VERTICAL STRUCTURE OF ACCRETION DISKS: A SIMPLIFIED ANALYTICAL MODEL

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ABSTRACT

A simplified model of the vertical structure of accretion disks is derived. Analytical expressions for the temperature and density structure, which represent a generalization of the gray model long known in the theory of classical stellar atmospheres, are presented. The formalism naturally explains similarities and differences between the structure of a disk and a stellar atmosphere. In particular, the influence of viscous dissipation and external irradiation of the disk by the central star, as well as of the finite optical thickness of the disk, may be easily accounted for and explained by the present model.

Subject headings: radiative transfer — stars: accretion — stars: atmospheres

1. INTRODUCTION

The knowledge of the vertical structure of accretion disks is a crucial factor that determines the quality of corresponding spectroscopic diagnostics. Most studies assume that the radial structure is given by the canonical model (Shakura and Sunyaev 1973; Lynden-Bell and Pringle 1974; Pringle 1981; for a recent review see Frank, King, and Raine 1985), which assumes cylindrically symmetric, stationary Keplerian disk. In contrast, the vertical structure is treated with a varying degree of sophistication.

The simplest method is to avoid calculation of vertical structure completely: the theoretical emergent spectrum is calculated either as a superposition of blackbody fluxes corresponding to the local effective temperature given by the canonical model (Lynden-Bell 1969), or by using model stellar atmosphere fluxes instead of blackbody fluxes (Mayo, Wickramasinghe, and Whelan 1980; Wade 1984, 1988). Obviously, both these approaches are valuable in the historical context, for they were able to provide the zero-order estimates of the disk radiation, but at present any serious attempt to derive physical properties of disks from their radiation must be based on a more refined physical description.

The next category of simple approaches comprises models constructed assuming optically thin, vertically homogeneous disks. The emergent spectrum is again calculated rather easily, because the optically thin approximation enables one to write analytical expressions for the emergent flux (Williams 1980; Tylenda 1981; Williams and Ferguson 1982; Williams and Shipman 1988). The obvious drawback of these methods is that a disk need not generally be optically thin.

There are several studies that calculate the vertical structure. Some of them do not treat the radiation field in detail, employing basically the diffusion approximation for radiation transfer (Meyer and Meyer-Hofmeister 1982, 1983; Cannizzo and Wheeler 1984; Cannizzo and Cameron 1988). Although these models have contributed enormously to our understanding of the formation and evolution of accretion disks, they are still not satisfactory from the point of view of spectroscopic diagnostics. The reason for this is that the diffusion approximation provides an acceptable description of the radiation field only at large optical depths, but the emergent radiation field originates at optical depths of the order of unity, where not only departures from the diffusion approximation but also departures from the local thermodynamic equilibrium (LTE) may be, and in many case actually are, very important.

Accurate spectroscopic diagnostics of accretion disks requires models of vertical structure, with self-consistent treatment of radiative transfer. Some preliminary and still very simplified models of this kind have already been published by Kift\(^2\) and Hubeny (1986), Shaviv and Wehrse (1986), and by Adam et al. (1988). Although these models certainly represent an improvement or at least a potential improvement in the disk spectroscopic diagnostics, the situation is still very far from being satisfactory. First, more models covering a wide range of input parameters should be calculated; second, and perhaps more importantly, all the existing models are only purely numerical solutions of a complicated set of coupled equations, which by themselves do not help us to achieve a real physical understanding of the structure of accretion disks.

Therefore, as in the classical stellar atmospheres, a gray model, which is constructed using some appropriately defined frequency-average opacities, may serve as a guide to the general nature of the results to be expected from more realistic calculations. Also, the gray model serves as an excellent starting approximation for a subsequent iterative method, such as, for instance, the complete linearization. Gray models of accretion disks were constructed by Kift\(^2\) and Hubeny (1986), who used them as starting solutions for their method, by Shaviv and Wehrse (1986), and by Adam et al. (1988).

All these authors solved the gray problem numerically. Nevertheless, it is possible to derive some simple analytic expressions that represent a generalization of corresponding classical stellar atmospheric results. This is the issue of the present paper.

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II. BASIC EQUATIONS

The cylindrically symmetric, stationary Keplerian disk is divided into a set of concentric rings, assuming that each ring behaves like an independent plane-parallel radiating slab. This reduces the basically three-dimensional problem to a set of one-dimensional problems.

The general equations governing the vertical structure of individual rings are the following: assuming that the radial component of gravity of the central star is balanced by the centrifugal force of the Keplerian rotation, and neglecting self-gravity of the disk, one may write the equation of vertical hydrostatic equilibrium as

$$\frac{dP}{dz} = -g(z)\rho ,$$

where $z$ is the geometrical distance from the central plane, $P$ and $\rho$ are the total pressure and density, respectively, and

$$g(z) = Qz , \quad Q = \frac{Gm^*}{R^3} ,$$

is the effective vertical gravity. $G$ is the gravitational constant, $m^*$ is the mass of the central star, and $R$ is the distance of the given disk ring from the central star. Here the neglected terms are of the order of $(z/R)^2$. In contrast to stellar atmospheres, the gravity now depends on the vertical coordinate $z$.

It is advantageous to introduce the mass-depth variable

$$m(z) = \int_{z}^{\infty} \rho \, dz .$$

The total column mass, $M = m(0)$, is related to the usual surface density $\Sigma$ through $M = \Sigma/2$. Equation (2.1) may then be simply written as $dP/dm = g$, but due to the dependence of $g$ on $z$, no simple integral of this equation can be written.

The canonical model gives for $M$ the following expression:

$$M = \dot{m}_* \left( 6\pi \bar{\nu} \right) \left[ 1 - \left( \frac{R_*}{R} \right)^{1/2} \right] ,$$

where $\dot{m}_*$ is the mass flux through the disk, $R_*$ is the radius of the central star, and $\bar{\nu}$ is the depth-averaged kinematic viscosity. In this paper, we consider the average viscosity to be given by (see Lynden-Bell and Pringle 1974; Williams and Ferguson 1982; Kflı and Hubeny 1986)

$$\bar{\nu} = \left( \frac{Gm^*}{R} \right)^{1/2} R e ,$$

where $Re$, the effective Reynolds number of the accretion flow, is assumed to be the depth-independent input parameter of the model.

The second basic equation is the energy balance equation,

$$D_{\text{mech}} = D_{\text{rad}} ,$$

which states that the energy dissipated in unit volume, $D_{\text{mech}}$, is equal to the net radiation loss per unit volume. Assuming that energy is released through the radial shear of the Keplerian motion, the former is given by

$$D_{\text{mech}} = (9/4)Qw\rho ,$$

where $w$ is the (generally $z$-dependent) kinematic viscosity. The net radiation loss is generally given by

$$D_{\text{rad}} = 4\pi \int_{0}^{\infty} [\eta(v, z) - \chi(v, z)J(v, z)] dv ,$$

where $\eta$ and $\chi$ are the monochromatic emission and absorption coefficients, respectively, and $J$ is the mean intensity of radiation.

In fact, equation (2.6) neglects convective energy transport; i.e., it is valid only in the convectively stable layers. In the convectively unstable layers, equation (2.6) should be replaced by

$$D_{\text{mech}} = D_{\text{rad}} + D_{\text{conv}} ,$$

where $D_{\text{conv}} = dF_{\text{conv}}/dz$, $F_{\text{conv}}$ being the vertical convective flux. In this paper, we will neglect convection and consider the energy balance in the form of equation (2.6).

Since the radiation enters the energy balance equation explicitly, the third basic equation of the problem is the radiative transfer equation, written as

$$\frac{\partial I(v, \mu, z)}{\partial z} = -\chi(v, z)I(v, \mu, z) + \eta(v, z) ,$$

where $I(v, \mu, z)$ is the specific intensity of radiation at frequency $v$, $\mu$ is the cosine of the polar angle.

The corresponding boundary conditions are as follows: the lower boundary condition expresses the symmetry of the disk with respect to the central plane,

$$I(v, \mu, m = M) = I(v, -\mu, m = M) .$$

The upper boundary condition reads

$$I(v, -\mu, m = 0) = I^{\text{ext}}(v, \mu) , \quad \mu > 0 .$$
where $I^{\text{ext}}$ is a prescribed incident (generally angle-dependent) intensity of external radiation. One usually assumes $I^{\text{ext}} = 0$, but in some cases there may be an appreciable external irradiation of the disk by the central star; this case will be briefly discussed in § IIId.

It should be noted that in the case of nonzero external irradiation, the intensity of radiation in the disk generally depends not only on the polar angle $\mu$, but also on the azimuthal angle $\phi$. The radiation intensity in the transfer equation (2.9) and its boundary conditions (2.10)-(2.11) has then to be understood as $I(v, \mu, \phi) = \int_0^{2\pi} I(v, \mu, \phi) d\phi/2\pi$.

These equations are complemented by definition equations for the absorption and emission coefficients, which in turn require the corresponding equations for the relevant atomic level populations. The appropriate equations are precisely the same as in the stellar atmospheres theory (Mihalas 1978), and they will not be described here.

The first and second moments of equation (2.9) are

$$
\frac{dH}{dm} = \frac{\chi_s}{\rho} \left( J_s - S_s \right), 
$$

(2.12)

$$
\frac{dK}{dm} = \frac{\chi_s}{\rho} H_s, 
$$

(2.13)

where $J_s, H_s, K_s$ are the usual moments of the specific intensity of radiation (Mihalas 1978). The source function $S_s$ is given in LTE by

$$
S_s \equiv \eta_s/\chi_s = \left( \kappa_s B_s + \sigma_s J_s \right) / \left( \kappa_s + \sigma_s \right),
$$

(2.14)

where $\kappa_s$ and $\sigma_s$ are the coefficients of true absorption and scattering (per unit volume), respectively, and $B_s$ is the Planck function.

The frequency-integrated moment equations (2.12) and (2.13) may be written as

$$
\frac{dH}{dm} = \kappa_J J - \kappa_B B, 
$$

(2.15)

$$
\frac{dK}{dm} = \kappa_H H, 
$$

(2.16)

where $J, H, K$ are the frequency-integrated moments $J_s, H_s, K_s$ of the specific intensity (i.e., for instance, $J = \int_0^\infty J_s dv$), $B = (\sigma/\pi)T^4$ is the frequency-integrated Planck function; $\sigma$ being the Stefan-Boltzmann constant. Further,

$$
\kappa_J = \int_0^\infty (\kappa_s/\rho) J_s dv/J, 
$$

(2.17)

$$
\kappa_B = \int_0^\infty (\kappa_s/\rho) B_s dv/B, 
$$

(2.18)

$$
\kappa_H = \int_0^\infty (\kappa_s/\rho) H_s dv/H, 
$$

(2.19)

are the usual absorption mean, Planck mean, and flux mean opacity, respectively (Mihalas 1978). All mean opacities are defined here as opacities per unit mass. Notice also that while the absorption and Planck means are defined through the true absorption coefficient, the flux mean is defined through the total extinction (absorption + scattering) coefficient.

The energy equation (2.6), using equations (2.7), (2.8), and (2.12), may be written

$$
\frac{dH}{dm} = -E\omega(m), 
$$

(2.20)

with

$$
E = (9/16\pi)Q, 
$$

(2.21)

where we indicate that viscosity is generally depth dependent. Note that in the case of classical stellar atmospheres, equation (2.20) simply reads $dH/dm = 0$. In other words, the total vertical radiation flux is not conserved in the accretion disks, in contrast to classical stellar atmospheres. The solution of equation (2.20) is

$$
H(m) = EM\bar{\omega}[1 - \theta(m)], 
$$

(2.22)

where

$$
\bar{\omega} = \int_0^M \omega(m) dm/M, 
$$

(2.23)

is the mean (depth-averaged) viscosity, and

$$
\theta(m) = \int_0^m \omega(m') dm'/\bar{\omega}M, 
$$

(2.24)

is a monotonically increasing function between $\theta(0) = 0$ and $\theta(M) = 1$; in the case of depth-independent viscosity $\theta(m) = (m/M)$.
III. TEMPERATURE STRUCTURE

a) Formal LTE Solution

From equations (2.15) and (2.20), it immediately follows that

$$B_m = \left[ \nu_f(m) J(m) + Ew(m)/\kappa_{\nu_f}(m) \right]. \tag{3.1}$$

The integrated mean intensity $J$ is given by the solution of equation (2.13), written here as

$$f_{\nu_f}(m) J(m) = f_{\nu_f}(0) J(0) + \int_0^m \frac{1}{\kappa_{\nu_f}(m')} dm', \tag{3.2}$$

where we have introduced the (global) Eddington factor,

$$f_{\nu_f} = \frac{K}{J}, \tag{3.3}$$

the optical depth associated with flux-mean opacity,

$$\tau_{\nu_f}(m) = \int_0^m \frac{\kappa_{\nu_f}(m') dm'}{J}, \tag{3.4}$$

and the auxiliary quantity

$$\tau_{\nu_f}(m) = \int_0^m \frac{\kappa_{\nu_f}(m') dm'}{J}, \tag{3.5}$$

which may be called the "viscosity-weighted" flux-mean optical depth. The last step is to express the surface mean intensity through known functions. The usual method is to introduce the second Eddington factor,

$$f_{\nu_f} = H(0)/J(0), \tag{3.6}$$

and to estimate its value using first the Eddington approximation $f_{\nu_f} = 1/\sqrt{3}$, and then to recalculate it by solving the transfer equation. Although this is formally correct for any $I^{ext}$ in the upper boundary condition (2.11), it is practically useful only for the case of no incident radiation $I^{ext} = 0$. The general case will be considered in § III.d.

Defining the effective temperature of the given ring of a disk as

$$T_{eff}^4 = \frac{\sigma}{4\pi} \frac{J}{T_{eff}}, \quad \text{i.e.,} \quad T_{eff} = \left( 4\pi EMw/\sigma \right)^{1/4}, \tag{3.7}$$

and using equations (3.1)-(3.7), we get

$$T^4(m) = \frac{3}{4} T_{eff}^4 \left[ \frac{\gamma_J(m)}{\gamma_H} \left[ \tau_{\nu_f}(m) - \tau_{\nu_f}(m) + \frac{2\mu}{\sqrt{3}} \right] + \frac{1}{3M\kappa_{\nu_f}(m)} \frac{w(m)}{w} \right], \tag{3.8}$$

where $\gamma_J$ and $\gamma_H$ are factors of the order of unity, defined by

$$\gamma_J = \frac{\kappa_J(3\kappa_{\nu_f} f_{\nu_f})}{\gamma_H} \tag{3.9}$$

and

$$\gamma_H = \sqrt{3f_{\nu_f}(0)/f_{\nu_f}}. \tag{3.10}$$

b) LTE Gray Solution

So far, no approximations except those of LTE have been made. Equation (3.8) is then an essentially exact LTE vertical temperature distribution in a disk. However, equation (3.8) represents only a formal solution, because the quantities $\gamma_J$, $\gamma_H$, $\tau_{\nu_f}$, and $\kappa_{\nu_f}$ are not a priori known.

In order to provide a deeper physical insight, we invoke several further approximations:

1. We put $\gamma_J = \gamma_H = 1$, which follows from two separate approximations, namely the Eddington approximation $f_{\nu_f} = 1/\sqrt{3}$, and the equality $\kappa_{\nu_f} = \kappa_{\nu_f}$, which is usually a good approximation (Mihalas 1978). Notice, however, that the equality $\gamma_H = 1$ is a good approximation only in the case of weak or no incident radiation (see § III.d).

2. We assume that $\tau_{\nu_f} = \tau_{\nu_f}$, being the Rosseland mean opacity. Next, we write $\kappa_{\nu_f} = \kappa_{\nu_f}/\tau_{\nu_f}$, where $\epsilon$ may roughly be understood as the average of $\kappa_{\nu_f}/\tau_{\nu_f}$.

3. Finally, we assume that the mean opacities and viscosity vary slowly with $m$, which yields $\tau_{\nu_f} \approx \left( \frac{1}{2} \right) \tau_{\nu_f}/\tau_{\nu_f}$ and $M\kappa_{\nu_f} \approx \epsilon_{\nu_f}$, where $\tau_{\nu_f}$ is the total Rosseland optical depth at the disk midplane (i.e., the total optical thickness of the disk is $2\tau_{\nu_f}$).

Equation (3.8) then reads (writing $\tau$ for $\tau_{\nu_f}$)

$$T^4 = \frac{3}{4} T_{eff}^4 \left[ \frac{1 - \frac{\tau}{2\tau_{\nu_f}}}{\sqrt{3}} + \frac{1}{3M\tau_{\nu_f}} \frac{w(m)}{w} \right]. \tag{3.11}$$

This result is a generalization of the well-known gray temperature distribution for stellar atmospheres and reveals several interesting features.

Consider first the case $\tau_{\nu_f} \gg 1$, i.e., the optically thick ring. Then for $1 \leq \tau \leq \tau_{\nu_f}$, the factor $\epsilon$ is likely to be of the order of unity. The last term of equation (3.11) is then negligible, and one obtains

$$T^4 = \frac{3}{4} T_{eff}^4 (1 + \sqrt{3}) \quad \text{for} \quad \tau > 1, \quad \tau_{\nu_f} \gg 1. \tag{3.12}$$
The Planck mean opacity is determined predominantly by frequency regions of high \( \kappa \), such as \( \text{H I}, \text{Mg I}, \text{C} \), \( \text{Al m}, \text{Si iv}, \text{C iv}, \text{N v}, \text{O vi} \), etc., where each line is an efficient coolant for a certain temperature range, as seen and Wehrse. These layers may in fact emit a considerable amount of radiation energy in strong resonance lines of abundant species, in contrast to stellar atmospheres. This phenomenon is important for moderately optically thick disks, because for very large optical thickness the regions \( \tau \approx \tau_{\text{tot}} \gg 1 \) practically do not influence the emergent radiation and hence are not very interesting from the point of view of spectroscopic diagnostics.

In the limit of a very small total optical depth, \( \tau_{\text{tot}} \ll 1 \), we get

\[
T = T_{\text{eff}} \left[ \frac{1}{\sqrt{3}} + \frac{w(\tau)}{4\tau_{\text{tot}} w} \right]^{1/4} \tag{3.13}
\]

which reveals two interesting facts. First, the local temperature may be much higher than the effective temperature and, second, the local temperature need not necessarily be constant, as it is usually assumed, owing to the presence of the terms \( \epsilon \) and \( w(\tau)/w \), which may depend strongly on depth.

### c) Surface Temperature

Letting \( \tau \to 0 \), and assuming first a depth-independent viscosity, we obtain from equation (3.11)

\[
T(0) = T_{\text{eff}} \left[ \sqrt{\frac{3}{4}} + \frac{1}{4(\epsilon \tau_{\text{tot}})} \right]^{1/4} . \tag{3.14}
\]

Since \( \tau_{\text{tot}} \) is basically given by the global properties of the disk (above all, by the mass accretion rate \( \dot{m} \) and the distance from the central star \( R \)) and is thus fixed for a given disk ring, the crucial factor that determines the surface temperature is \( \epsilon \).

At first sight, one may argue that \( \epsilon \) goes to zero for \( \tau \to 0 \). Indeed, \( \kappa \) decreases with \( \tau \) more quickly than does \( \sigma \), because \( \sigma \) is proportional to \( n_e \), the electron density, while \( \kappa \) is roughly proportional to \( n^2 \) in LTE. The total opacity is then more and more dominated by electron scattering, \( \epsilon \) goes to zero, and consequently the temperature increases indefinitely! Physically, this temperature increase is caused by the fact that energy is dissipated at all depths (recall the assumption of depth-independent viscosity), but the gas at small optical depths predominantly scatters; to a much lesser extent, it absorbs or emits radiation, so it possesses no efficient mechanism to reradiate the dissipated energy.

This is precisely the argument raised by Shaviv and Wehrse (1986) and Adam et al. (1988), although they used somewhat different language. They also suggested a way to avoid an indefinite temperature rise, which can be easily seen in our formalism: the critical last term of equation (3.11) contains \( w/\dot{w} \), so that letting \( w(m)/\dot{w} \) decrease with \( m \) more rapidly than \( \epsilon \) prevents the "thermal catastrophe" of the disk.

However, the argument of Shaviv and Wehrse and Adam et al. is not quite correct. The crucial point is that the intuitive interpretation of \( \epsilon \) as an average of the ratio of pure absorption to total extinction is misleading. The correct definition of \( \epsilon \) follows from equation (13), namely, that \( \epsilon = \kappa_\nu M/M_{\text{tot}} \). The Planck mean opacity is determined predominantly by frequency regions of high opacity, i.e., by strong resonance lines. Physically, this means that regions with \( \tau \ll 1 \) are not such poor emitters, as argued by Shaviv and Wehrse. These layers may in fact emit a considerable amount of radiation energy in strong resonance lines of abundant species, such as \( \text{H I}, \text{Mg II}, \text{C II}, \text{Al m}, \text{Si iv}, \text{C iv}, \text{N v}, \text{O vi} \), etc., where each line is an efficient coolant for a certain temperature range, as seen in the case of normal stellar chromospheres and coronae.

The above considerations were meant as a simple demonstration that the temperature need not necessarily increase to infinity for models with depth-independent viscosity. It is quite clear that in the superficial layers, the LTE approximation is absolutely inadequate and should be replaced by a more realistic NLTE approach. In any case, the question remains whether the line cooling will be efficient enough to remove the thermal instability itself, or whether a decrease of viscosity with depth will still be needed. This can be solved only by future detailed calculations.

### d) Influence of External Irradiation

In the case where the disk is irradiated by an external radiation, the temperature structure given by equation (3.8) is formally correct, but the equality \( \gamma_H \approx 1 \), used in §§ IIIb and IIIc, may no longer be a useful approximation. To be able to deal with this situation, let us first slightly modify the two-stream version of the Eddington approximation. Let us assume that \( I(\mu) = I^* \) for \( \mu > 0 \), and that \( I(\mu) = I^- \) for \( -\mu_0 \leq \mu \leq 0 \). This approximation is relevant for the case of a disk irradiated by the central star; \( \mu_0 \) is the cosine of the characteristic angle, which is roughly given by \( n/2 \) minus the angle by which the radius of the central star is seen from the particular ring of the disk. Notice that the standard two-stream approximation corresponds to \( \mu_0 = 1 \) (see also Hummer 1982). The first three moments of the specific intensity now read

\[
J = (I^* + \mu_0 I^-)/2 , \quad H = (I^* - \mu_0^2 I^-)/4 , \quad K = (I^* + \mu_0^2 I^-)/6 , \tag{3.15}
\]

To obtain the appropriate value of \( \mu_0 \), we write at the surface

\[
\frac{1}{2} \mu_0 I^- = J_{\text{ext}} \equiv \frac{1}{2} \int_0^1 dv \int_0^1 d\mu \int_0^{2\pi} d\phi \mu I_{\text{ext}}(v, \mu, \phi)/2\pi , \tag{3.16}
\]

\[
\frac{1}{2} \mu_0^2 I^- = H_{\text{ext}} \equiv \frac{1}{2} \int_0^1 dv \int_0^1 d\mu \int_0^{2\pi} d\phi \mu I_{\text{ext}}(v, \mu, \phi)/2\pi , \tag{3.17}
\]
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i.e., the most suitable value of \( \mu_0 \) is given by

\[
\mu_0 = 2H_{\text{ext}}/J_{\text{ext}}.
\]

Substituting equations (3.16) and (3.17) into equation (3.15), we obtain at the surface

\[
J(0) = 2H(0) + \bar{\mu}_0 J_{\text{ext}}, \quad \text{with} \quad \bar{\mu}_0 = 1 + \mu_0.
\]

If we now define a modified second Eddington factor as

\[
f_H = H(0)/[J(0) - \bar{\mu}_0 J_{\text{ext}}],
\]

then a good approximation is \( f_H = \frac{1}{3} \), as follows from equation (3.19). This is slightly different from the usual value \( f_H = 1/\sqrt{3} \), which is caused by the fact that we have used the two-stream approximation in the most straightforward version, instead of the so-called "\( \sqrt{3} \)" version. The latter would yield for \( f_H \) defined by equation (3.20) again \( f_H \approx 1/\sqrt{3} \).

Substituting equations (3.16)-(3.20) into equations (3.2) and (3.1), we obtain an equation quite analogous to equation (3.8); the only modification consists in the replacement

\[
\gamma_H/\sqrt{3} \rightarrow \gamma_H/\sqrt{3} + 4\pi f(x)\bar{\mu}_0 J_{\text{ext}}/(\sigma T_{\text{eff}}^4).
\]

Putting for illustration

\[
\bar{\mu}_0 J_{\text{ext}} = WB(T_*),
\]

where \( T_* \) is the effective temperature of the central star and \( W \) is the dilution factor (a function of the radius of the central star, the distance from the central star, and generally of the shape of the disk), we finally obtain an interesting expression:

\[
T^4(m) = \frac{3}{4} T_{\text{eff}}^4 \left\{ \gamma_H \frac{1}{\sqrt{3}} + \gamma_H/\sqrt{3} + 4\pi f(x) W \left( T_* / T_{\text{eff}} \right)^4 \right\} + \frac{1}{3M K_B w} \int \frac{w(m)}{T_*},
\]

which represents a generalization of equation (3.8). In contrast to other studies that consider an irradiation of the disk by the central star (Friedjung 1985; Adams and Shu 1986; Kenyon and Hartmann 1987), the present one shows explicitly that the effect of irradiation depends not only on the radial distance from the central star, but also on the vertical distance from the central plane.

Indeed, if \( W \ll 1 \) and \( T_{\text{eff}} \approx T_* \), the additional term is unimportant, and the temperature distribution in the disk is practically unaffected by the external irradiation. If, in contrast, \( (T/J_{\text{eff}})^4 \ll 1 \), the irradiation is important. In the extreme case, the overall temperature distribution is dominated by external radiation, and we obtain from equation (3.23)

\[
T \approx W^{1/4} T_* \quad \text{for} \quad \tau < W(T_*/T_{\text{eff}})^4,
\]

i.e., the local temperature in the disk is influenced everywhere from the surface down to optical depths which depend on the dilution factor (and hence on the radial distance from the central star) and on the effective temperature of both the disk and the central star.

Finally, let us consider, for illustration, the simplest case of an irradiation of a geometrically very thin disk by the central star. The star radiates as a blackbody with no limb darkening. Adopting the same model as Friedjung (1985; the reader is referred to this paper for more details of the algebra involved), we obtain

\[
J_{\text{ext}} = \int_{0}^{\sin^{-1} y} \sin \phi d\phi \int_{\pi/2}^{\pi/2} d\psi I^* = \frac{I^*}{4 \pi} \left[ 1 - (1 - y^2)^{1/2} \right],
\]

\[
H_{\text{ext}} = \int_{0}^{\sin^{-1} y} \sin^2 \phi d\phi \int_{\pi/2}^{\pi/2} d\psi I^* = \frac{I^*}{8} \left[ \sin^{-1} y - y(1 - y^2)^{1/2} \right],
\]

where \( I^* = B(T_*) \) is the frequency-integrated intensity of the stellar radiation, \( \psi \) is the azimuthal angle on the stellar surface, and

\[
y = R_*/R.
\]

For large distances from the central star, \( y \) is small, \( \sin^{-1} y \approx y \approx y^3/6 \), and then

\[
J_{\text{ext}} \approx (I^*/8) y^2, \quad H_{\text{ext}} \approx (I^*/12) y^3,
\]

and then

\[
\mu_0 \approx \left( \frac{3}{5} \right) y, \quad W \approx y^2/8.
\]

This equation, together with equation (3.24), shows that the actual temperature in the disk at small optical depths \( \tau < W(T_*/T_{\text{eff}})^4 \), depends on the radial distance as \( R^{-1/2} \). At first sight, this result might seem surprising, because other studies of the effects of external irradiation (Friedjung 1985; Kenyon and Hartmann 1987) conclude that the disk temperature in the case of a hot central star depends on the radial distance as \( R^{-3/4} \). However, this discrepancy may be easily explained. The present approach determines the actual temperature in the disk, which has no immediate connection to the actual amount of flux radiated by the disk, while the above authors determined the effective temperature, which only measures the total flux but has no immediate connection to the actual temperature within the disk. The implicit assumption underlying both studies mentioned above is that all flux incident on the disk surface is absorbed by the disk material and then reradiated away. Another implicit assumption is that the disk material is heated homogeneously at all depths.
In contrast, the present approach avoids these assumptions and determines explicitly how the disk material is heated locally by the external irradiation. It shows that the disk is heated more at the surface where the material is fully exposed to the external radiation and less and less toward greater optical depths. However, the present approach does not address the question of what is the modified effective temperature, i.e., the total radiation flux (intrinsic + reradiated) emitted by the disk. This may be done easily by solving the transfer equation numerically. Finally, it should be stressed that the present simple approach, equation (3.23), breaks down for the case of an optically very thin disk, because the Eddington factor $f_k$ is then also modified. This case would require a more detailed study and will be considered in a separate paper.

The reason why the actual temperature at the exposed layers is proportional to $R^{-1/2}$ is that the quantity which actually matters in the energy balance equation is the mean intensity (i.e., the radiation energy density; proportional to $R^{-2}$) rather than the flux (proportional to $R^{-3}$). This is easily seen from equation (3.1): the Planck function, and hence the local temperature, is given through the mean intensity. In other words, the energy balance equation has to be understood either as an equality involving the absolute value of the mean intensity (eqs. [2.6]–[2.8]), or as a derivative of the flux.

IV. HYDROSTATIC EQUILIBRIUM

a) Isothermal Density Structure

Solution of the hydrostatic equilibrium equation (2.1) is more complicated for disks than for stellar atmospheres, basically because of depth-dependent gravity acceleration. Nevertheless, it is possible to derive a simple approximate analytical model which gives a physical insight and yields a very good starting approximation for the subsequent iteration procedure.

Neglecting turbulent pressure, the total pressure $P = P_g + P_r$, where the gas pressure $P_g$ is written as

$$P_g = c_g^2 \rho,$$  \hspace{1cm} (4.1)

and $P_r$ is the radiation pressure, $P_r = (4\pi/c)K$. Using equations (2.16) and (2.22), the radiation pressure gradient is written as

$$\frac{dP_r}{dz} = -\frac{\sigma}{c} \kappa_H T_{eq}^4 \rho [1 - \theta(z)].$$  \hspace{1cm} (4.2)

The simplest approximation consists of (i) assuming $c_g$, which has the meaning of the (isothermal) sound speed associated with the gas pressure [not to be confused with the total (isothermal) sound speed $c_s$, which is defined as $(P/\rho)^{1/2}$], to be independent of depth, and (ii) the flux mean opacity $\kappa_H$ is also depth independent. The hydrostatic equilibrium equation (2.1) is then written as an equation for density, namely

$$\frac{d\rho}{dz} = -\frac{2}{H_g^2} \rho \left( \frac{2H_r}{H_g} \right) \rho [1 - \theta(z)],$$  \hspace{1cm} (4.3)

where

$$H_g = \frac{(2c_g^2/Q)^{1/2}}{\kappa_H},$$  \hspace{1cm} (4.4)

and

$$H_r = \frac{(\sigma/c)T_{eq}^4 \kappa_H/Q}{Q},$$  \hspace{1cm} (4.5)

may be called the gas pressure scale height, and radiation pressure scale height, respectively. Notice that $H_r$ has the meaning of the disk semithickness in the case of dominant radiation pressure, and analogously for $H_g$ (see Shakura and Sunyaev 1973).

The main complication is the function $\theta(z)$ (see eq. [2.24]). However, it may be written, in a good approximation, as

$$\theta(z) = \begin{cases} 
1 - (z/H) & \text{for } z < H , \\
0 & \text{for } z \geq H .
\end{cases}$$  \hspace{1cm} (4.6)

where

$$H = M/\rho_0 ,$$  \hspace{1cm} (4.7)

may be the called density scale height; $\rho_0 = \rho(0)$ is the density at the central plane. Notice that unlike $H_g$ and $H_r$, the determination of $H$ is not trivial; see equation (4.11).

Equation (4.3) can now be easily solved, yielding

$$\rho(z) = \rho_0 \exp \left[ -\frac{z^2}{H_g^2} \left( 1 - \frac{H_r}{H} \right) \right] \text{ for } z \leq H ,$$  \hspace{1cm} (4.8)

and

$$\rho(z) = \rho_0 \exp \left[ -\left( \frac{z - H}{H_g} \right)^2 \exp \left[ -\frac{H - H_r}{H_g} \right]\frac{H - H_r}{H_g} \right] \text{ for } z \geq H .$$  \hspace{1cm} (4.9)

The density scale height now follows from the condition that $\int_0^H \rho(z)dz = M$. Introducing dimensionless parameters

$$h = H/H_g , \quad r = H_r/H_g ,$$  \hspace{1cm} (4.10)
we obtain the following algebraic equation for \( h \):

\[
h = \frac{\sqrt{\pi}}{2} \left( \frac{h}{h - r} \right)^{1/2} \left[ 1 - \text{erfc} \left( \left( h(h - r) \right)^{1/2} \right) \right] + \text{erfc} \left( h - r \right) \exp \left[ -(h - r)^2 \right],
\]

(4.11)

where the complementary error function is defined by \( \text{erfc} \left( x \right) = \left( \frac{2}{\sqrt{\pi}} \right) \int_{x}^{\infty} \exp \left( -t^2 \right) dt \). Equation (4.11) is easily solved by the Newton-Raphson method, with the initial estimate \( h_0 = r + (1/r) \) for \( r > 1 \), and \( h_0 = \sqrt{\pi}/2 \) for \( r \leq 1 \).

Having determined the density, equation (2.3) then yields the column mass,

\[
\frac{m(x)}{M} = \frac{\sqrt{\pi}}{2} \frac{1}{h} \exp \left[ -(h - r)^2 \right] \text{erfc} \left( x - r \right) \quad \text{for} \quad x \geq h,
\]

(4.12)

and

\[
\frac{m(x)}{M} = \frac{\sqrt{\pi}}{M} \frac{1}{2[\text{erfc} (h(h - r) ]} \left( \text{erfc} \left( \left( x(h - r) \right)^{1/2} \right) \right) - \text{erfc} \left( \left( h(h - r) \right)^{1/2} \right) \quad \text{for} \quad x \leq h,
\]

(4.13)

where \( x \) is the geometrical distance from the central plane expressed in units of gas pressure scale height,

\[
x = z/H_g.
\]

(4.14)

In the case of negligible radiation pressure, \( H_r \ll H_g \), the density scale height is simply \( H \approx \sqrt{\pi}/2)H_g \), and the density and column mass are now simple functions of \( z \), namely,

\[
\rho(z) \approx \rho_0 \exp \left[ -z/H_g^2 \right],
\]

(4.15)

and

\[
m(z) \approx M \text{erfc} \left( z/H_g \right),
\]

(4.16)

with \( \rho_0 \) given by equation (4.7); i.e., \( \rho_0 \approx (2/\sqrt{\pi})M/H_g \).

### b) Second-Order Form of the Hydrostatic Equilibrium Equation

The expressions derived in the previous subsection are valid only in the isothermal case. However, if the sound speed, as a function of depth, is known from the previous iteration, the pressure can be updated by solving the hydrostatic equilibrium equation with the sound speed held fixed. The dependence of \( g \) on \( z \) is accounted for by expressing equation (2.1) as \( \frac{dP}{dm} = Qz \), differentiating once more over \( m \), and using the exact expression \( \frac{dz}{dm} = -1/\rho \), which yields

\[
\frac{d^2P}{dm^2} = -\frac{c_s^2 Q}{P},
\]

(4.17)

where the total sound speed \( c_s \) is taken as a given function of depth.

The corresponding boundary conditions are as follows. The second-order lower boundary condition follows from expressing \( P(m) \) near the central plane in the form of a Taylor expansion:

\[
P(m) = P(M) + (m - M)P'(M) + \frac{1}{2}(m - M)^2 P''(M).
\]

(4.18)

Now, \( P'(M) = 0 \), and the desired lower boundary condition follows from substituting for the second derivative of pressure from equation (4.17) to (4.18).

The upper boundary condition is more complicated. Denoting the column mass at the first (outermost) depth point \( m_1 \), and assuming that the temperature is constant for \( m < m_1 \), we obtain from equations (4.9) and (4.13)

\[
m_1 = H_g \rho(z_1) f \left( \frac{z - H_r}{H_g} \right),
\]

(4.19)

where

\[
f(x) = \left( \sqrt{\pi}/2 \right) \exp \left( x^2 \right) \text{erfc} \left( x \right).
\]

(4.20)

The boundary condition simply follows from expressing \( \rho(z_1) = P_1/c_s^2 \), where now \( H_g, H_r, \) and \( c_s \) are taken as known quantities at the first depth point. Finally, we note that for \( x \), large \( f(x) \) may be approximated as \( f(x) \approx (1 - 1/2x^2)/(2x) \).

### V. CONSTRUCTION OF THE GRAY MODEL

#### a) Numerical Procedure

The global input parameters of the overall disk model are \( R*, m_*, m_a, \) and the Reynolds number \( Re \). A given ring is then identified by a specific \( R \), the distance from the central star. From these parameters, one readily calculates the alternative input parameters \( T_{\text{eff}}, Q, M, \) and \( \dot{\omega} \) by equations (3.7), (2.2), (2.4) and (2.5), respectively.

The construction of the LTE gray model of a given ring now proceeds basically in three steps:

1. Basic initialization, which consists first of establishing the mass-depth scale \( m_d, d = 1, \ldots, ND \), with \( m_1 \) sufficiently small, and \( m_{ND} = M \). Usually, the mass depth points are spaced logarithmically equidistantly between \( m_1 \) and \( m_{ND} \); the distance between
the next-to-last and the last depth point is chosen to be sufficiently small in order to ensure sufficient numerical accuracy of corresponding lower boundary conditions of the radiative transfer and hydrostatic equilibrium equations, as is also common in the classical stellar atmospheric modelling (Mihalas 1978). The next step consists of determining the corresponding geometrical coordinate \( z(m_d) \), which is done by inverting equation (4.12) or (4.13), viz.,

\[
x = r + \text{inverfc} \left\{ \frac{m}{M} \frac{2h}{\sqrt{\pi}} \exp \left[ \frac{(h - r)r}{r} \right] \right\} \quad \text{for} \quad x \leq h ,
\]

\[
x = \left( \frac{h}{h-r} \right)^{1/2} \text{inverfc} \left[ \frac{2[h(h-r)]^{1/2} m - m_H}{\sqrt{\pi}} + \text{erfc} \left[ \frac{(h-h)^{1/2}}{r} \right] \right] \quad \text{for} \quad x \geq h,
\]

where \( m_H = m(h) \), which is given by equation (4.12) or (4.13) for \( x = h \); \( x \) is the dimensionless vertical distance given by equation (4.14); and \( \text{inverfc}(x) \) is the inverse complementary error function. An accurate numerical procedure for its evaluation has been developed by Strecok (1968). I have found that it can be reasonably well fitted by the formula

\[
\text{inverfc}(x) = \left[ -\ln \left( 2x - x^2 \right) \right]^{1/2} (a \ln^2 x + b \ln x + c) ,
\]

with \( a = -1.7726701 \times 10^{-4}, b = 7.4871471 \times 10^{-3} \), \( c = 0.88623 \).

The sound speed associated with gas pressure is generally given by

\[
c_s^2 = \frac{k}{\mu m_H N} \frac{N}{N_n} T ,
\]

where \( k \) and \( m_H \) are the Boltzmann constant and the mass of hydrogen atom, respectively; \( \mu \) is the mean molecular weight, defined by

\[
\mu = \sum \frac{\alpha_A A_E}{Z_E} \sum\left\{ Z_E A_E \right\} ,
\]

where \( \alpha_A \) and \( A_E \) are the atomic mass (in units of hydrogen atom mass), and the abundance (by number, relative to hydrogen) of chemical element \( E \). The summation in equation (5.5) extends over all chemical elements which are considered. \( N \) and \( n_q \) are the total particle number density and the electron density, respectively. For a pure hydrogen gas, the factor \( N/(N - n_q) \) attains values between 1 (for completely neutral gas) and 2 (for completely ionized gas).

In the initialization step, \( c_s \) is determined by equation (5.4), with \( T = T_{\text{eff}} \), and with some suitable estimate for \( N/(N - n_q) \). Once the geometrical distances \( z_s \) are determined, density and gas pressure follow from equations (4.8), (4.9), and (4.1).

2. For each depth, starting with \( d = 1 \), the following iteration loop is performed: \( a \) first the increment of the Rosseland mean and Planck mean opacities is estimated (taken equal to the increment at the previous depth \( d - 1 \); the initial values at the first depth point are given by the input values); then the optical depths corresponding to depth \( d \) are determined. \( b \) The temperature from equation (3.11) (or eq. [3.23]) is calculated. \( c \) Given the current values of temperature and pressure, the total particle number density \( N \) is determined, and then, by solving the set of Saha equations together with the number and the charge conservation equations, the electron number density is calculated. \( d \) The temperature and electron density, all the relevant atomic level populations are calculated, as are new Rosseland and Planck mean opacities and optical depths. \( e \) Return to step \( (b) \) and the loop \( (b)-(d) \) is repeated several times, until \( (T_{\text{new}} - T_{\text{old}})/T_{\text{old}} \) is sufficiently small. As a result of step 2, new values of \( T(m_d), \rho(m_d) \), and \( P(m_d) \), and then \( c_s(m_d) \), are determined.

3. Having determined the total sound speed, equation (4.17) is solved numerically for the total pressure; then for each depth, steps \( 2c - 2e \) are repeated several times, until convergence in temperature is achieved. If needed, the entire step 3 may be performed several times.

\[ b) \text{ Iterative Improvements} \]

The procedure described in the previous subsection calculates the true LTE gray model, because only the approximate \( T(r) \) relation (3.11) (or eq. [3.23]) has been used. It is clear that this model can be improved in principle by using the more exact \( T(r) \) relation (3.8) (with the modification of eq. [3.21], if appropriate). There are two most straightforward approaches to achieve this.

The first approach may be called the lambda iteration method: given the structure of the ring, the radiative transfer equation may be solved frequency by frequency, and then improved values of \( y_j, y_H, T_H, \) and \( T_e \) are determined. Step 3 described above is then performed again, replacing the approximate \( T(r) \) relation (3.11) by equation (3.8). This method is very simple, but, as it is well known, it possesses several severe drawbacks which follow from the very nature of the lambda iteration procedure (Mihalas 1978, p. 173).

Another method, which is still very simple to implement but which removes many problems connected with the lambda iteration, is a generalization of the Unsöld-Lucy method. Denoting the current estimates of the moments and the Planck function as \( J_0, K_0, \) and \( B_0 \), expressing the exact \( J = J_0 + \Delta J \), etc., and expressing \( \Delta B = (4\pi/\nu)T^2 \Delta T \), equations (2.15) and (2.16) yield after some straightforward algebra

\[
\Delta T(m) = \frac{3\pi}{4\sigma T(m)^3} \left\{ \gamma_j \left[ \frac{h_B}{\nu/3} + \int_0^m \kappa_H(m') \Delta H(m') dm' \right] - \frac{1}{3\kappa_H(m)} \frac{dH(m)}{dm} \right\} ,
\]

where \( \gamma_j, y_H, \) and \( \kappa_H \) follow, as before,

\[
\Delta H = H_{\text{mech}} - H_0 ,
\]

with \( H_{\text{mech}} \) given by equation (2.22).
Unlike the classical stellar atmosphere case, where the Unsöld-Lucy procedure has been found quite effective in constructing LTE models, the present generalization may still suffer from convergence problems for some disk models. The basic reason is that in the case of a disk, quantities $\gamma_f$ and $\gamma_H$ play a more important role in the energy balance equation than the corresponding quantities in the stellar atmospheric case (recall that the atmospheric counterpart of our eq. [3.1] is a simple equality $B = J$). In general, the basic drawback of all iterative methods is that the energy balance and radiative transfer are not solved simultaneously.

VI. CONCLUSION

The aim of this paper was to derive a simplified model of the vertical structure of accretion disks. This model may be used either by itself for an exploratory spectroscopic diagnostics of accretion disks or as a starting approximation for a more involved numerical method, as, for instance, the complete linearization method already used by Kriz and Hubeny (1986), to obtain more realistic models.

However, the basic aim of this paper was to demonstrate that the simple, analytical model derived here represents a very useful tool for a physical understanding of the vertical structure of accretion disks and its relation to the classical stellar atmospheric structure, which is by no means evident from purely numerical, albeit more exact, modeling.

The model presented here obviously represents only a first step toward a more reliable spectroscopic diagnostics of accretion disks. Some preliminary results of comparison between the analytical model and more exact models have been discussed elsewhere (Hubeny 1989). A more detailed comparison and discussion of various models, as well as a calculation of an extensive grid of disk models, is currently underway and will be reported in future papers.

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