STATISTICAL CONCEPTS IN RADIATIVE TRANSFER THROUGH INHOMOGENEOUS MEDIA

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ABSTRACT

We extend the theory of radiative transfer in inhomogeneous media to handle transfer for scale lengths small compared to the scale size of the inhomogeneity. We call this the microscopic domain of inhomogeneous radiative transfer. We introduce a concept called the vector intensity distribution to characterize the statistical properties of radiation in various species of medium. We express radiative transfer in an inhomogeneous atmosphere in terms of the evolution of this vector intensity distribution and its various moments along the optical path.

Subject heading: radiative transfer

1. INTRODUCTION

Astronomy needs a formal statistical theory of radiative transfer in complex media whose physical structure is characterized in statistical terms. This need is felt in many areas of astrophysics. Our particular motivation has arisen from the wish to reconcile far-infrared observations of the extreme solar chromosphere. In this context Lindsey (1987) and Braun and Lindsey (1987) were led to explore a simple case of radiative transfer in a multicomponent medium in which radiating (and absorbing) structures were distributed randomly along the ray path. This approach was extended and formalized Jefferies and Lindsey (1988, hereafter Paper I); the present paper is a further extension.

In cases, such as in local thermodynamic equilibrium (LTE), where the source function is known in terms of local physical variables, radiative transfer is greatly simplified. Such cases, as we shall see, allow the possibility of a general statistical approach. The solution to the radiative transfer equation can be represented by the familiar integral

$$I = \int S_v(T)e^{-\kappa z}dz,$$

where $S_v$ and $\kappa_v$ are, respectively, the source function and the absorption coefficient (so that $S_v\kappa_v$ is the emissivity) and $z$ is the optical depth. When the integrand of this equation is known as a function of position, $z$, the calculation of the intensity, $I$, is a simple matter. However, when we must characterize a complex medium statistically, we come to the problem of understanding the corresponding statistical properties of the intensity. That is the problem addressed in Paper I which will be further considered here in a more general context. As in our earlier study, we shall confine our attention to local thermodynamic equilibrium, wherein $S_v$ is simply the Planck function. In the more general, non–LTE case, $S_v$ depends on the local radiation field. This introduces a higher order of difficulty for statistical radiative transfer and we shall defer consideration of that case.

We are, then, interested in LTE radiative transfer in an inhomogeneous medium. By the word “inhomogeneous” we shall mean that the medium consists of different regions (or components) that are not of a single kind (i.e., are not “homogeneous”). We shall assume that there exist a finite number of separate types of medium, each confined to its particular set of regions or volume elements. Within the volume elements of any one species of medium the physical conditions may change in a well-defined continuous manner from point to point. Similarly, over the space occupied by the atmosphere as a whole the probability of occurrence of a specific type of component changes in a continuous manner. However, we will assume that different types of medium are totally separated by boundary surfaces, and that across such a boundary surface physical conditions change discontinuously. We characterize the physical conditions of each species of medium by an opacity, $\kappa_v$, and a source function, $S_v$. Which type of medium actually occurs at a particular point on a specified optical path through the atmosphere is considered a random variable, to be characterized statistically.

If we were able to specify the actual positions along the ray path where the boundary surfaces occur, then the radiative transfer problem would again be solved simply by integration of equation (1). Many such calculations have been made for special cases and some studies have extended into the non–LTE domain, but all are inherently deterministic. Our interest, on the other hand, centers on the case where we know the locations of the structures only in a statistical sense, and can therefore pursue only statistical information on the emergent intensity, $I_e$.

In characterizing an inhomogeneous medium this way, we find it useful to distinguish between two types of characteristic distance: (1) The scale sizes characterizing the structures of which the medium is made up, in particular, the largest separations over which the types of medium at two different points remain significantly correlated. (2) The distance scales over which the physical properties of various types of medium and the statistical occurrence probabilities of the different types of medium change.

Paper I considers an atmosphere that is characterized by several species of medium, each with its own spatial dependence of physical properties and probability of occurrence at
the surface. The statistical properties of the medium then remain effectively constant over a distance \( dz \) that can be much greater than the scale size of inhomogeneous structures in the atmosphere—as for example the distance between the points \( P \) and \( Q \) in Figure 1a.

There are, however, many interesting situations that do not allow this simplification and where we are forced to work with a segment length, \( dz \), that is small enough that the species of medium in two adjacent segments are strongly correlated. Consider, for example, a photospheric magnetic flux tube (see Fig. 1b), with an optical path extending upward at some angle from the hot inside wall of the tube. If the flux tube is greater in diameter than the e-folding height of opacity in the photosphere, the optical path is likely to remain in the interior of the flux tube through many, or all, overlying layers in the atmosphere. The small e-folding height forces us to work with relatively small segment lengths \( dz \), and the type of medium in adjacent segments becomes strongly correlated. Then the procedure in Paper I begins to fail, and a microscopic approach becomes necessary. This situation arises in attempts to calculate center-to-limb brightness profiles for radiation emanating from the chromosphere.

The purpose of this paper is to explore a microscopic statistical approach to LTE radiative transfer, i.e., to look at radiative transfer statistically on size scales that may be small compared to that of the inhomogeneous structure. As in Paper I, we will consider an atmosphere that can be characterized by an integral number of different species of medium, to each of which we assign an integer index, \( i \). At various places along the optical path transitions occur from one species of medium to another. Rather than trying to express the probability that a given species of medium is present in a given interval \( dz \) (since this now depends on the species in adjoining segments) we shall look at the probability, given that a particular species of medium is present at one point, that in the elementary distance, \( dz \), along the optical path, a transition will occur to a different species of medium. This probability can depend on both types of medium involved. In this paper we will assume that it is independent of how far back on the optical path the last transition occurred; this implies an exponential distribution in structure thicknesses, as in Poisson statistics. This approach may seem to preclude certain elementary cases of interest. It clearly does not cover, for example, media in which all structures are of the same thickness. Although we will not deal with these cases here, the theory will provide tools useful for later consideration of such cases.

It should be understood that the theory we are developing here is not a replacement for the macroscopic theory (Paper I), but is rather an extension of the theory to handle the microscopic side of the inhomogeneous radiative transfer problem. For certain cases, such as the flux tubes of Figure 1b, the entire problem may be basically microscopic, and the procedure of Paper I will not apply. For other cases, the microscopic situation is trivial, and the problem reduces to that of Paper I. There will be intermediate cases, where some consideration on both microscopic and macroscopic scales is useful. In those cases a microscopic treatment will generally provide the basic parameters necessary to apply the microscopic procedure of Paper I.

II. THE BASIC THEORY

We wish to express the differential evolution of the statistical properties of the intensity along a path distance, \( dz \), that is...
sufficiently small that the species of medium found at two points, P and Q, separated by distance $dz$ along the optical path are strongly correlated, as opposed to being independent. Indeed, by “differential evolution” we imply that we shall work on a scale for which the medium at point Q is most likely the same species as at point P, and that the probability of encountering a transition in species between two such points is a differential quantity proportional to $dz$.

Normally by “statistical properties of the intensity” we mean the statistical distribution of the intensity. However, here we find it useful to introduce a generalization of the intensity distribution that includes not only the intensity as a random variable, but also the species of medium found at a particular point. The utility of such a generalization becomes evident in the course of the development of the theory.

a) The Vector Intensity Distribution

We characterize each point, $P$, on the optical path by a vector intensity distribution $f$ where the number of dimensions of the space spanned by $f$ is equal to the number of components in the medium. Each species of medium, thus each dimension of the space spanned by $f$, is indexed by an integer, $i$. The $i$th component of $f$ is not a real number, but rather a scalar distribution, $f_i(I)$, in the intensity, $I$. The probability that at a particular point on the optical path the medium is of species $i$ and that the specific intensity is between $I$ and $I + dI$ is expressed by $f_i(I)dI$. The task of the theory is to determine how $f$ evolves. This is conveniently expressed in terms of certain operators that act on $f$ along the optical path.

b) Evolution Operators

Two basic processes act to evolve the vector intensity distribution, $f$: (1) absorption of radiation by the current species of medium and replenishment by radiation emitted by the medium, and (2) the possibility of transitions between different species of medium. Both of these are conveniently expressed by operators on the vector intensity distribution, as we shall see. In general these two operators do not commute; however, for small path lengths, $dz$, they are commutative to first order in $dz$. In order to express these operators in useful terms, we shall proceed by determining how each acts alone, in the absence of the other.

c) Smashers and Opaque Emitting Material

We consider first the effect on the distribution $f$ of opaque emitting material and exclude the possibility of transitions between different species. Basically, the effect of a uniform medium of opacity $\kappa_i$ and source function $S_i$ on an intensity $I$ over a distance $\Delta z$ is to transform the intensity $I$ to $I'$ according to the rule

$$I' = \alpha_i I + \beta_i S_i,$$

where

$$\alpha_i = e^{-\kappa_i}; \quad \tau_i = \kappa_i \Delta z; \quad \alpha_i + \beta_i = 1.$$  

The components of the new distribution, $f'$, must satisfy the conservation condition

$$f'(I)dI' = f(I)dI,$$  

and in consequence, by simple substitution, we find that $f$ transforms as

$$f' = \frac{1}{\alpha_i} f \left( \frac{1 - \beta_i S_i}{\alpha_i} \right)$$  

The effect of absorption and re-emission is to “compress” the $i$th component, $f_i$, of the distribution inward toward $I = S_i$, while extending it vertically so as to preserve the area under it. This combined operation of laterally compressing and vertically extending the distribution is what we call smashing the distribution. This is illustrated in Figure 2. In the limit as $\tau_i$ approaches infinity $f'(I)$ approaches the Dirac $\delta$-function, $\delta(I - S_i)$.

An important case is that of an infinitesimal smashing:

$$f'_i = f_i + df_i = (1 + \kappa_i dz)f_i(I + \kappa_i(I - S_i)dz).$$  

Expanding $f_i$ to first order in $dz$, we see that

$$df_i = \left[ \kappa_i(I - S_i) \frac{df_i}{dI} + \kappa_i f_i \right] dz.$$  

For future discussion we shall find it convenient to borrow the operator formalism of quantum mechanics to express the evolution of the vector intensity distribution, $f$, along the optical path. To do this, we define $T$ to be the operator that applies the following linear transformation

$$Tf = \delta f - \kappa_i(I - S_i) \frac{df}{dI} + \kappa_i f_i$$  

This operator ($1 + T dz$) is an infinitesimal “smasher.” It smashes each component of $f$ individually according to the source function and opacity of its particular species of medium. Under the action of the operator $T$ each component of $f$ evolves independently of all other components as physically must be the case. This behavior is assured by the Kronecker delta in equation (8).

![Fig. 2.—Effect of a uniform absorbing and emitting medium on a single component, $f_i$, of the vector distribution function. The $i$th component $f_i$ of the initial distribution (solid curve) is squeezed inward toward the vertical line centered at the source function, $S_i$, on the abscissa. This squeezed distribution (dotted curve) is extended upward, so that the total area under it is preserved. We say that we are smashing the vector distribution, $f$, when we do this to each of its components. Each species of medium, of course, smashes its particular component of $f$ its own separate way, depending on its particular source function and opacity.](image)
Successive applications of \((1 + T \, dz)\) give us the evolution of \(f\) over a macroscopic distance, \(\Delta z\). Once again in analogy with the formalism of quantum mechanics, the evolution over a macroscopic distance, \(\Delta z\) may be characterized by an operator, \(S\), of the form
\[
S = \exp (\Delta z \, T) \tag{10}
\]
applied to \(f\). Like \(T\), \(S\) is a pure smasher with off-diagonal elements all zero.

d) Mixers and Transitions

We now consider how to represent, in operational terms, transitions from one species of medium to another, assuming for the present that the intensity distribution is not evolved by absorption or emission in the distance, \(dz\), over which we give the transition a chance to occur. If we force a transition from medium \(j\) into medium \(i\), the \(j\)th component of the distribution, \(f_j\), vanishes and its contents are transferred to \(f_i\). Now, let the probability of a transition to species \(i\), given that the current species is \(j\), be \(p_{ij} \, dz\). The possibility of such a transition contributes a fraction \(p_{ij} \, dz\) of the \(j\)th component of \(f\) to the \(i\)th component, at the expense of the \(j\)th component.

We can represent all possibilities for transitions from any type of medium to any other by an array of transition rates, \(p_{ij}\), for each pair \((i, j)\). We disallow transitions from any species of medium into itself, thus \(p_{ii} = 0\). All of the various transition possibilities simply mix different components of \(f\) into one another at rates proportional to \(p_{ij}\) as we proceed along the optical path. This too can be expressed by a simple linear transformation of \(f\):
\[
df = Pf \, dz . \tag{11}
\]
In terms of the individual components of \(f\),
\[
df_i = \sum_j P_{ij} f_j \, dz , \tag{12}
\]
where for \(j \neq i\),
\[
P_{ij} = p_{ij} \tag{13}
\]
and the diagonal elements of \(P\) are given by
\[
P_{ii} = -\sum_k P_{ki} . \tag{14}
\]
The diagonal elements of \(P\) express the loss in the distribution of each component of \(f\) due to transitions to all other species of medium, hence the sum. For later use we note that
\[
\sum_k P_{ki} = 0 . \tag{15}
\]
We call the evolution of \(f\) due to the possibility of transitions between different types of material mixing. The operator \((1 + P \, dz)\) is an infinitesimal mixer. In analogy with equation (10), we can express pure mixing over a macroscopic path length, \(\Delta z\), by the operator
\[
N = \exp (\Delta z \, P) . \tag{16}
\]
e) The Radiation Transport Equation

To evolve the vector distribution, \(f\), in an inhomogeneous medium, we continually apply infinitesimal smashing and mixing operators alternately with infinite discrimination along the optical path. The infinitesimal smashers and mixers commute to first order and therefore can be applied in any order:
\[
(1 + T \, dz)(1 + P \, dz) = 1 + (T + P) \, dz , \tag{17}
\]
and on adding the infinitesimal components of the evolution operators and dividing by \(dz\) we obtain the following equation for the evolution of \(f\) along the optical path:
\[
\frac{df}{dz} = (T + P) \, f . \tag{18}
\]
Expressed in terms of the individual components of the vector \(f\), equation (18) becomes
\[
\frac{df_i}{dz} = \kappa_i (I - S_i) \frac{df_i}{dz} + \sum_j (\kappa_i \delta_{ij} + P_{ij}) f_j . \tag{19}
\]
This is the formal radiation transport equation for the vector distribution of intensity in an inhomogeneous medium. It stands in exact analogy with the elementary radiative transfer equation for the evolution of intensity, \(I\), in a homogeneous medium, namely,
\[
\frac{dI}{dz} = \kappa (I - I) . \tag{20}
\]
Those familiar with classical particle transport theory will recognize the analogy of equation (19) to the Boltzmann particle transport equation, which expresses the evolution in time of the distribution of various species of particles in phase space.

III. Moments

Among the properties of the intensity distribution, we are particularly interested in its moments and how they evolve along the optical path. For this we need to look, at the moments of the individual components of \(f\). We first define the \(n\)th vector moment, \(M^{(n)}\), by
\[
M^{(n)} = \int I(0) f \, dI . \tag{21}
\]
The \(i\)th component, \(M_i^{(n)}\), of \(M^{(n)}\) is given by
\[
M_i^{(n)} = \sum_j P_{ij} M_j^{(n-1)} + \sum_j (\kappa_i \delta_{ij} + P_{ij}) M_j^{(n)} . \tag{22}
\]
With these definitions it follows readily from equation (19) that
\[
\frac{d}{dz} M_i^{(n)} = \kappa_i (I - S_i) M_i^{(n-1)} + \sum_j (\kappa_i \delta_{ij} + P_{ij}) M_j^{(n)} . \tag{23}
\]
The moment, \(M^{(n)}\), of the total intensity distribution is given by
\[
M^{(n)} = \sum_i M_i^{(n)} . \tag{24}
\]
Because of equation (15), the sum over \(P_{ij}\) on the right side of equation (23) vanishes when summed over \(i\), leaving
\[
\frac{d}{dz} M^{(n)} = \kappa_i \sum_j [S_i M_j^{(n-1)} - M_i^{(n)}] . \tag{25}
\]
The zero-order moments of the individual components follow from equation (23) as
\[
\frac{d}{dz} M_i^{(0)} = \sum_j P_{ij} M_j^{(0)} . \tag{26}
\]
The zeroth moment \(M_i^{(0)}\) is simply the probability at any point that the medium there is of species \(i\).
One of the consequences of equation (15) is that the determinant of the \( P \) matrix is zero. It therefore must have at least one eigenvector whose eigenvalue is zero. Each of the other eigenvectors must have a negative real part or be zero. If the \( P \) matrix is constant along the optical path, the vector, \( \mathbf{M}^{(0)} \), of zeroth moments will come to a stationary stable equilibrium state. If there is only one eigenvector with zero eigenvalue, that will be the single stable equilibrium approached by the zeroth moment if \( P \) is constant.

For \( n = 0 \) equation (25) becomes

\[
\frac{d}{dz} M^{(0)} = 0 .
\]

The total zeroth moment is simply the sum total of the probabilities and is therefore constant, normalized to unity.

The equations governing the first moments in the various species of medium, come from setting \( n = 1 \) in equation (23), i.e.,

\[
\frac{d}{dz} M^{(1)} = \kappa_i S_i M^{(0)} + \sum_j (P_{ij} - \kappa_i \delta_{ij}) M^{(1)}_j .
\]

Similarly, equation (25) for the mean intensity becomes

\[
\frac{d}{dz} M^{(1)} = \sum_i \kappa_i (S_i M^{(0)}_i - M^{(1)}_i) .
\]

This will look more familiar if we define the expectations

\[
\langle \kappa S \rangle \equiv \sum_i \kappa_i S_i M^{(0)}_i ,
\]

and

\[
\langle \kappa I \rangle \equiv \sum_i \kappa_i M^{(1)}_i ,
\]

so that equation (29) becomes

\[
\frac{d}{dz} \langle I \rangle = \langle \kappa (S - I) \rangle ,
\]

resembling equation (20).

IV. EXAMPLES

We illustrate the numerical application of the foregoing theory with some simple cases. We consider first the distribution function for radiation passing through a two-component medium. We let both media have the same opacity but different source functions. We also let the transition rates, \( p_{12} \) and \( p_{21} \), be equal.

It is instructive first to consider the behavior of the intensity itself along the optical path. It varies depending on transitions back and forth between the two types of medium in the way illustrated in Figure 3. In a particular type of medium, the intensity drifts exponentially toward the source function for that medium. If transitions are frequent over an optical path length of unit opacity, (i.e., \( p_{12}/\kappa > 1 \)), the intensity will generally never approach either source function. It will, rather, wander back and forth in a localized random walk, remaining confined in a narrow range between the limits defined by the two source functions (Fig. 3a). If, on the other hand, transitions occur infrequently over a unit optical path length (i.e., \( p_{12}/\kappa \approx 1 \)), the intensity will then spend a considerable fraction of the optical path near each source function, resulting in a distribution sharply concentrated near the two source functions (Fig. 3b).

We can characterize the evolution of the ideal intensity distribution along the optical path by solving the radiation transport equation (19). This is easily done to first order in \( dz \) by slightly smashing, then slightly mixing the initial distribution alternately many times to simulate the cumulative effect of both process working continuously and simultaneously together. Three cases are illustrated in Figure 4, ranging from frequent to infrequent transition rates. The medium is entered at \( z = 0 \) in species 1 with an intensity of zero. The entire distribution is, therefore, piled into \( f_1 \) at \( I = 0 \). In each case the distribution evolves along the \( z \)-direction, spreading outward from a spike at \( I = 0 \) to relax in due course to an equilibrium distribution spanning the interval between the source functions (\( S_1 \) and \( S_2 \)) of the two species. The first moments, normalized by their respective zeroth moments, are plotted in the \( I-z \) plane above the distribution maps.

![Figure 3](https://example.com/fig3.png)

**Fig. 3**.—Simulations of the detailed variations of intensity along the optical path. In panel a we show the case for frequent mixing. This results in a relatively narrow intensity distribution, plotted at the right edge of the box. In panel b we show the case for infrequent mixing. Here the intensity is concentrated at the limits set by the source functions, causing a double-spiked distribution, again plotted at the right edge of the box.
Fig. 4.—Three cases showing evolution of the total intensity distribution along the optical path for atmospheres containing two species of medium. Parallel evolution of the normalized first moments, $M_1^{(1)}/M_1^{(0)}$ and $M_2^{(1)}/M_2^{(0)}$, are plotted in the planes above each distribution. In all cases, both media have the same opacity but different source functions, $S_1 = 30$, and $S_2 = 70$. The transition probabilities are equal: $p_{12} = p_{21}$. In (a) we show the case for which transition probabilities are substantially greater than the opacity: $p_{12}/\kappa = 2$. In (b) $p_{12}/\kappa = 0.75$. In (c) $p_{12}/\kappa = 0.5$, and the emergent radiation tends strongly to be characterized by one or the other source function.
In Figure 4a we show the case for relatively frequent mixing, $p_{12}/\kappa = p_{21}/\kappa = 2$. Noting that $1/p_j$ is the characteristic size of elements of type $j$ for a two-component medium, we see that individual volume elements of either species of medium tend to be optically thin in this case. The distribution evolves to a profile that looks somewhat like a truncated Gaussian. In Figure 4b we show the evolution of the total intensity distribution for $p_{12}/\kappa = 0.75$; and in Figure 3c we show the case for relatively thick structures, $p_{12}/\kappa = \frac{1}{2}$. For optically thin structures (Fig. 4a), the moments come to equilibrium well inside the boundaries defined by the source functions. For optically thick structures they expand outward toward these boundaries.

In Figure 5 we examine the equilibrium distributions (i.e., the limit as $z \to \infty$) for different ratios of transition rate ($p$) to opacity ($\kappa$). Figure 5a shows the equilibrium distributions for ratios ranging from $p/\kappa = 5$ (optically thin structures) to $p/\kappa = 0.5$ (optically thick structures). The form of the equi-

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**Figure 5**—Equilibrium intensity distributions for media with different transition rates: in panel a are shown distributions for media with ratios $p_{12}/\kappa$ ranging from 5 to 0.5. In panel b we show the distribution resulting from $p_{21}/\kappa = 1.0$ while $p_{12}/\kappa = 0.85$. The jagged appearance of profiles for smaller values of $p_{12}/\kappa$ results from numerical errors.
librium distribution for large ratios approaches a narrow Gaussian, appropriate to a random-walk problem. For ratios less than unity spikes emerge at the source functions, quickly becoming extremely narrow and high as the ratio decreases past 0.5.

Thus far we have only considered cases where the transition rates, \( p_{12} \) and \( p_{21} \), are equal. If they are not equal, the equilibrium distribution will be sloped toward the source function of the species with the lower transition rate into the other species. Figure 4b shows a case where \( p_{12}/\kappa \) is 0.85 while \( p_{21}/\kappa \) is unity.

It is simple to extend the application of the theory beyond two species of medium. In general, however, the size of the \( P \) matrix increases as the square of the number of species and so, eventually, does the amount of numerical work required. In Figure 6 we show the evolution of the intensity distribution function for a medium composed of three species. Transitions are permitted between media 1 and 2 and between 2 and 3 with equal rate (\( p/\kappa = 0.4 \)), but not directly between media 1 and 3. Thus, we are requiring that media 1 and 3 always be separated by a transition layer (medium 2). As before, we enter the medium in species 1 at zero intensity. The distribution quickly diffuses into species 2, and from thence into species 3, eventually to relax into an equilibrium configuration where it is shared equally by species 1 and 3, somewhat at the expense of species 2.

V. COMPARISON WITH TRANSPORT IN THE MACROSCOPIC DOMAIN

Is it possible that the macroscopic approach developed in Paper I is actually equivalent to the microscopic approach developed here, at least in certain respects? We may obtain at least a partial insight through direct comparison of solutions in the two cases—the simplest being that for which the various media parameters (opacities, source functions, transition rates for the microscopic problem, and occupancy probabilities for the macroscopic) are constant with position along the ray path. In this section we shall adopt the terminology of Paper I when discussing the macroscopic problem.

In a medium of two components, each of which has constant properties along the ray path, it is easy to show directly from equations (23) and (26) that, when \( z \) becomes arbitrarily large,

\[
M_1^{(1)} = \frac{p_{12}^* \kappa_1 (\kappa_2 + p_{12}) S_1 + \kappa_2 p_{21} S_2}{(\kappa_1 \kappa_2 + p_{21} \kappa_2 + p_{12} \kappa_1)},
\]

and

\[
M_2^{(1)} = \frac{p_{21}^* \kappa_2 (\kappa_1 + p_{21}) S_2 + \kappa_1 p_{12} S_1}{(\kappa_1 \kappa_2 + p_{21} \kappa_2 + p_{12} \kappa_1)},
\]

with

\[
p_{12}^* = \frac{p_{12}}{p_{12} + p_{21}}.
\]

The expected value of the intensity (the sum of these two moments) then approaches the value

\[
M^{(1)} = \frac{p_{12}^* \kappa_1 (\kappa_2 + p_{12} + p_{21}) S_1 + p_{21}^* \kappa_2 (\kappa_1 + p_{12} + p_{21}) S_2}{p_{12}^* \kappa_1 (\kappa_2 + p_{12} + p_{21}) + p_{21}^* \kappa_2 (\kappa_1 + p_{12} + p_{21})}.
\]
This is the effective source function for the medium. The effective source function for the macroscopic case, as given in Paper I, is (again for constant parameters along the ray path)

$$M^{(1)} = p_1 S_1(1 - e^{-t_1}) + p_2 S_2(1 - e^{-t_2})$$

where the \( t_i \) are the optical path lengths of a single cell and the \( p_i \) are the occupancy probabilities of the cells into which the ray path is divided.

To compare these two results, we need a way to associate the parameters \( p_{ij} \) of the microscopic formulation with those—\( p_i \) and \( t_i \) (or equivalently the cell size, \( \Delta z \)—of the macroscopic formulation. A natural way to do so follows if we require that the expectation thicknesses, \( \Delta_i \), be the same for both formulations. For the microscopic case the thicknesses are simply given as

$$\Delta_i = \frac{1}{p_i},$$

where \( j \) is whichever number (1 or 2) \( i \) is not. For the macroscopic case it is easy to see that the mean structure thickness for medium \( i \) is, simply,

$$\Delta_i = \frac{\Delta z}{1 - p_i},$$

where \( \Delta z \) is the cell thickness and \( p_i \) is the occupancy probability for medium \( i \). Equation (39) shows that \( \Delta_i \) is the harmonic mean of the characteristic structure thicknesses, i.e.,

$$\frac{1}{\Delta z} = \frac{1}{\Delta_1} + \frac{1}{\Delta_2} = p_{12} + p_{21},$$

where the final form follows from equation (38).

The occupancy probabilities, \( p_i \), are implicitly related to the \( p_{ij} \) via equation (40). Thus, since \( p_i = \Delta_i/(\Delta_1 + \Delta_2) \), we find, after equation (35), that \( p_i = p_i^0 \). The same result follows on solving equation (26) and identifying \( p_i \) with the asymptotic value of \( M^{(0)}(z) \), the zero-eigenvector of the \( P \) matrix.

With the identification given above it is easy to show that the asymptotic solutions (36) and (37) are equivalent for a wide variety of cases. We may, similarly, compare the microscopic and macroscopic solutions for finite values of \( z \) by solving directly the appropriate equations for the expected values of the intensity. Figure 7 shows such a comparison where the expectation intensity \( \langle I \rangle \) is plotted as a continuous curve while for the macroscopic case it is, of course, given only at the cell boundaries, since it is only defined there. In Figure 7a the source functions of media 1 and 2 are chosen to have values of 1 and 2, respectively. We let the transition rates \( p_{12} = p_{21} = 0.4 \). The occupancy probabilities for the matching macroscopic problem are equal, both \( \frac{1}{2} \), and the cell size is 1.25, following equation (40). In this case the macroscopic approximation is very good; in fact, it is exact. In Figure 7b we plot the comparison for the considerably different case of a medium of sparse but optically thick structures that are cool (with a source function of unity) embedded in a hot optically thin ambium (with a source function of 10). Here we take transition rates \( p_{12} = 0.25 \) and \( p_{21} = 0.1 \). The matching macroscopic problem has occupancy probabilities of \( p_1 = 2/7 \) and \( p_2 = 5/7 \) with a cell size of 20/7. In this case differences begin to emerge. Our matching of parameters does not result in equal effective source functions (see eqs. [36] and [37]). We could relax our condition matching the expectation structure thicknesses in favor of matching the effective source functions and fix this particular discrepancy. The asymptotes would then match; however, it is evident from Figure 7b that the mean intensity approaches its asymptote differently in the macroscopic problem than in the microscopic, and we generally expect these differences to remain. A more complete comparison of these two problems would be useful, but is not attempted here. Similarly, we defer the more complicated comparison of the two problems for media of more than two components.

VI. FUTURE DIRECTIONS

The purpose of this paper has been to define the problem of statistical radiative transfer in the microscopic domain and to illustrate the basic concepts. Many applications exist. We have begun to apply it to model center-to-limb contrasts of solar plage throughout the visible and infrared continuum. We are

![Fig. 7](image-url)
particularly interested in applying the method to new sub-
millimeter observations of solar plage and the chromospheric
supergranular network we have recently made on the James
Clerk Maxwell Telescope on Mauna Kea.

While solar observations provide an excellent field in which
to apply inhomogeneous radiative transfer theory, applications
are obviously far from limited to the Sun. The theory will apply
equally well in analyzing emission from such other highly inho-
mogeneous media as the interstellar medium and H II regions,
as long as the assumption of LTE is appropriate, which is
specifically the case for the infrared continuum.

Many areas not touched on here are very much open to
further development. Three examples are the following

a) A Continuous Selection of Medium Types

We have developed the theory for atmospheres with a finite
number of types of medium, indexed simply by integers. It is
possible to relax this restriction to allow media spanning a
continuum of different types. This continuum need not be con-
fined to a single dimension.

b) Different Distributions in Structure Sizes

In our treatment, transition probabilities between different
types of medium are represented by simple transition rates,
with no dependence on how long the optical path has been in a
certain type of medium. This results in an exponential distribu-
tion in the segment lengths for any given type of medium. This
restriction can be loosened by skillful use of the $P$ matrix. By
dividing a single type of medium into a sequence of different
virtual types, and specifying transition rates from one virtual
type of medium to the next, the distribution components for
the different virtual types of medium serve as a memory tool for
various path distances from entrance into the actual structure.
In practice a large variety of structure size distributions could
be generated with only a few off-diagonal elements of the $P$
matrix being nonzero. Thus, use of the full $P$ matrix in the way
we apply it here would be unnecessary, indeed probably quite
wasteful. This is certainly a ripe area for further theoretical
development.

c) Non-LTE

Much of the formalism we have developed here for LTE
radiative transfer will surely prove useful in a more general
non-LTE theory, where the local absorption and emission at a
point are not uniquely described by physical conditions at that
point, but depend on the ambient radiation field and so on
conditions in neighboring atmospheric regions.

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