth of the solar convection zone. Tables 2A and 2B list some illustrative values of $\beta$, $\Omega$, and $N$ for these two values of $u$. We can then use equations (28), (29), and (30) to find out $\eta$, $\alpha$, and $G$ in all the corresponding cases. In Figures 1a and 1b, we have already plotted $G$ versus $\alpha$ and $\eta$ versus $\alpha$ for the 10,000 km case by the dashed lines. We have seen that these dashed curves are somewhat shifted with respect to curves for the free dynamo wave, enabling one to have larger values for $\alpha$ and $\eta$.

We now present $G$ versus $\alpha$ and $\eta$ versus $\alpha$ plots for $u = 0.5$ by the dashed lines in Figures 2a and 2b, whereas the solid lines give results for a free dynamo with $k_x$ corresponding to the same thickness (as given by eq. (15)). It is seen that the curves corresponding to the free dynamo wave and corresponding to the results with boundaries are closer together in the case of a thicker layer, implying that the effects of the boundary are felt more as we make the dynamo layer thinner. We note also that a thicker dynamo tends to make the values of $\alpha$ and $\eta$ in the $\alpha^\omega$ regime quite a bit larger—somewhat closer to the anticipated values in the body of the convection zone.

Figure 1b shows that the asymptotic value of $\eta$ in the $\alpha^\omega$ limit can be increased by about a factor of 30 on the imposition of boundary conditions on a 10,000 km slab with respect to the value for the corresponding free dynamo waves. One would like to understand the physical reasons behind this large change produced by the boundaries. A bigger question also remains. We have pointed out that our upper boundary conditions are only approximate. One may wonder if our results could be off by large factors because of a modest error in the boundary conditions. Though we do not attempt to provide a full answer to this question, a partial answer can be obtained in the following way. Choudhuri (1984) had presented results for $\alpha^\omega$ dynamos with $u = 0.5$ under different boundary conditions. We can compare those results with our present results for $u = 0.5$ in the $\alpha^\omega$ limit. Table 3 gives the values of $\Omega$, $N$, and $\eta$ for a dynamo in a slab with $u = 0.5$ under different conditions, whereas the profiles of the toroidal field under those conditions are shown in Figure 3. Our solution in this paper is indicated by S, whereas W corresponds to the free dynamo wave. I and F are solutions taken from Choudhuri (1984), which had the same bottom boundary conditions as what we use here and had top boundary conditions as indicated in Table 3. We see in Figure 3 that S and I correspond to fairly smooth fields with small gradients so that a larger diffusivity $\eta$ would not produce too large decays. Hence, the necessary diffusivities turn out to be larger in these cases, and a larger value of $\alpha G$ is needed to work against these diffusivities. The field profiles corresponding to F and W show more variations, since the azimuthal field is made to go to zero at the top boundary.
Table 3

<table>
<thead>
<tr>
<th>Top Boundary Conditions</th>
<th>( \Omega )</th>
<th>( N )</th>
<th>( \eta )</th>
<th>( \alpha G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>W: Free waves</td>
<td>10.9</td>
<td>238</td>
<td>1.0\times10^{11}</td>
<td>1.8\times10^{-6}</td>
</tr>
<tr>
<td>S: ( A_y = 0, \partial B_y/\partial x = 0 )</td>
<td>3.4</td>
<td>48</td>
<td>3.2\times10^{11}</td>
<td>3.7\times10^{-6}</td>
</tr>
<tr>
<td>I: ( A_y ) matches potential field ( \partial B_y/\partial x = 0 )</td>
<td>2.3</td>
<td>17</td>
<td>4.7\times10^{11}</td>
<td>2.9\times10^{-6}</td>
</tr>
<tr>
<td>F: ( A_y ) matches potential field ( B_y = 0 )</td>
<td>4.9</td>
<td>68</td>
<td>2.2\times10^{11}</td>
<td>2.5\times10^{-6}</td>
</tr>
</tbody>
</table>

Fig. 3.—The profiles of \( B_y \) for an \( \omega \) dynamo with \( u = 0.5 \) under different boundary conditions as listed in Table 3.

Consequently, \( \eta \) and \( \alpha G \) have to be smaller in these cases. We discuss the implications of the upper boundary not being impervious more fully in the Appendix. There we also consider the question whether the poloidal field could leak through the solar convection zone, and we argue that any leakage should be confined to the lower layers of the convection zone. We also point out in the Appendix that the values of different dynamo parameters for the free wave case and for the impervious boundaries case can be taken as limits defining a range within which the actual values are expected to lie.

Before proceeding to a final discussion of the allowed values for the dynamo parameters in the next section, we present some plots to illustrate the nature of the generated fields.

Figures 4a and 4b show contours of \( B_y \) and \( A_y \) in the x-z plane for \( u = 0.5 \) with \( \beta = 0.5 \) and \( \beta = 6.0 \). The points corresponding to these two values of \( \beta \) are indicated in Figure 2a. We see that \( \beta = 0.5 \) corresponds to a point where departures from the \( \omega \) limit just begin, whereas \( \beta = 6.0 \) pushes us toward the other limit of \( \alpha \) dynamos. The positive value contours are denoted by solid lines, negative value contours are denoted by dashed lines, and the zero contour is denoted by the dotted line. We have taken \( \alpha \) to be positive. If \( \alpha \) were negative, then the positive and negative values for \( A_y \) have to be interchanged. We may also remind the reader that the contours of \( A_y \) trace out the poloidal field lines.

VI. CONCLUSION

We are finally in a position to discuss the ranges of values allowed for the different dynamo parameters and the signifi-
tances of those values. We already mentioned in § III that we tend to get values of \( \alpha \) and \( \eta \) which are smaller than what we expect on the basis of crude mixing length theory. For turbulence with length scale \( l \) and velocity scale \( v \) taking place in a frame rotating with angular velocity \( \Omega \), the turbulent diffusivity \( \eta \) and the \( \alpha \)-coefficient should be given by the expressions

\[
\eta = \gamma_1 v l, \tag{38}
\]

\[
\alpha = \gamma_2 \Omega l, \tag{39}
\]

where \( \gamma_1 \approx \gamma_2 \approx \frac{1}{3} \) for isotropic homogeneous turbulence (Parker 1979). The dynamo action, however, is supposed to take place in a layer which is convectively stable and where turbulence is expected to occur intermittently in the regions where plumes may have penetrated from above. Since turbulent regions make up only a fraction of the volume, the coefficients \( \gamma_1 \) and \( \gamma_2 \) must be weighted by a volume filling factor.

We know next to nothing about the properties of turbulence in the overshoot region. For purposes of rough estimation, let us take

\[
\gamma_1 = \gamma_2 = \frac{1}{6}, \tag{40}
\]

which is probably correct within a factor of 2 or 3.

We now find the allowed regions in the \( \alpha G \) parameter space for a dynamo operating in a layer of thickness 10,000 km with the right period and the right \( \theta \)-wavelength. The results for the free dynamo wave (with vertical wavelength corresponding to 10,000 km) and for the dynamo satisfying our boundary conditions are plotted in Figure 6 as curves I and II. These curves were already shown in Figure 1a. We have pointed out in the last section and in the Appendix that the actual values may be expected to lie between these two curves, presumably close to curve II. We now impose several other restrictions to shrink the available parameter space further. A constant shear of \( 10^{-5} \) \( \text{s}^{-1} \) across a layer of thickness 10,000 km gives rise to a total velocity difference of 100 \( \text{m s}^{-1} \). It seems unlikely that the velocity difference across a thin layer at the bottom of the convection zone would be larger. In the near future, it will probably be possible to use accurate helioseismology data to estimate the value of the shear. For the time being, let us take \( 10^{-5} \) \( \text{s}^{-1} \) as the maximum allowed value of the shear. The line III indicates this limit in Figure 6. To make \( \alpha \) and \( \eta \) sufficiently small within the framework of the mixing length theory, we note from equations (38) and (39) that we have to make the length scale \( l \) small. Let us take \( l \approx 100 \) km as the lowest limit we are prepared to accept. One may object that this is already a rather unrealistically low value. But the calculations show that if we wish to make the solar dynamo operate in a thin layer, we must allow for length scales of the order of a few hundreds of kilometers. From equations (39) and (40), we find that a length scale of 100 km corresponds to \( \alpha \approx 3 \text{ cm s}^{-1} \). This limit is indicated in Figure 6 by the line IV.

To put a limit on the \( \alpha G \) parameter space from the right-hand side, one could conceivably argue that the poloidal field should not be too large or that the shear term giving rise to the periodicity should not become small compared to the other terms. Let us put a limit from somewhat different considerations. Dividing equation (38) by equation (39), we find

\[
\frac{\alpha}{\eta} = \frac{\Omega}{v}. \tag{41}
\]

If the turbulent velocity were too small, then the fluid motions would not have been able to twist the magnetic fields to produce the dynamo action. It is expected that the magnetic energy in the dynamo region should be roughly in equipartition with the fluid kinetic energy. This implies that the
magnetic field in the dynamo region has the following relation with the turbulent velocity:

\[ B \sim \sqrt{4\pi \rho \nu} \, . \]

Although we do not know what the value of the magnetic field in the dynamo region is, we know that it has to be at least several thousand gauss if we want to account for all the flux that we observe on the solar surface (Parker 1987). Putting a lower limit of 5000 G on the magnetic field gives us

\[ \nu > 3 \times 10^{3} \, \text{cm s}^{-1}, \]

when we use the density at the bottom of the convection zone taken from standard models (Spruit 1974). Using this limit in conjunction with equation (41), we conclude that

\[ \frac{\alpha}{\eta} < 10^{-9} \, \text{cm}^{-1}. \]

According to Figure 1b, we may allow a value of \( \eta \) upto \( 3 \times 10^{10} \, \text{cm}^{2} \text{s}^{-1} \) so that

\[ \alpha < 30 \, \text{cm s}^{-1}. \]

This limit is shown by the line V in Figure 6. This line lies toward the right of where curve I bends and more or less coincides with the place where curve II begins its more gentle bend. We have discussed in §III that the bends in these curves correspond to where the poloidal field starts becoming large and where the shear term starts falling short of other terms in the equations. It is interesting that the limit obtained from very different considerations roughly coincides with the bends in the curves, where we would have put the limit if we demanded that the poloidal field should not become too large or the shear term should remain at least comparable to the other terms.

We have at last isolated an allowed region in the \( aG \) parameter space which is shaded in Figure 6. Using the allowed range for \( \alpha \) and the fact that the allowed regions may be bounded by values for free dynamo waves and for dynamos with boundaries, one can easily find out the allowed region for a dynamo in a 10,000 km layer in the \( aG \) plot in Figure 1b. It is to be noted that all the parameters \( \alpha, G, \eta \), and \( \eta \) are restricted to within about a factor of 10 or so. For illustrative purposes, consider the values

\[ \alpha \sim 10 \, \text{cm s}^{-1}, \]
\[ \eta \sim 10^{10} \, \text{cm}^{2} \text{s}^{-1}, \]

lying somewhere in the middle of the range. Using equations (38) and (39), we find that these values correspond to

\[ l \sim 300 \, \text{km}, \]
\[ \nu \sim 30 \, \text{m s}^{-1}. \]

The length scale is certainly small compared to the different scale heights and the depth of the overshoot layer. Most numerical simulations of compressible convection aim at studying the large-scale motions and do not throw much light on the possibility of small-scale turbulence due to the limitation of the finite grid size (Gilman 1986). So we know virtually nothing about the small-scale turbulence in the solar convection zone. We may note that the recent calculations of Choudhuri and D'Silva (1990) also hint at the existence of turbulence at scales of hundreds of kilometers at the bottom of the convection zones. It seems that such turbulence is essential for suppressing the Coriolis force felt by the rising magnetic flux in the convection zone. A turbulent velocity of 30 m s\(^{-1}\) is not unreasonable, especially since it corresponds to an equipartition magnetic field of 5000 G. However, one normally thinks of such velocities associated with large-scale convective motions. If the turbulence obeys the Kolmogorov spectrum, one would expect the velocities associated with the small-scale turbulence to be smaller. One can question whether it is justified to combine a small length scale of 300 km with a turbulent velocity of 30 m s\(^{-1}\) normally expected to be associated with larger scales. We do not know the answer to this question. We merely point out that if we make the solar dynamo operate in a thin layer of 10,000 km thickness, then we are forced to these conclusions.

The aim of this paper was to explore what may be involved if we want the solar dynamo to operate in a thin layer. We have not tried to build any detailed model. We have taken the dynamo region to be rectangular. Perhaps a much worse approximation was to take all the dynamo coefficients constant within the dynamo layer. It is well known that the \( \alpha \)-coefficient has opposite signs in the two hemispheres leading to dynamo waves which interfere at the equator. Using the helioseismology data presented by Brown et al. (1989), Gilman, Morrow, and DeLuca (1989) point out that the shear also varies considerably with latitude. In fact, they claim that the shear has the right sign only from 0° to 30° latitudes, suggesting that the dynamo action may be confined to these low latitudes. A more realistic model of the solar dynamo certainly has to take account of these facts.

We refrain from giving a very categorical answer to the question whether everything is okay for a dynamo that has to operate in a thin layer of 10,000 km and gives rise to all the observed properties. We have seen that such a dynamo has to be of the \( \alpha \times \omega \) type. Using the results for free dynamo waves and for dynamos with boundaries, we have been able to restrict the values that different dynamo parameters may have to within about a factor of 10. Mixing length arguments based on these values suggest reasonable turbulent velocities, but rather small length scales. Another property of an \( \alpha \times \omega \) dynamo is that the poloidal field is no longer negligible as in the \( \alpha \omega \) dynamo. Since the allowed parameter space more or less coincides with the region where the ratio of the poloidal to the toroidal field changes rather quickly, it is difficult to make definitive predictions about the magnitude of the poloidal field. It is probably not insignificant, but still not as large as the toroidal field. This poloidal field is not expected to leak beyond the lower layers of the convection zone. We also seem to conclude that the shear term giving rise to the periodicity may hover around being comparable to the other terms in our equation, suggesting a possible connection with the observation that the periodicity of the solar dynamo may be on the brink of being chaotic (Feynman and Gabriel 1989). Until we have a deeper understanding of turbulence and the properties of convective overshooting, it may not be scientifically justified to draw firmer conclusions.

This work was done while I was on leave from the Indian Institute of Science for a summer. I am grateful to Gene Parker for inviting me to spend the summer in Chicago and for supporting my stay. Apart from the economic aspects of the support, several stimulating conversations with Gene during the course of this work influenced my mode of thinking substantially. It was Gene who first brought the work of Feynman...
and Gabriel (1989) to my attention and suggested the possible connection between their work and mine. I learned a lot from discussions with Peter Gilman and Ed DeLuca, although my points of view often did not agree with theirs. Finally, my thanks go to Arieh Konigl and C.-M. Ko for providing moral support by their interest in this work. This work was supported in part by the National Aeronautics and Space Administration under grant NGL 14-001-001.

APPENDIX

THE TOP BOUNDARY CONDITIONS AND THE LEAKAGE OF POLOIDAL FIELD

We have pointed out that it is difficult to find the exact top boundary conditions in our problem. We have presented calculations using the same impervious boundary conditions which were used by DeLuca and Gilman (1986). These conditions made the top and bottom boundaries symmetric, which enabled us to split the solutions in the even and odd modes so that the dispersion relation could be obtained by setting a $2 \times 2$ determinant equal to zero. For asymmetric boundary conditions, such splitting of modes will not be possible, and one would get a much more complicated dispersion relation from a $4 \times 4$ determinant. Since we are interested mainly in the orders of magnitude of various quantities, we have not considered it worthwhile to pursue such calculations. However, calculations with asymmetric boundary conditions for the considerably simpler $\omega\omega$ dynamo problem was presented in Choudhuri (1984), and we have compared in § V our present results in the $\omega\omega$ limit with the results of Choudhuri (1984).

Our boundary condition (10) would imply that the poloidal field does not leak at all outside the dynamo region. On the other hand, Choudhuri (1984) matched the poloidal field with an outside potential field corresponding to a situation of a free leakage of the poloidal field. The actual situation is something intermediate between the two, where the region above the top boundary has resistivity comparable to the dynamo region, which should allow the poloidal field to leak to some extent. Let us show that this leaked poloidal field should fall off with a scale height much smaller than the depth of the convection zone and hence should not remain significant up to the solar surface. Since there is no source term in the region above the dynamo layer, we expect from equation (4) that $A_y$ will satisfy

$$ \left( \frac{\partial}{\partial t} - \eta \nabla^2 \right) A_y = 0, $$

in that region. For a solution of the form $\exp \left[ i(\omega t - k_x x - k_z z) \right]$, we have

$$ \left( \frac{k_x}{k_z} \right)^2 = -1 - i\Omega, $$

where $\Omega$ becomes the same dimensionless number we introduced in § IV if we take the diffusivity $\eta$ to be the same as the diffusivity in the dynamo layer. For a thin layer, $\Omega$ is much larger than 1, so that

$$ \left( \frac{k_x}{k_z} \right)^2 \approx -i \Omega. $$

The scale height of the falling of the poloidal field is now given by

$$ \lambda = \left| \frac{1}{\text{Im} (k_z)} \right| \approx \frac{1}{k_z} \sqrt{\frac{2}{\Omega}}. $$

To have a rough estimate, we can use the value $\Omega = 38$ obtained in our calculations with the impervious top boundary, though strictly speaking, this is not self-consistent. Taking the value of $k_z$ from equation (7), the scale height $\lambda$ turns out to be about 26,000 km, which is almost an order of magnitude less than the depth of the convection zone in the standard models. Hence, the poloidal field created in the overshoot region can leak appreciably only to the lower layers of the convection zone and should fall off by several orders of magnitude before reaching the photosphere. In fact, with the poloidal field falling so rapidly, one may now face the opposite problem of how to account for the weak poloidal field that is observed on the solar surface. It should be noted that the diffusivity in the higher layers of the convection zone is much higher, and there the poloidal field falls off more slowly. So our theoretical considerations seem to be consistent with the observations of weak poloidal fields on the photospheric surface.

The actual boundary condition for the poloidal field at the top surface should be something intermediate between forcing $A_y$ to go to zero and allowing it to match a potential field outside. As for the toroidal field, we see in Figure 3 that it does not vary much near the upper surface in cases S and I, in which the boundary was taken to be impervious. On the other hand, $B_y$ is made to go to zero in case F. Which of these situations come closer to reality? We know that any azimuthal field that enters the convection zone above the stable dynamo layer will start rising due to magnetic buoyancy. It is difficult to figure out how that will influence the azimuthal field in the layers immediately underlying. Choudhuri (1989) presents numerical simulations of the evolution of flux tubes which have parts anchored in the stable layers and parts within the convective zone. Although the parts within the convection zone start rising rapidly, the parts within the stable layers remain relatively unaffected. If the azimuthal field in the upper layers remained totally unaffected by the loss of flux from the layers immediately above, then an impervious boundary condition would have been correct. However, the azimuthal field in the top layers should feel some affect of this flux loss, and, consequently, we may expect a reasonable dip in the profile of $B_y$ in the topmost layers of the dynamo region.
How do we now deal with this complex situation? In view of the many uncertainties present, we can only hope to arrive at rough estimates of various dynamo parameters. So we work out the results for the free dynamo wave which involves a fairly variable profile of $B_y$ and results for the symmetric boundary conditions case which involves a rather flat profile of $B_y$. If we find out the values of different quantities for these two limiting cases, we may expect the real values to lie within these limits. However, if we look at the values of $\eta$ and $\pi G$ in Table 3, we find that the value of $\eta$ in case I happens to lie outside the limits set by S and W. If one were to change the top boundary condition for $B_y$ in the I case in a suitable fashion so as to produce a reasonable dip such as we expect in reality, one would hope that $\eta$ may again decrease and may not be too much out of our limits. Hence, if we use the free dynamo wave and the dynamo with symmetric impervious boundaries to limit our parameter space, it is highly unlikely that we shall arrive at drastically wrong conclusions.

REFERENCES


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