ON THE BREMSSTRAHLUNG EFFICIENCY OF NONTHERMAL HARD X-RAY SOURCE MODELS

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Abstract. It has often been stated, but never rigorously proven, that interpreting observed hard X-ray emission in terms of a thick-target source gives a lower limit to the flux of electrons which have to be injected into the source. The truth of this statement, for the burst-integrated emission, is rigorously established here. Also an explicit inversion for the injected electron flux in terms of the photon spectrum is given, for the case where all electrons traverse a single value of column density. This generalises the previous results for the thick- and thin-target limits.

The use of the standard thick-target formalism for the interpretation of instantaneous (as opposed to burst-integrated) photon fluxes is also discussed. By considering the specific case of the thick-target trap model, it is shown that use of this formalism can either underestimate or overestimate the injected electron flux, at different times in the same event, but that integration of the inferred electron fluxes over the event nevertheless yields the true, burst-integrated electron flux.

1. Introduction

Brown (1971) showed analytically how to infer the number and energy distribution of flare nonthermal electrons from their observed (bremsstrahlung) hard X-ray emission. By assuming that all electrons stop collisionally before leaving the source (the 'thick-target' limit), in a time less than instrumental time resolution, he expressed the flux of electrons injected into the source as an integral functional of the photon flux and its first and second derivatives with respect to photon energy. Subsequent application of the resulting formalism to real data led to the conclusion that the total energy initially released as fast electrons may amount to several tenths of the flare energy (Hoyng, Brown, and van Beek, 1976; Lin and Hudson, 1976). More recently, it has been claimed that the total electron energy (at least in one flare) may even exceed the total energy otherwise manifested (Tanaka and Zirin, 1985). Clearly, this high efficiency of particle acceleration has important implications for flare-energy transport and for the flare-energy release (e.g., MacKinnon, 1986).

Irrespective of the actual relative magnitudes of electron lifetime and instrumental time resolution, it is immediately clear that a thick-target source makes the maximum possible use of a given injected population of electrons, in terms of total (burst-integrated) X-ray yield. What is not quite so clear is that the total electron energy content is minimized by interpreting the observed X-ray fluence in terms of a thick-target source. Indeed, the kinetic energy content of the instantaneous population (rather than injected flux) of electrons implied by observed X-ray fluxes is insignificant compared to the total flare energy (Kane and Anderson, 1970) and one can, in consequence, find statements in the literature to the effect that 'thin-target' sources (in which electron energy changes

negligibly in the source) may be more efficient than thick-target ones (e.g., Zirin, 1988). Because the Coulomb collisional energy loss rate decreases with increasing electron energy, the effective electron distribution in a thick-target source is harder than the distribution which was injected into it. So to explain a fixed photon spectrum, a thick-target source requires an injected electron spectrum which has a predominance of electrons at low energies, as compared to that required for a thin-target source. Since the total electron energy content is governed by the form of the distribution at low energies, this also leads us to ask if a source which was not 'completely thick' (in the sense that all electrons stop in it) could be contrived which was a more efficient X-ray producer than a thick-target source. In fact the answer to this question is 'no', as we show in the next section.

Brown and Emslie (1988) asked the question: 'which sorts of photon spectra are compatible with which sorts of assumed source?' In the course of establishing the result outlined above, we generalize one of their results, concerning thick- and thin-target sources, to the case where the source can be characterized by a single column density.

It is of interest also to address the error incurred by assuming that electron lifetimes are shorter than instrumental time resolution, if this is not in fact the case. We do not address this question in full generality. However, we do show that the use of Brown's (1971) formalism to analyse hard X-rays from a coronal magnetic trap underestimates the injected electron flux early in the burst and overestimates it late in the burst, but nevertheless gives a true result for the burst-integrated electron flux.

2. Lower Limit on the Burst-Integrated Electron Flux

An electron of kinetic energy \( E \) (keV) at point \( r \) in the atmosphere loses energy at a rate (Trubnikov, 1965)

\[
\frac{dE}{dt} = -Kn(r)E^{-1/2},
\]

where \( K = 4.9 \times 10^{-9} \) and \( n(r) \) is the proton density at point \( r \) (we assume a fully-ionized hydrogen plasma). Equation (1) is valid for \( E < 160 \) keV (Bai and Ramaty, 1979). We use it in the following discussion for definiteness, but generalisation of all the subsequent results to an arbitrary energy loss rate is immediate, provided this can be expressed as a function of \( E \) only (Brown and MacKinnon, 1985). Using (1) and \( |v(E)| = (2E/m_e)^{1/2} \) (\( v(E) \) is electron velocity and \( m_e \) is electron mass) we find that an electron of initial energy \( E_0 \), which has traversed column density \( N \), has energy \( E \) given by

\[
E^2 = E_0^2 - 2K'N,
\]

where \( K' = K(m_e/2)^{1/2} \). Note that \( N \) is the column density traversed by the electron parallel to its instantaneous velocity, not parallel to some fixed direction, such as the local vertical to the solar surface. To make this more concrete, let the electron be produced
at time $t = 0$, and let $E$ be its energy at time $t$. Then

$$N = \int_0^t n(r'(t')) |v(t')| \, dt' .$$

For comparison, if $\hat{z}$ is the unit direction vector of the local vertical, the vertical column density traversed by the electron would be

$$N_{\text{vert}} = \int_0^t n(r'(t')) v(t') \cdot \hat{z} \, dt' .$$

The reason for introducing column density in this way is the following. The total bremsstrahlung emitted by an electron depends on the actual column density traversed by it, not on the column density parallel to some arbitrary direction. If we inject a population of electrons into a source of fixed, vertical column density $N_{\text{vert}}$, the statistical vagaries of collisional scattering will ensure that they leave this source having traversed a range of column densities $N$ (although electron energy decreases monotonically according to (1), evolution of the instantaneous direction of motion of the electron is a statistical process). Thus it would appear that detailed study of electron transport is necessary to calculate the bremsstrahlung from such a situation. But since we are concerned here only with comparing an arbitrary source with the thick-target case (for which no detailed study of transport is necessary), it is sufficient to incorporate the statistical element introduced by transport into the arbitrariness of the source thickness, as we do below.

Note from (2) that an electron stops completely when it has traversed a column density $E_0^2 / 2K'$.

Now let $f(E_0, N, t)$ be the flux ($s^{-1} \text{keV}^{-1} \text{cm}^2$) of electrons of initial energy $E_0$, injected at time $t$, which traverse column depth $N$ during their subsequent history in the source, and put

$$F(E_0, N) = \int_0^\infty f(E_0, N, t) \, dt .$$

Denote by $Q(\varepsilon, E_0, N)$ the total (time-integrated) flux ($\text{keV}^{-1}$) of photons of energy $\varepsilon$ (keV) produced by an electron of initial energy $E_0$, which traverses a column density $N$. Then the total fluence ($\text{keV}^{-1}$) of X-rays $\mathcal{F}(\varepsilon)$, of energy $\varepsilon$, is given by

$$\mathcal{F}(\varepsilon) = \int_{\varepsilon}^{\infty} \int_{N}^{E_0^{2}/2K'} f(E_0, N, t) Q(\varepsilon, E_0, N) \, dN \, dE_0 \, dt$$

$$= \int_{\varepsilon}^{\infty} \int_{N}^{E_0^{2}/2K'} F(E_0, N) Q(\varepsilon, E_0, N) \, dN \, dE_0 .$$

(3)
Following Brown (1971), and using (1), we find that $Q$ is given by

$$Q(\varepsilon, E, N) = \frac{1}{K'} \int_{E_1}^{E_0} E \frac{d\sigma}{d\varepsilon} (\varepsilon, E) \, dE ,$$  \hspace{1cm} (4)

where $d\sigma/d\varepsilon$ is the cross-section differential in photon energy for emission of a photon of energy $\varepsilon$ by an electron of energy $E$, and $E_1 = \max \{ \varepsilon, [E_0^2 - 2K'N]^{1/2} \}$.

For the electron energies here, $d\sigma/d\varepsilon$ is well approximated by the nonrelativistic Bethe–Heitler result (e.g., Heitler, 1954):

$$\frac{d\sigma}{d\varepsilon} = \frac{\sigma_0}{\varepsilon E} \ln \left\{ \frac{1 + (1 - \varepsilon/E)^{1/2}}{1 - (1 - \varepsilon/E)^{1/2}} \right\}$$  \hspace{1cm} (5)

($\sigma_0$ a constant), and specific results may be obtained for this cross-section, but the argument holds for a general cross-section, so we write

$$\frac{d\sigma}{d\varepsilon} = \frac{1}{\varepsilon E} q(\varepsilon, E) .$$  \hspace{1cm} (5a)

Putting (4) and (5a) in (3), and changing the order of integration, we find

$$\mathcal{J}(\varepsilon) = \varepsilon \mathcal{J}(\varepsilon) = O(G; \varepsilon) ,$$  \hspace{1cm} (6a)

where

$$G(E) = \int_{E_1}^{E_0} \int_{(E - E_0^2 - 2K'E_0^2)^{1/2}}^{E_0^{1/2}} F(E_0, N) \, dN \, dE_0$$  \hspace{1cm} (6b)

is the total number of electrons which ever attain energy $E$ somewhere in the source, and the linear integral operator $O$ is given by

$$O(g; \varepsilon) = \frac{1}{K'} \int_{\varepsilon}^{\infty} q(\varepsilon, E) g(E) \, dE$$  \hspace{1cm} (6c)

for any sufficiently well-behaved function $g$.

Explicit inversion of $O$ has been given for the cross-section (5) by Brown (1971), and for the Kramers’ approximate cross-section by Brown and Emslie (1988); for arbitrary $q$ we may still formally invert (6a) to get $G$:

$$G(E) = O^{-1} \{ \mathcal{J}; E \} .$$  \hspace{1cm} (7)
The upshot of this is that the observed photon spectrum $J(\varepsilon)$ tells us the quantity $G$, which is simply related to the total electron flux only in the thick- and thin-target limits. The quantity of interest here is the total injected electron flux at any energy $E$:

$$F(E) = \int_0^{E^2/2K'} F(E, N) \, dN.$$  \hspace{1cm} (8)

A thick-target source is one in which all electrons stop. So

$$F_n(E, N) = 2F_0(E)\delta(N - E^2/2K'),$$

where $F_0(E)$ is the total flux of electrons of energy $E$, $\delta$ is the Dirac delta function and the factor of two comes from the behaviour of $\delta$ when its zero is an endpoint of the interval of integration (e.g., Heitler, 1954). Thus, substituting in (8) and (6b),

$$F_n(E) = F_0(E) = -G'(E).$$

Now, for a source of arbitrary thickness, differentiating (6b) gives

$$F(E) = -G'(E) + \frac{E}{K'} \int_{E}^{\infty} \left( E_0 - \frac{E_0^2 - E^2}{2K'} \right) dE_0 \geq -G'(E) = F_n(E).$$

Thus we have shown that the assumption of a thick-target source yields a strict lower limit to the total flux of electrons $F(E)$ at every electron energy $E$. Consequently, functionals of $F$ such as the total electron energy are also bounded below by the thick-target result. Furthermore, these results are true for any cross-section.

3. Source Characterized by a Single Column Density

Consider the rather artificial case where the source is characterized by a single value of column density $N_0$, which all electrons either traverse or are stopped in (corresponding to $E_0^2 > 2K'N_0$, and $E_0^2 < 2K'N_0$, respectively). Then

$$F(E, N) = F_0(E)\delta[N - \min(N_0, E^2/2K')] [1 + H(N_0 - E^2/2K')].$$

The factor involving the Heaviside step function $H$ results from again taking care when the $\delta$ function has argument zero at an endpoint of the range of integration. $G$ becomes

$$G(E) = \int_{E}^{(E^2 + 2K'N_0)^{1/2}} F_0(E_0) \, dE_0.$$  \hspace{1cm} (9)

Differentiating (9), we get

$$F_0(E) = \frac{E}{(E^2 + 2K'N_0)^{1/2}} F_0[(E^2 + 2K'N_0)^{1/2}] - G'(E).$$  \hspace{1cm} (10)
Now, we can evaluate (10) at $E = (E^2 + 2K'N_0)^{1/2}$, and substitute the resulting expression back in (10) to get

$$F_0(E) = \frac{E}{(E^2 + 4K'N_0)^{1/2}} F_0[(E^2 + 4K'N_0)^{1/2}] - \frac{E}{(E^2 + 2K'N_0)^{1/2}} G'[(E^2 + 2K'N_0)^{1/2}] - G'(E).$$

So, by repeated use of (10) in this manner, and using $F_0(E) \to 0$ as $E \to \infty$ (for any physically meaningful $F_0$), we get

$$F_0(E) = -G'(E) - E \sum_{m=1}^{\infty} \frac{G'((E^2 + 2mK'N_0)^{1/2})}{(E^2 + 2mK'N_0)^{1/2}}. \quad (11)$$

Here the thick-target limit ($N_0 \to \infty$) is recovered immediately, and the thin-target result, given for a power-law photon spectrum by Brown (1976) may be recovered as follows. If the photon spectrum is a power law, so (with cross-section (5)) is $G$ (Brown, 1971). Then the sum in (11) may be expressed as the generalized Riemann-zeta function. For the case $E_0^2/2K'N_0 \gg 1$, use in (11) of the asymptotic expansion of this function (Erdelyi et al., 1954) recovers Brown's (1976) result.

Brown and Emslie (1988) showed that any photon spectrum which can be interpreted as resulting from a thick target (i.e., for which $G'(E) < 0 \forall E$) can also be interpreted as resulting from a thin-target source. Equation (11) extends this result by showing that a photon spectrum interpretable as a thick target may also be interpreted as a source of an arbitrary column density $N_0$.

Some loose insight into the physical reason for the maximal efficiency of the thick target may be gleaned from Equation (9). The integral of $F_0$ from $E$ to $(E^2 + 2K'N_0)^{1/2}$ must be equal to $G(E)$, for all $E$. $G(E)$ is fixed (by observations). Clearly $F_0$ which satisfies this requirement for $N_0 \to \infty$ can be everywhere smaller than for any finite $N_0$, just because the range of integration becomes infinite. Physically this is equivalent to the fact that the thick-target source makes maximum use of every electron injected into it.

### 4. Inference of Instantaneous Electron Fluxes

The upshot of Section 2 is as follows: the burst-integrated electron flux inferred from the burst-integrated photon flux is minimized, at every energy of interest, if we assume that the source is a thick target. In fact one often finds that total electron fluxes are calculated by: making Brown's (1971) assumption that electrons stop in times shorter than instrumental time resolution; calculating an electron flux at each time; and integrating the electron fluxes over the duration of the burst (e.g., Hoynig, Brown, and van Beek, 1976). Then it is of interest to ask: what error is incurred in the inferred electron flux (and thus in the total electron energy) if electron lifetimes are in fact longer...
than instrumental time resolution? Both the total, burst-integrated electron flux and the instantaneous electron fluxes are of interest here, since the latter are often used in discussion of other flare emissions (e.g., Gunkler et al., 1984).

We do not attempt to give a general answer here. Instead we illustrate how error may arise in the instantaneous electron fluxes by appealing to the 'trap' model (Takakura and Kai, 1966; Melrose and Brown, 1976), in which electrons are trapped in the corona by magnetic convergence. Their lifetimes there are of order 10–100 s. This model may give a reasonable description of extended hard X-ray bursts (Vilmer, Kane, and Trottet, 1982).

Denote by $N(E, t)$ the number of electrons per keV of energy $E$ in the trap at time $t$, and by $S(E, t)$ the rate at which they are being injected (keV$^{-1}$ s$^{-1}$). Then, if no electrons are present before $t = 0$, we have (Melrose and Brown, 1976)

$$N(E, t) = \frac{E^{1/2}}{K n_0} \int_E^{E_0} S(E', t') \, dE', \quad (12)$$

where $n_0$ is the trap density (cm$^{-3}$), assumed uniform, and

$$E'^{3/2} - E^{3/2} = \frac{3}{2} K n_0 (t - t');$$

$$E_0 = [E^{3/2} + \frac{3}{2} K n_0 t]^{2/3}.$$

In (12) we have assumed perfect trapping in the corona. Escape is difficult to treat correctly (MacKinnon, 1988, 1989), and the present calculation is for illustrative purposes only.

The X-ray flux at time $t$ is then

$$I(\varepsilon, t) = n_0 \int_{\varepsilon}^{\infty} v(E) \frac{d\sigma}{d\varepsilon} N(E, t) \, dE =$$

$$= \frac{1}{K' \varepsilon} \int_{\varepsilon}^{\infty} q(\varepsilon, E) \int_E^{E_0} S(E', t') \, dE' \, dE'. \quad (13)$$

We might erroneously try to interpret the flux (13) by supposing that all electrons stop effectively instantaneously in a thick target. Then comparison of (6) and (13) shows that the inferred injected flux $S_*(E, t)$ and the actual flux $S$ would be related via

$$\int_E^{\infty} S_*(E', t) \, dE' = \int_E^{E_0} S(E', t') \, dE'.$$
and, therefore, differentiating w.r.t. \( E \),

\[
S_*(E, t) - S(E, t) = -\frac{E^{1/2}}{Kn_0} \int_{E_0}^E \frac{\partial S}{\partial t'}(E', t') \, dE'
\]

(14)

(we have assumed \( S(E, 0) = 0 \forall E \)). Result (14) shows that \( S_* \) underestimates \( S \) in the rise phase of the burst, and overestimates it in the decline.

However, now consider the error in the burst-integrated flux by integrating (14) over time, changing the order of integration in the right-hand side, and changing variable of integration from \( t \) to \( t' \):

\[
\int_0^\infty [S_*(E, t) - S(E, t)] \, dt = -\frac{E^{1/2}}{Kn_0} \int_0^\infty \int_0^\infty \frac{\partial S}{\partial t'}(E', t') \, dt' \, dE' =
\]

\[
= -\frac{E^{1/2}}{Kn_0} \int_E^\infty [S(E', \infty) - S(E', 0)] \, dE' = 0 .
\]

The integral of \( S_* \) over the burst duration gives the correct result, even although its instantaneous values are erroneous.

5. Concluding Remarks

We embarked on this work with the aim of seeing if uncritical application of Brown's (1971) results could have led to the total electron energy being over- or under-estimated. This possibility now seems almost excluded. Interpretation of an observed photon fluence on the assumption of a thick-target source does indeed yield a lower limit to the burst-integrated electron flux at every energy. Moreover, no error in this time-integrated flux results from assuming that the electrons stop instantaneously, even if they are in fact trapped in a region of low density. Although established here in a particular, idealised case, it seems likely that this last result should hold in general (For example, consider a situation where some fraction of all the electrons injected stopped immediately, and the rest remained trapped. Then the same result would follow.) A more general proof might proceed by including all statistical aspects of electron evolution in the injected flux, as was done in order to establish the maximal efficiency of the thick-target source. We do not pursue this here.

We have deliberately not addressed the inference from X-rays of the population of electrons instantaneously present. While this may be useful in certain circumstances (e.g., the trap model; Melrose and Brown, 1976), it seems to us an unwise procedure in general. One learns nothing about how quickly the electrons are having to be replenished at a given moment, which is what determines the total energy requirement.
One final caveat, obvious but important: all of this paper has addressed fast electrons in a 'cold' medium. In the light of the foregoing analysis, it seems all the more certain that such an X-ray source implies a high efficiency of particle acceleration. This theoretically undesirable consequence can be avoided only if all electrons in the source have similar energies. This is a qualitatively different situation from the class of models considered here, and it demands a qualitatively different analysis.

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References