A FORMULATION OF NON-IDEAL LOCALIZED (OR BALLOONING) MODES IN THE SOLAR CORONA

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Abstract. The stability equations for localized (or ballooning) modes in the solar atmosphere are formulated. Dissipation due to viscosity, resistivity, and thermal conduction are included using the general forms due to Braginskii (1965). In addition, the effect of gravity, plasma radiation, and coronal heating are included. The resulting equations are one-dimensional and only involve derivatives along the equilibrium magnetic field. Thus, the stabilising influence of photospheric line-tying, which is normally neglected in most numerical simulations, can be studied in a simple manner. Two applications to sound wave propagation and thermal instabilities in a low-beta plasma are considered with a view to determining realistic coronal boundary conditions that model the lower, denser levels of the solar atmosphere in a simple manner.

1. Introduction

The MHD equations that describe the solar coronal plasma exhibit a wide variety of instabilities including ideal and resistive instabilities as well as Rayleigh–Taylor and radiatively driven thermal instabilities. Normally, these instabilities are studied in isolation without attempting to investigate how they are coupled to and modified by other modes. MHD instabilities are thought to be responsible for releasing magnetic energy during a solar flare and thermal instabilities have been suggested as a means for forming a prominence or filament. Thermal instabilities have been studied by several authors. Field (1965) extended the pioneering work of Parker (1953) by analyzing an infinite, uniform medium. Several modes were identified, including the thermal or radiative mode that is driven by the radiative loss function in the thermal energy balance. Although this mode is relatively slow, compared with the Alfvén and sound travel times, it provides a simple mechanism for explaining the formation of cool prominences. However, the details of such a formation have yet to be worked out.

There have been many extensions to this basic instability. The analysis of the thermal stability of coronal loops, in which the plasma is now of finite length and the basic state is nonuniform, has been studied by several authors (e.g., Antiochos, 1979; Habbal and Rosner, 1979; Hood and Priest, 1980a; and Wragg and Priest, 1982). The various conclusions are mainly due to the different boundary conditions that have been selected.

The coupling of the radiative mode to MHD modes, and in particular the tearing

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mode, due to a sheared, infinite magnetic field has been investigated by Van Hoven, Steinolfson, and Tachi (1983), Steinolfson (1983), and Van Hoven, Tachi, and Steinolfson (1984), with particular reference to enhancing the tearing mode time-scale as a mechanism for explaining the fast energy release during a solar flare. Including the effects of compressibility and viscosity, Tachi, Steinolfson, and Van Hoven (1985) have extended this work. Alternatively, the radiative-tearing mode has been used by An (1985, 1986) and An et al. (1985) to analyse coronal condensations. In all of the above cases, the basic plasma has been assumed infinite in extent and no attempt has been made to allow for the important effect of the dense photosphere (except a trial function approach by An, 1984, for very long loops).

The dense photosphere has a very strong stabilizing effect on standard ideal and resistive MHD instabilities (see, for example, Hood and Priest, 1979, 1980b; Hood, 1983; Einaudi and Van Hoven, 1983; and Cargill, Hood, and Migliuolo, 1986). The main effect, as far as tearing modes are concerned, is that there is no longer a singular surface for global disturbances, unless the line-of-sight magnetic fields have a polarity reversal. This prevents a global tearing mode from forming and as all the above radiative-tearing calculations require such a singular surface their importance to solar situations is not at all clear. Resistive effects and photospheric line-tying have been studied by Velli and Hood (1986) for localized, interchange modes. It is discovered that there is no threshold for resistive instabilities but that an instability always exists, changing its growth rate from diffusion when the plasma is ideally stable to a fractional power of resistivity when ideal marginal stability is approached. Van der Linden, Goossens, and Hood (1987, 1988) showed how viscosity provides a strong stabilising effect on these modes.

Hollweg (1985) has suggested that viscosity could well be the main dissipative effect in the solar corona and so the general viscous stress tensor (Braginskii, 1965) is included in this formulation. In general, including line-tying with the above dissipation terms converts the stability analysis from a one-dimensional to a two-dimensional problem. This complication means that complex numerical methods must be used to solve the coupled partial differential equations. Unfortunately, the underlying physics is sometimes hidden because of the complexity in obtaining solutions. To circumvent this, the ballooning approximation is used and localized instabilities are examined. The first detailed description of localized (or ballooning) modes, in laboratory plasmas, was given by Connor, Hastie, and Taylor (1979) with a more rigorous treatment by Dewar and Glasser (1983). Hood (1986) used this ordering in the solar coronal context to study the ideal MHD stability of line-tied magnetic fields. While these instabilities may not give rise to large-scale disruptions and condensations, they allow the effect of line-tying to be studied in detail. In addition, these instabilities automatically generate short length scales and so they may well be important in coronal heating problems and in explaining the small-scale structures in prominences.

In Section 2, the general MHD equations are presented and the basic state is specified. In the next section these equations are linearized and the final set of differential equations is given in a form fit for numerical computations, with a detailed derivation.
in the Appendix. Section 4 analyses the effect of line-tying on the radiative mode and the conclusions are in the last section.

2. MHD Equations and Equilibrium

The advantage of the localized (or ballooning) mode approach is that many extra non-ideal effects can be included without severely complicating the analysis. Only derivatives along field lines remain and so photospheric line-tying can easily be included. For this discussion, the MHD equations are taken as

\[ \rho \frac{Dv}{Dt} = -\nabla p + j \times B + \rho g - \nabla \cdot (W) , \]  \hspace{1cm} (2.1)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 , \]  \hspace{1cm} (2.2)

\[ \eta j = E + v \times B , \]  \hspace{1cm} (2.3)

\[ \frac{\rho}{\gamma - 1} \frac{D}{Dt} \left( \frac{p}{\rho^\gamma} \right) = \nabla \cdot (\kappa \nabla T ) - \mathcal{P} , \]  \hspace{1cm} (2.4)

\[ p = \rho \frac{\mathcal{R}}{\mu} T , \]  \hspace{1cm} (2.5)

with

\[ j = \frac{\nabla \times B}{\mu} , \]  \hspace{1cm} (2.6)

\[ \frac{\partial B}{\partial t} = -\nabla \times E , \]  \hspace{1cm} (2.7)

and the convective time derivative defined by \( D/Dt \equiv \partial/\partial t + v \cdot \nabla \). \( \eta \) and \( \kappa \) are the resistivity and thermal conductivity tensors respectively, \( W_{ij} \) is the viscous stress tensor (Braginskii, 1965), and \( \mathcal{P} \) is the energy gain-loss function that includes plasma radiation, coronal and ohmic heating. In addition, \( \nabla \cdot B = \nabla \cdot j = 0 \).

There are five viscosity coefficients, but in a strong magnetic field, \( \eta_2 = 4 \eta_1 \); \( \eta_4 = 2 \eta_3 \) and the viscosity tensor takes the form

\[ W_{ij} = 3 \eta_0 \left( \frac{1}{3} \delta_{ij} - \frac{B_i B_j}{B^2} \right) \left( \frac{B_k B_m}{B^2} v_{km} - \frac{\nabla \cdot v}{3} \right) - \]

\[ - \eta_1 \left[ 2v_{ij} - \delta_{ij} \nabla \cdot v + \frac{1}{B^2} \left( \delta_{ij} B_k B_m v_{km} - 2v_{im} B_mB_j - \right) \right] \]
\[ -2v_{ij}B_k B_i + B_i B_j \nabla \cdot \mathbf{v} + \frac{B_i B_j}{B^2} B_k B_m v_{km} \] -

\[ -8 \eta_1 \left[ v_{im} B_j B_m + v_{kj} B_i B_k - 2B_i B_j \frac{B_k B_m}{B^2} v_{km} \right] \frac{1}{B^2} \]

\[ -\eta_3 \left[ v_{im} b_{jm} + v_{jk} b_{ik} - \frac{v_{km}}{B^2} (B_i B_k b_{jm} + B_j B_m b_{ik}) \right] \]

\[ -4 \eta_3 \frac{v_{km}}{B^2} (b_{ik} B_j B_m + b_{jm} B_i B_k) \], \quad (2.8) \]

where

\[ v_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (2.9) \]

and

\[ b_{ij} = \varepsilon_{ijk} \frac{B_k}{B}; \quad (2.10) \]

\( \varepsilon_{ijk} \) is an antisymmetric unit tensor and summation over repeated indices is assumed. \( \eta_0 \approx 10^{-17} T^{5/2} \) kg m \(^{-1}\) s \(^{-1}\) (Lifshitz and Pitaevskii, 1981; Hollweg, 1985) comes from the contribution to viscosity parallel to the magnetic field, \( \eta_1 = \frac{3}{10} (\eta_0 / (\omega \tau)^2) \) is due to perpendicular viscosity and \( \eta_3 = \frac{1}{2} (\eta_0 / \omega \tau) \) to finite Larmor radius effects (see Lifshitz and Pitaevskii, 1981). Here \( \omega \tau \) is the (dimensionless) product of proton-cyclotron frequency \( \omega \) and proton-collision time \( \tau \). When all quantities are measured in SI units, we have \( \omega \tau \approx 1.6 \times 10^{15} BT^{3/2} / n \log A \), where \( \log A \) is the Coulomb logarithm, which is mostly taken as 22 in solar corona applications (Braginskii, 1965), and \( n \) is the number density per m \(^3\).

The left-hand side of Ohm’s law, Equation (2.3), can be split into components parallel and perpendicular to the equilibrium magnetic field as

\[ \eta_\parallel \mathbf{j}_\parallel + \eta_\perp \mathbf{j}_\perp \], \quad (2.11) \]

where for strong magnetic fields, \( \omega \tau \gg 1 \),

\[ \eta_\perp \approx 2 \eta_\parallel \approx 0.13 \times 10^3 \log A T^{-3/2} \text{ kg m}^{-2} \text{ C}^{-1} \text{ s}^{-1}. \]

Similarly, the thermal conduction term can be rewritten as

\[ \nabla \cdot (\kappa \nabla T) = B \cdot \nabla \left( \frac{\kappa_\parallel}{B^2} B \cdot \nabla T \right) + \nabla \cdot \left( \kappa_\perp \frac{B \times (\nabla T \times B)}{B^2} \right), \quad (2.12) \]

where

\[ \kappa_\parallel \approx 1.8 \times 10^{-10} \frac{T^{5/2}}{\log A} \text{ W m}^{-1} \text{ K}^{-1} \quad (2.13) \]
and
\[ \kappa_{\perp} \approx 2 \times 10^{-31} \frac{n^2}{T^3 B^2} \kappa_{||} \]  
(from Braginskii, 1965).

To obtain the equilibrium equations we assume $v$ and all time derivatives vanish. The coordinate system used is shown in Figure 1. Assuming that all variables are independent of the distance $y$, the magnetic field can be written as
\[ \mathbf{B} = \nabla A \times \mathbf{e}_y + B_y(A)\mathbf{e}_y. \]

![Figure 1](image.png)

Fig. 1. The Cartesian coordinate system ($x, y, z$) and the orthogonal flux coordinate system ($A, l, y$). The $y$-axis is perpendicular to the plane of the figure and pointing backwards. $\nabla A$ is the direction perpendicular to the magnetic flux surfaces and $\nabla l$ is along the poloidal magnetic field.

The magnetohydrostatic equation (2.1) then has components
\[ \frac{\partial}{\partial A} \left( \mu p \right) - \mu \rho g_A = -B_y \frac{dB_y}{dA} - \nabla^2 A \tag{2.14} \]

and
\[ \frac{\partial p}{\partial l} - \rho g_p = 0, \tag{2.15} \]

where
\[ g_A = \frac{g \cdot \nabla A}{|\nabla A|^2}, \quad g_p = \frac{g \cdot B_p}{B_p}, \quad \mathbf{B}_p = \nabla A \times \mathbf{e}_y, \]

and $l$ is the distance along the poloidal field in the direction
\[ \nabla l = \frac{\nabla A \times \mathbf{e}_y}{B_p}. \]

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The energy equation reduces to
\[ B_p \frac{\partial}{\partial l} \left( \frac{\kappa \parallel B_p}{B^2} \frac{\partial T}{\partial l} \right) = \mathcal{L}, \tag{2.16} \]
since the contribution from \( \kappa \perp \) is much smaller than that from \( \kappa \parallel \), and all equilibrium quantities are independent of \( y \). The loss function consists of plasma radiation, a coronal heating function and Ohmic dissipation. However, Ohmic dissipation, in general, does not significantly contribute to the equilibrium heating. Hence, for the equilibrium state,
\[ \mathcal{L} = p^2 \chi T^{a-2} - h, \]
where \( \chi \) and \( a \) are defined by, for example, Hildner (1974) and Rosner, Tucker, and Vaiana (1978). The heating function \( h \) may depend on \( p, T, B \) and even the distance along the field lines, but its specific form need not be discussed here. For the ballooning ordering, Ohmic dissipation only appears in the first-order correction equations, and so is now neglected. Thus, we lose one of the methods of coupling radiation and resistive instabilities and also the super heating instabilities (Steinolfson, 1983).

In deriving the stability equations in the next section, it is assumed that all equilibrium quantities are known, although obtaining solutions to (2.14), (2.15), and (2.16) can be a complicated problem in its own right. However, if the plasma beta is small, then the left-hand side of (2.14) is negligible and the right-hand side defines a force-free magnetic field. This then defines the geometry and Equations (2.15) and (2.16) can be solved along each field line to build up the equilibrium. The first-order corrections to the field could then be evaluated from (2.14) by substituting \( p \) and \( \rho \) into the expression on the left-hand side.

3. Stability Equations for Localized Modes

Equations (2.1)–(2.7) are linearized and, assuming temporal variations of the form \( e^{st} \) for the perturbations, give
\[
\begin{align*}
\rho s \mathbf{v} &= -\nabla p_1 + \mathbf{j}_1 \times \mathbf{B} + \mathbf{j} \times \mathbf{B}_1 - \rho_1 \mathbf{g} e_z - \nabla \cdot (W), \tag{3.1} \\
s \rho_1 + v \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} &= 0, \tag{3.2} \\
\eta \parallel \mathbf{j} \parallel + \eta \perp \mathbf{j} \perp &= \mathbf{v} \times \mathbf{B} + \mathbf{E}_1, \tag{3.3} \\
\frac{1}{\gamma - 1} (s p_1 + v \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v}) &= -\nabla \cdot \mathbf{Q}_1 - \frac{\partial \mathcal{L}}{\partial T} T_1 - \frac{\partial \mathcal{L}}{\partial p} p_1 - \frac{\partial \mathcal{L}}{\partial B} \frac{\mathbf{B} \cdot \mathbf{B}_1}{B}, \tag{3.4} \\
\frac{p_1}{\rho} &= \frac{\rho_1}{T} + \frac{T_1}{T}. \tag{3.5}
\end{align*}
\]
sB₁ = −∇ × E₁, \hspace{1cm} (3.6)
∇ · B₁ = 0 , \hspace{1cm} (3.7)
∇ · j₁ = 0 , \hspace{1cm} (3.8)

where all perturbation quantities except v = v₁ and W = W₁ are labelled by a subscript 1. Q₁ is the linearized heat flux. In Equation (3.1) we have assumed that the gravitational force is directed opposite to the z-axis, and we have neglected the perturbation of gravity. To satisfy (3.7), we may write the perturbed magnetic field as

B₁ = ∇ × a₁ , \hspace{1cm} (3.9)

i.e., in terms of a magnetic vector potential, and then

\[ j₁ = \frac{1}{\mu} \left( \nabla \times (\nabla \times a₁) \right). \] (3.10)

Assuming that the modes are localized about a particular magnetic flux surface, all perturbation quantities take the form

\[ f(r) = f(A) \exp \left( i \frac{S(r)}{\varepsilon} \right), \]

where \( \varepsilon \) measures the ratio of the perturbation to equilibrium length scales and is assumed small. Thus, the disturbances have a rapid variation in and across each magnetic flux surface but have a slow variation along each field line if

\[ B \cdot \nabla S = 0 . \] (3.11)

A simple solution to (3.11) is

\[ S = y - \int \frac{B_y}{|\nabla A|} \, dl = y - \int \frac{B_y}{B_p} \, dl . \] (3.12)

From (3.12) it is straightforward to show that

\[ \nabla A \cdot (\nabla S \times B) = B^2 . \] (3.13)

Before proceeding to the final form of the stability equations some useful identities are presented. Define

\[ K_s \equiv \frac{1}{B^4} \left( B \times \nabla S \right) \cdot K , \] (3.14)

where the equilibrium field line tension vector is

\[ K = (B \cdot \nabla) B . \] (3.15)
In Cartesian (or slab) geometry, this term is equal to zero. Next we have

$$\mathbf{j} \cdot \nabla S = \frac{\partial p}{\partial A} - \rho g_A,$$

(3.16)

where $\mathbf{j}$ is the equilibrium electrical current density. Finally,

$$\nabla \cdot \left( \frac{(\nabla S \times \mathbf{B})}{B^2} \right) = \Lambda = 2K_s + \mu \frac{\mathbf{j} \cdot \nabla S}{B^2}.$$

(3.17)

To continue with the derivation of the final form of the linearized equations it is useful to uncurl the induction Equation (3.6) to give

$$\mathbf{E}_1 = -s \mathbf{a}_1 - \nabla \phi_1,$$

(3.18)

where the electrostatic potential $\phi_1$ is related, to leading order, to the velocity component in the direction $\mathbf{B} \times \nabla S$, as can be deduced from Ohm's law (3.3). The basic assumptions used in the derivation are that $s$, $\eta_0$, $\kappa_i$, $\mathbf{B}_1$, $p_1$, $\rho_1$, $E_1$, and $T_1$ are all $O(1)$ quantities while $\mathbf{j}_1$ is $O(\varepsilon^{-1})$, $\phi_1$, $\mathbf{a}_1$, and $\eta_3$ are $O(\varepsilon)$ and finally $\eta_1$, $\eta_2$, $\eta_3$, and $\kappa_\perp$ are $O(\varepsilon^2)$. It is useful to split up the perturbation quantities into perpendicular and parallel components, using $\mathbf{B}$, $\nabla S$ and $\mathbf{B} \times \nabla S$ as the directions of basis vectors.

A detailed derivation, following Connor and Hastie (1985) but for a single fluid model and including gravitational effects and a loss function, $\mathcal{L}_i$ is given in the appendix. The final equations are given by

$$\mathbf{B} \cdot \nabla \phi_1 = -s \left( 1 + \frac{\eta_1 |\nabla S|^2}{\mu \varepsilon^2 s} \right) a_\parallel B,$$

(3.19)

$$\mathbf{B} \cdot \nabla \left( \frac{\mathbf{v}_\parallel}{B} \right) = F_1,$$

(3.20)

$$\mathbf{B} \cdot \nabla \left( \frac{|\nabla S|^2 a_\parallel}{\mu \varepsilon B} \right) = F_2,$$

(3.21)

$$\mathbf{B} \cdot \nabla (\varphi) = -s \rho v_\parallel B - \frac{i}{\varepsilon} a_\parallel B (\mathbf{j} \cdot \nabla S) + \rho g_p B_p + \frac{\omega}{B} \mathbf{B} \cdot \nabla B -$$

$$-4 \frac{\eta_1 B |\nabla S|^2}{\varepsilon^2} v_\parallel + \frac{\eta_3 \phi}{\varepsilon^2} \mathbf{B} \cdot \nabla \left( \frac{|\nabla S|^2}{B} \right) +$$

$$+ 2 \frac{\eta_3 |\nabla S|^2}{B \varepsilon^2} \mathbf{B} \cdot \nabla \phi_1 - \frac{i v_\parallel}{\varepsilon} (\nabla S \times \mathbf{B}) \cdot \nabla \eta_3 +$$

$$+ 3 \frac{i \eta_3 v_\parallel}{\varepsilon B^2} (\nabla S \times \mathbf{B}) \cdot \nabla B^2 + \frac{6 i \eta_3 v_\parallel (\mathbf{j} \cdot \nabla S)}{\varepsilon},$$

(3.22)
\[ \mathbf{B} \cdot \nabla \mathcal{F} = \frac{\gamma p}{\gamma - 1} F_1 + F_3, \quad (3.23) \]

\[ \mathbf{B} \cdot \nabla T_1 = \frac{1}{\kappa_\parallel} B^2 F_4, \quad (3.24) \]

\[ \mathcal{P} = p_1 + \frac{2}{3} \omega, \quad (3.25) \]

\[ \omega = -2 \eta_0 F_1 + \eta_0 \nabla \cdot v_\perp - 3 \frac{\eta_0 v_\parallel B \cdot \nabla B}{B^2} + \frac{3 i \eta_0 \phi_1 K_s}{\varepsilon}, \quad (3.26) \]

\[ \frac{\mathbf{B} \cdot \mathbf{B}_1}{\mu} = -p_1 + \frac{\omega}{3} - \frac{\eta_3 |\nabla S|^2}{\varepsilon^2 B} \phi_1, \quad (3.27) \]

\[ \nabla \cdot v_\perp = -\frac{i}{\varepsilon} \phi_1 A - \frac{s}{B^2} \left( 1 + \frac{\eta_\perp |\nabla S|^2}{\varepsilon^2 \mu S} \right) \mathbf{B} \cdot \mathbf{B}_1, \quad (3.28) \]

\[ \frac{\rho_1}{\rho} = \frac{p_1}{p} - \frac{T_1}{T}, \quad (3.29) \]

\[ F_1 = -\nabla \cdot v_\perp - s \left( \frac{p_1}{p} - \frac{T_1}{T} \right) - \frac{v_\parallel B_p}{\rho B} \frac{\partial \rho}{\partial l} + i \phi_1 \left( \frac{\partial \rho}{\partial A} + \frac{B_y \nabla S \cdot \nabla A}{B^2 B_p} \frac{\partial \rho}{\partial l} \right), \quad (3.30) \]

\[ F_2 = -\frac{|\nabla S|^2 s \rho}{\varepsilon B^2} \phi_1 + i \Delta p_1 + i (\mathbf{j} \cdot \nabla S) \frac{\mathbf{B} \cdot \mathbf{B}_1}{B^2} + \]

\[ + i \phi_1 \left( \frac{g_A + g_p}{B^2 \nabla S \cdot \nabla A} \right) - \frac{i \omega}{3} \left( \frac{2 i \mathbf{j} \cdot \nabla S}{B^2} + \frac{\nabla S \times \mathbf{B} \cdot \nabla B}{B^3} \right) + \]

\[ + \frac{|\nabla S|^4}{\varepsilon^3 B^2} \eta_1 \phi_1 + 2 \frac{\eta_3 |\nabla S|^2}{\varepsilon B} F_1 + 2 \frac{v_\parallel \varepsilon B}{B \nabla} \mathbf{B} \cdot \nabla \left( \frac{\eta_3 |\nabla S|^2}{B} \right) + \]

\[ - \frac{v_\parallel \eta_3}{\varepsilon B} \mathbf{B} \cdot \nabla \left( \frac{|\nabla S|^2}{B} \right) - 2i \frac{|\nabla S|^2}{\varepsilon^2 B} \left( \frac{\eta_3 A}{B} + \frac{\nabla S \times \mathbf{B} \cdot \nabla \eta_3}{B^2} \right) \phi_1 - \]

\[ - \frac{\eta_3 s |\nabla S|^2}{\varepsilon B^3} \left( 1 + \frac{\eta_\perp |\nabla S|^2}{\varepsilon^2 \mu S} \right) (\mathbf{B} \cdot \mathbf{B}_1), \quad (3.31) \]

\[ F_3 = \frac{1}{\gamma - 1} \left( s p_1 + \frac{\partial \rho}{\partial l} \frac{B}{B} - \frac{i}{\varepsilon} \phi_1 \left( \frac{\partial \rho}{\partial A} + \frac{B_y \nabla S \cdot \nabla A}{B^2 B_p} \frac{\partial \rho}{\partial l} \right) \right) \times \]

\[ \times \frac{\gamma p}{\gamma - 1} \nabla \cdot v_\perp + \frac{\partial \mathcal{L}}{\partial T} T_1 + \frac{\partial \mathcal{L}}{\partial \rho} p_1 + \frac{\partial \mathcal{L}}{\partial B} \frac{\mathbf{B} \cdot \mathbf{B}_1}{B} + \frac{\kappa_\perp |\nabla S|^2}{\varepsilon^2} T_1 + \]

\[ + \frac{i a_\parallel}{\varepsilon B} \nabla S \times \mathbf{B} \cdot \nabla \left( \frac{\kappa_\parallel \mathbf{B} \cdot \nabla T}{B^2} \right) - \frac{\mathbf{B} \cdot \mathbf{B}_1}{B^2} \mathcal{L}, \quad (3.32) \]
\[ F_A = \mathcal{J} - \kappa_{\parallel 1} \frac{\mathbf{B} \cdot \nabla T}{B^2} + \kappa_{\parallel} \frac{\mathbf{B} \cdot \nabla T}{B^4} (\mathbf{B} \cdot \mathbf{B}_1) - \frac{i\kappa_{\parallel}}{\varepsilon B^3} a_{\parallel} \nabla S \times \mathbf{B} \cdot \nabla T, \quad (3.33) \]

where

\[ \kappa_{\parallel 1} = \frac{\partial \kappa_{\parallel}}{\partial T} T_1 + \frac{\partial \kappa_{\parallel}}{\partial \rho} \rho \left( \frac{p_1 - T_1}{p} T \right). \]

Thus, the stability equations are (3.19)–(3.24), supplemented by (3.25)–(3.33). Equations (3.19) and (3.21) are essentially the Alfvén wave equations coupled to both the slow mode equations (3.20) and (3.22) and the energy equations (and, hence, radiative mode) (3.23) and (3.24).

4. Applications

4.1. Idealized Equilibrium

Since the stability equations for localized modes only involve derivatives along the magnetic field, they are essentially one-dimensional and relatively straightforward to solve numerically. However, to illustrate the use of the equations, we restrict attention to a simplified equilibrium model of the corona and chromosphere. In each region, the temperature and density are assumed uniform and gravity is neglected. The magnetic field is potential, due to a line current at an arbitrary height but taken for illustration at a transition region level. The plasma beta is small everywhere and the two regions are linked by a transition region through which the temperature and density vary in a smooth but rapid manner. Thus, in cylindrical coordinates, as illustrated in Figure 2, the equilibrium is

\[ \mathbf{B} = \left( 0, \frac{B_0 b}{r}, 0 \right), \]

\[ p = p_0, \quad \beta = \frac{2\mu p_0}{B_0^2} \ll 1, \]

\[ T = \begin{cases} 
T_u & \text{for } -\frac{\pi}{2} + \delta \theta \leq \theta \leq \frac{\pi}{2} - \delta \theta, \\
T(\theta) & \text{for } \frac{\pi}{2} - \delta \theta \leq \theta \leq \frac{\pi}{2} + \delta \theta, \\
T_1 & \text{for } \frac{\pi}{2} + \delta \theta \leq \theta \leq \frac{3\pi}{2} - \delta \theta, 
\end{cases} \]

\[ p = 2\rho \mathcal{A} T, \]

(4.1)
where \( \tilde{\mu} \) has been taken as 0.5. The energy equation is \( \mathcal{L} = 0 \) in regions 1 and 3, but

\[
\frac{1}{r^2} \frac{d}{d\theta} \left( \kappa_r \frac{dT}{d\theta} \right) = \mathcal{L}
\]

(4.2)

in region 2. It is assumed that the transition region is narrow compared with coronal length scales and chromospheric scale height. The actual structure in region 2, the transition region, will not be discussed here in detail. Thermal conduction is assumed negligible everywhere, except in region 2.

There are several obvious weaknesses in the above model. For example, gravity has been neglected in the chromosphere, and the optically thick losses are only modelled by characteristic timescales, that is setting \( \partial \mathcal{L}/\partial T \) and \( \partial \mathcal{L}/\partial p \) as constants in region 3. However, it is not our purpose to obtain a detailed solar atmosphere but rather to see how well certain coronal disturbances connect through the transition region to the chromosphere. Previous work, by Hood (1986), on the choice of coronal boundary conditions that model photospheric line-tying, has shown that unstable localized modes satisfy the rigid wall conditions with \( \xi_{||} = \xi \cdot \mathbf{B}/B \), the parallel component of the displacement) vanishing at the coronal interface. The position of the transition region is allowed to move and is located at \( \theta = \pi/2 + \xi_{||}(\pi/2)/r \).

4.2. Stability equations

The stability equations are simplified by neglecting all viscosity, resistivity, and thermal conduction tensor components and gravity so that Equations (3.19)–(3.24) reduce, in regions 1 and 3, to

\[
\frac{1}{r} \frac{d}{d\theta} \xi_{||} = - \frac{p_1}{p} + \frac{T_1}{T},
\]

(4.3)
\[ \frac{1}{r} \frac{d}{d\theta} = -s^2 \rho \xi, \tag{4.4} \]

and

\[ \frac{s}{\gamma - 1} p_1 + \frac{\gamma p}{\gamma - 1} \frac{s}{r} \frac{d}{d\theta} \xi + \frac{\partial \mathcal{L}}{\partial T} T_1 + \frac{\partial \mathcal{L}}{\partial p} p_1 = 0. \tag{4.5} \]

Here, we have assumed that \( \beta \ll 1 \) (\( \beta = 2 \mu_0 B^2 \)), a typical plasma beta) and that the magnetic field disturbances are small \( (O(\beta)) \). Across the interface of region 2, the usual jump relations are taken, namely,

\[ \begin{align*}
\langle p_1 \rangle &= 0, \\
\langle \xi \rangle &= 0,
\end{align*} \tag{4.6} \]

to connect regions 1 and 3.

Equations (4.3)–(4.5), with (4.6), are now solved for two cases: (i) sound waves alone and (ii) sound and radiative modes together.

### 4.3. Sound Waves

The radiative terms are now neglected by setting \( \frac{\partial \mathcal{L}}{\partial T} = \frac{\partial \mathcal{L}}{\partial p} = 0 \). The basic equations are now (4.4) and (4.5), with (4.3) defining the temperature variations. Define the sound travel times \( \tau_{s1} = r/c_{s1} \) and \( \tau_{s3} = r/c_{s3} \), where the sound speeds \( c_{s1} \) and \( c_{s3} \) in regions 1 and 3 are given by \( \gamma p/\rho_1 \) and \( \gamma p/\rho_3 \) respectively. Scaling the frequency, \( \sigma(\sigma^2 = -s^2) \) against the coronal sound travel time \( \tau_{s1}^{-1} \), the even solutions are

\[ \xi = \begin{cases} 
\cos \sigma \theta & \text{for region 1}, \\
\alpha \cos \frac{\sigma}{v} (\pi - \theta) & \text{for region 3},
\end{cases} \]

where \( v^2 = \tau_{s1}^2/\tau_{s3} = \rho_1/\rho_3 \) is the ratio of the coronal and chromospheric densities. Using the matching conditions (4.6), the dispersion relation can be expressed as

\[ \cos \frac{\sigma \pi}{2} \sin \frac{\sigma \pi}{2v} + v \sin \frac{\sigma \pi}{2} \cos \frac{\sigma \pi}{2v} = 0. \tag{4.7} \]

In the limit \( v \ll 1 \), three expansions can be obtained. If \( m \) and \( n \) are integers, then the frequency and the interface amplitude are given by

\[ \begin{cases} 
\sigma_1 = 2n + 1 + \frac{2v}{\pi} \cot \left( \frac{2n + 1}{2v} \pi \right) + O(v^2) \\
\xi_1 \left( \pi \right) = (-1)^n \frac{2v}{\pi} \cot \left( \frac{2n + 1}{2v} \pi \right) + O(v^2)
\end{cases} \quad \text{for } 2n + 1 \neq 2nv, \tag{4.8} \]
\[
\begin{align*}
\text{II} & \quad \left\{ \begin{array}{ll}
\sigma_2 = 2m\nu - \frac{2\nu^2}{\pi} \tan(vmn\pi) + O(\nu^3) \\
\xi_{\|} \left( \frac{\pi}{2} \right) = \cos(vmn\pi) + O(\nu^3)
\end{array} \right. \\
& \quad \text{for } 2m\nu \neq 2n + 1, \quad (4.9)
\end{align*}
\]

and
\[
\begin{align*}
\text{III} & \quad \left\{ \begin{array}{ll}
\sigma_3 = 2n + 1 + \nu\lambda_1 \frac{2}{\pi} + O(\nu^2) \\
\xi_{\|} \left( \frac{\pi}{2} \right) = (-1)^n \nu\lambda_1 + O(\nu^2)
\end{array} \right. \\
& \quad \text{for } 2n + 1 = 2m\nu, \quad (4.10)
\end{align*}
\]

where \(\lambda_1\) satisfies \(\lambda_1 \tan \lambda_1 = 1\), with the first value of \(\lambda_1 \approx 0.86\). Series I and III correspond to coronal modes with disturbances that are restricted to the corona and satisfy the rigid wall line-tying conditions. Series II, on the other hand, corresponds to chromospheric modes with a substantial displacement across the transition region except in the neighbourhood of coronal modes as shown in Figure 3. The frequency, as a function of mode number, is shown in Figure 3(a) and the transition region amplitude

![Graph](attachment:image.png)

Fig. 3a. The frequency \(\sigma\) as a function of mode number \(M\) for the chromospheric sound waves that do not correspond to rigid wall line-tying conditions (referred to in the text as series II).

in Figure 3(b). Thus, if \(\nu\) is small and the chromospheric mode number is not large then low-frequency motions can indeed drive a parallel flow across the interface. This means that if the corona can be driven sufficiently slowly then the rigid wall conditions can be violated. However, as the mode number, \(m\), increases, the amplitude of the parallel flow
Fig. 3b. The transition region amplitude versus frequency for the same modes as in Figure 3(a).

decreases until, at a typical coronal frequency, the rigid wall conditions are again satisfied.

However, increasing the mode number still further, it can be seen from Figure 3(b) that the amplitude should increase once again. It would appear that coronal values for the frequency, that is between the coronal normal mode values, should be able to support a parallel flow at the transition region but for these disturbances the mode number in the chromosphere is very large \((m \sim O(v^{-1}))\) and so it is likely that these motions will be subject to substantial damping. Only the low \(m\) modes will remain, but how are they generated?

Disturbances generated in the corona will travel down to the transition region at the coronal sound speed. Both a reflected and a transmitted wave will be generated. Defining the incident wave as \(I = f(t - r\theta)\), where \(t\) is measured in units of the coronal sound travel time and \(f(x)\) is an arbitrary function, the reflected \((R)\), and transmitted \((T)\) waves are given by

\[
R = \frac{v - 1}{v + 1} f(t + r\theta)
\]

and

\[
T = \frac{2v}{v + 1} f(t - \frac{r\theta}{v}).
\]

Since \(v \ll 1\), it is obvious that the amplitude of the parallel velocity at the transition region is only or order \(v\). Thus, the rigid wall conditions are satisfied. Only if the disturbances are generated in the chromosphere or if the corona is driven at a low frequency it is possible for rigid wall line-tying to be violated.
4.4. Radiative Instability

The loss terms in (4.5) are now included to investigate the influence of the radiatively stable chromosphere. The growth rate, $s$, is made dimensionless against the coronal radiative time-scale. The equations reduce to

$$\frac{d^2 \xi_\theta}{d \theta^2} + \Gamma^2 \xi_\theta = 0 , \quad (4.11)$$

where

$$\Gamma_1^2 = - \frac{\delta_1^2 \delta^2}{v^2} \frac{(s + \mathcal{L}_{T1} + \mathcal{L}_{p1})}{(s + \mathcal{L}_{T1}/\gamma)} \text{ in the corona} , \quad (4.12)$$

$$\Gamma_2^2 = - \frac{\delta_1^2 \delta^2}{v^2} \frac{(s + \alpha_1)}{(s + \alpha_2)} \text{ in the chromosphere} .$$

$\delta_1 = \tau_1^1/\tau_{rad}^1$ is the ratio of the coronal sound travel time to radiative time-scale and $v^2 = \rho_1/\rho_3$ is the ratio of the densities in region 1 and 3. $\mathcal{L}_{T1} = d \ln T$ and $\mathcal{L}_{p1} = d \ln T/d \ln p$ in the corona and in region 3, $\alpha_1 = \mathcal{L}_{T3} + \mathcal{L}_{p3}$, $\alpha_2 = \mathcal{L}_{T3}/\gamma$ are adjustable due to our ignorance of a reliable analytic form for the optically thick radiative transfer effects. However, $\Gamma_2$ is not very sensitive to the actual values of $\alpha_1$ and $\alpha_2$ as long as they are of similar magnitude. $\Gamma_2^2$ will remain approximately equal to $- \delta_1^2 \delta^2 / v^2$.

The solutions to (4.11) are simply $\cos \Gamma_1 \theta$ and $\cos \Gamma_2 (\pi - \theta)$ corresponding to an even mode and $\sin \Gamma_1 \theta$ and $\sin \Gamma_2 (\pi - \theta)$ for the odd mode. The dispersion relations are then

$$\Gamma_1 \cos \Gamma_1 \frac{\pi}{2} \sin \Gamma_2 \frac{\pi}{2} + \Gamma_2 \cos \Gamma_1 \frac{\pi}{2} \sin \Gamma_2 \frac{\pi}{2} = 0 \quad (\text{even modes}) \quad (4.13)$$

and

$$\Gamma_1 \sin \Gamma_1 \frac{\pi}{2} \cos \Gamma_2 \frac{\pi}{2} + \Gamma_2 \sin \Gamma_1 \frac{\pi}{2} \cos \Gamma_2 \frac{\pi}{2} = 0 \quad (\text{odd modes}) . \quad (4.14)$$

Notice that the temperature and pressure perturbations are out of phase with the velocity. As an illustration, take $k_2 = -i \Gamma_2 = s \delta_1 / v \approx 1.8 \delta_1 / v$, for the radiative mode with $\alpha = -1$ so that $\mathcal{L}_{T1} = -3$ and $\mathcal{L}_{p1} = 2$, and notice that if the chromospheric radiative time-scale was dominant, then $k_2$ would be multiplied by $(\alpha_1 / \alpha_2)^{1/2}$, which we would expect to be of order unity. Consider the odd modes for which the parallel displacement (or equivalently parallel velocity) is given by $\sin \Gamma_1 \theta$. In two limits, we can obtain approximate solutions to the radiative mode, given by

$$I: \quad \Gamma_1 \ll k_2 = 1.8 \delta_1 / v ,$$

$$\Gamma_1 = 2n - \frac{2}{\pi} \frac{2n}{k_2} \tanh \left( k_2 \frac{\pi}{2} \right) , \quad (4.15)$$
\[ s = \left( 1.8 - \frac{2.592}{(2n)^2} \delta_1 \right) \tau^{-1}_{\text{rad}}, \]
\[ \xi_{||} \left( \frac{\pi}{2} \right) = (-1)^n \frac{2n}{k_2} \tanh \left( k_2 \frac{\pi}{2} \right) \ll 1, \]

and

\[ \Gamma_1 \gg k_2 = 1.8 \delta_1 / \nu, \]
\[ \Gamma_1 = 2n + 1 + \frac{2}{\pi} k_2 \left( 2n + 1 \right) \tanh \left( k_2 \frac{\pi}{2} \right)^{-1}, \]
\[ s = \left( 1.8 - \frac{2.592}{(2n + 1)^2} \delta_1 \right) \tau^{-1}_{\text{rad}}, \]
\[ \xi_{||} \left( \frac{\pi}{2} \right) = (-1)^n \left\{ 1 - \frac{1}{2} k_2 \left( 2n + 1 \right) \tanh \left( k_2 \frac{\pi}{2} \right)^{-2} \right\} \approx (-1)^n, \]

for integer values of \( n \).

These approximations can be shown to give a good agreement with the numerical solutions of (4.14). From the amplitudes in (4.15) and (4.16), the rigid-wall conditions hold if \( \Gamma_1 \ll k_1 \) and the flow-through conditions if \( \Gamma_1 \gg k_2 \). Obviously the latter condition will hold if the mode number \( n \) is large enough. However, large \( n \) modes rapidly oscillate in the corona and should be strongly damped by, for example, thermal conduction. The low \( n \)-modes are the important ones and the choice of boundary conditions will depend on the value of \( k_2 \), since \( \Gamma_1 \) is order unity for low \( n \) numbers. For \( k_2 \gg 1 \), rigid wall conditions hold. Now from the definition of \( k_2, \delta_1, \) and \( \nu \), this condition implies that the coronal radiative time-scale is much shorter than the chromospheric sound travel time. Thus, if the chromosphere cannot adjust to the changes in density (pressure perturbations are very small) then the corona sees the transition region as a rigid wall.

On the other hand, if the coronal radiative time-scale is much longer than the sound travel time in the chromosphere, then the flow-through conditions are the relevant boundary conditions.

Because the growth rate is close to the classical radiative value (1.8 in our case), the pressure perturbations are always small. The phase difference between temperature and velocity implies that the rigid wall conditions are consistent with vanishing perturbed conductive flux, whereas the flow-through conditions suggest that the perturbed temperature is zero.

5. Conclusions

There have been several previous papers in which ideal MHD modes have been coupled to various dissipative effects. In this paper all the important dissipation terms, viscosity,
resistivity and thermal conductivity have been included along with plasma radiation (to
drive radiative modes), gravity and coronal heating. Using the ballooning ordering, a
set of six coupled first-order differential equations have been derived that only involve
variations along the equilibrium field. Thus, the important effect of the dense
photosphere can easily be included in a simple manner allowing the influence of
line-tying to be studied in detail.

By selecting a subset of the available parameters, the influence of each physical effect
can be studied individually and in relation to others.

The stability equations for localized modes can be applied to a wide variety of
different circumstances. Already ideal and resistive MHD modes have been investigated
for coronal magnetic fields (Hood, 1986; Velli and Hood, 1986) and viscosity has been
included by Van der Linden, Goossens, and Hood (1987, 1988). These equations also
contain a description of Rayleigh–Taylor modes as well as radiatively driven thermal
modes.

Two examples of the use of these equations were discussed. Firstly the propagation
of sound waves along the magnetic field was studied and by including a dense
chromosphere the boundary conditions at the transition region were deduced. For
typical coronal frequencies, the rigid wall line-tying conditions are perfectly valid. This
confirms the findings of Hood (1986) for unstable modes. However, low-frequency modes
(almost certainly chromospheric in origin) can violate the rigid wall conditions allowing
a parallel flow across the transition region.

Secondly, the radiative thermal instability, in the absence of thermal conduction, was
studied. The modes exhibit an accumulation point at the uncoupled, isobaric radiative
growth rate of $sp = -(\gamma - 1)T(\partial L^2/\partial T)$. However, this limit is only attained for
extremely large coronal wave numbers and so the resulting rapid oscillations will be
damped out by, for example, thermal conduction. The first few harmonics should still
survive but, even for these modes, the growth rate remains close to the accumulation
value. For odd velocity perturbations (even temperature) about the summit, the funda-
mental mode has a growth rate larger than $1.7\tau_{\text{rad}}^{-1}$ whereas the accumulation point is
$1.8\tau_{\text{rad}}^{-1}$. The pressure perturbations are essentially zero in keeping with the usual
radiative mode.

However, the important conclusions concern the choice of boundary conditions, at
the base of the corona, that will model the response of the chromosphere to coronal
disturbances. The results depend on the relative size of the chromospheric wave number
(really a damping length since the disturbances are evanescent in the chromosphere) to
the coronal value. The value of the chromospheric wave number does not depend
strongly on whether or not chromospheric radiation or sound waves provide the
dominant time-scales, since in the limit of either mechanism being dominant, the value
is proportional to the ratio of the chromospheric sound travel time to the coronal
radiative time-scale. If this ratio is large then the usual rigid wall conditions hold since
the chromosphere cannot respond quickly enough to coronal density changes. The
temperature boundary conditions are best simulated by vanishing temperature gradient,
although thermal conduction may affect this conclusion by smoothing out any jumps
in temperature. On the other hand, if the ratio is small, the flow through boundary conditions are the relevant choice and the perturbed temperature will vanish at the top of the transition region.

Obviously more detailed modelling of the transition region and chromosphere and the inclusion of thermal conduction are necessary before the correct choice of boundary conditions can be finally resolved (Antiochos et al., 1985; McClymont and Craig, 1985).

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Appendix

This derivation closely follows the two-fluid model of Connor and Hastie (1985). The linearized MHD equations are (3.1)–(3.10) with (3.3) replaced by

$$\eta_\parallel \mathbf{j}_\parallel + \eta_\perp \mathbf{j}_\perp = \mathbf{v} \times \mathbf{B} - \mathbf{s} \mathbf{a}_1 - \nabla \phi_1, \quad (A.1)$$

on using (3.18). The first equation comes from taking the scalar product of (A.1) with $\mathbf{B}$, so that

$$\mathbf{B} \cdot \nabla \phi_1 + s \mathbf{a}_1 \cdot \mathbf{B} + \eta_\parallel \mathbf{j}_\parallel \cdot \mathbf{B} = 0.$$ 

All three terms are $O(\varepsilon)$ and the leading contribution to $\mathbf{j}_\parallel$ comes from (3.10) as

$$\varepsilon^2 \mathbf{j}_\parallel = -\frac{1}{\mu} \nabla S \times (\nabla S \times \mathbf{a}_1) + O(\varepsilon)$$

$$\quad = -\frac{\nabla S(\mathbf{a}_1 \cdot \nabla S)}{\mu} + \frac{|\nabla S|^2}{\mu} \mathbf{a}_1 + O(\varepsilon).$$

Hence, defining

$$\mathbf{a}_1 \equiv a_\parallel \frac{\mathbf{B}}{B} + a_\perp,$$

we have

$$\mathbf{B} \cdot \nabla \phi_1 + s \left(1 + \frac{|\nabla S|^2}{s\varepsilon^2 \mu} \eta_\parallel \right) a_\parallel B = 0. \quad (A.2)$$
Next consider (3.8). $j_{||1}$ is already known but an expression for $j_{\perp 1}$ can be obtained from $B \times (3.1)/B^2$. Thus (3.8) gives

$$B \cdot \nabla \left( \frac{|\nabla S|^2}{\mu \varepsilon^2 B} \right) + \nabla \cdot \left( s \rho \frac{B \times v}{B^2} \right) + \nabla \cdot \left( f \times B \right) + \nabla \cdot \left( \frac{B \times \nabla \cdot W}{B^2} \right) = 0 ,$$

(A.3)

where

$$f = -\nabla p_1 + j \times B_1 - \rho_1 g e_z .$$

(A.4)

Care must be taken in evaluating the last term in (A.3) since the leading contribution automatically cancels. The first term is $O(1/\varepsilon)$ and the second is of the same order, namely

$$\frac{i}{\varepsilon} \nabla S \cdot \left( s \rho \frac{B \times v}{B^2} \right) = \frac{|\nabla S|^2 s \rho \phi_1}{\varepsilon^2 B^2} ,$$

(A.5)

on using (A.1). The next term contributes as

$$\nabla \cdot \left( \frac{B \times f}{B^2} \right) = \frac{i}{\varepsilon} \Delta p_1 + \frac{i}{\varepsilon} (j \cdot \nabla S) \frac{B \cdot B_1}{B^2} - \frac{i}{\varepsilon} \rho_1 g \frac{\nabla S \cdot (B \times e_z)}{B^2} ,$$

(A.6)

and the last term requires more simplification. From (2.8), $\nabla \cdot W$ can be split into three contributions:

$$\nabla \cdot W = \nabla \cdot W_0 + \nabla \cdot W_1 + \nabla \cdot W_3 ,$$

where $W_{0ij}$, $W_{1ij}$, and $W_{3ij}$ involve $\eta_0$, $\eta_1$, and $\eta_3$ terms, respectively. Define

$$\omega \equiv -3 \eta_0 \left( \frac{B \cdot (B \cdot \nabla) v}{B^2} - \frac{1}{3} \nabla \cdot v \right) ,$$

(A.7)

then

$$W_{0ij} = -\left( \frac{B_i B_j}{B^2} - \frac{1}{3} \delta_{ij} \right) \omega ,$$

and

$$\nabla \cdot W_0 = \frac{B}{B^2} \left( B \cdot \nabla \omega \right) - \frac{1}{3} \nabla \omega + \omega \left( \frac{\nabla \cdot B}{B^2} - 2 \frac{B \cdot \nabla B}{B^3} \right) .$$

(A.8)

Similarly,

$$\nabla \cdot W_1 = -\frac{|\nabla S|^2}{\varepsilon^2} \left( \frac{(v \times B) \times B}{B^2} - 4 \frac{B \cdot v}{B^2} B \right) .$$

(A.9)
The expression for $\nabla \cdot W_3$ is not straightforward and requires some tedious algebra. It is useful to consider the parallel and perpendicular components of $v$ separately. Using the ballooning ordering in (A.1), so that $\nabla \to (i/\varepsilon)\nabla S + \nabla$, 

$$v_\perp = -\frac{i}{\varepsilon} \phi_1 \frac{\nabla S \times B}{B^2} - \frac{\nabla \phi_1 \times B}{B^2} + v_\perp^*, \quad (A.10)$$

where the first term is $O(1)$, while the second term and $v_\perp^*$ are $O(\varepsilon)$.

$v_\perp^*$ can be obtained from Ohm’s law and the leading contribution to the momentum equation

$$p_1 + \frac{\mathbf{B} \cdot \mathbf{B}_1}{\mu} = i\varepsilon \frac{\nabla S \cdot (\nabla \cdot W)}{|\nabla S|^2}, \quad (A.11)$$

as

$$v_\perp^* = -s \frac{a_1 \times B}{B^2} - \frac{i\eta_1}{\varepsilon B^2} \left(p_1 - \frac{i\varepsilon \nabla S \cdot (\nabla \cdot W)}{|\nabla S|^2}\right) \nabla S + O(\varepsilon^2). \quad (A.12)$$

Using (A.11) is equivalent to eliminating the stable fast mode wave. For (A.3) we can derive

$$\nabla \cdot \left(\frac{B \times \nabla \cdot W}{B^2}\right) = -2 i \frac{|\nabla S|^2}{B \varepsilon^3} \left(\eta_3 A + \frac{\nabla S \times B}{B^2} \cdot \nabla \phi_1\right) +$$

$$+ \frac{i \eta_3}{B \varepsilon^3} |\nabla S|^2 (v_\perp^* \cdot \nabla S) - \frac{v_\parallel \eta_3}{B \varepsilon^3} \mathbf{B} \cdot \nabla \left(\frac{|\nabla S|^2}{B}\right) +$$

$$+ 2 \frac{\eta_3}{B \varepsilon^2} \frac{|\nabla S|^2}{B^2} \mathbf{B} \cdot \nabla \left(\frac{v_\parallel}{B}\right) + 2 \frac{v_\parallel}{\varepsilon^2 B} \mathbf{B} \cdot \nabla \left(\frac{\eta_3 |\nabla S|^2}{B}\right) +$$

$$+ \frac{|\nabla S|^4}{\varepsilon^4 B^2} \eta_1 \phi_1 + \frac{i \omega}{3} \left(\frac{2 j \cdot \nabla S}{B^2} + \frac{1}{B^3} (\nabla S \times \mathbf{B}) \cdot \nabla B \right), \quad (A.13)$$

where $v_\parallel = (\mathbf{v} \cdot \mathbf{B})/B$ and $v_\perp^* \cdot \nabla S$ is derived later. (A.3), (A.5), (A.6), and (A.13) provide the second stability equation.

Now take the scalar product of $\mathbf{B}$ and the momentum equation (3.1) to obtain

$$\mathbf{B} \cdot \nabla p_1 + \rho s v_\parallel B = \mathbf{B} \cdot (j \times \mathbf{B}_1) + \rho_1 g \cdot \mathbf{B} - \mathbf{B} \cdot (\nabla \cdot W). \quad (A.14)$$

The first term on the right-hand side is readily found to reduce to

$$\mathbf{B} \cdot (j \times \mathbf{B}_1) = -\frac{i}{\varepsilon} a_\parallel B (j \cdot \nabla S), \quad (A.15)$$
\[ \mathbf{B} \cdot (\nabla \cdot \mathbf{W}) = \frac{2}{3} \mathbf{B} \cdot \nabla \omega - \frac{\omega}{B} \mathbf{B} \cdot \nabla B + 4 \frac{\eta_1 B |\nabla S|^2}{\varepsilon^2} v_\parallel + 6 \frac{i \eta_3 v_\parallel (\mathbf{j} \cdot \nabla S)}{\varepsilon} + \]
\[ + \frac{\eta_3 \phi_1}{\varepsilon^2} \mathbf{B} \cdot \nabla \left( \frac{|\nabla S|^2}{B} \right) + \frac{2 \eta_3 |\nabla S|^2}{\varepsilon^2 B} \mathbf{B} \cdot \nabla \phi_1 - \]
\[ - 2 \frac{i}{\varepsilon} \nabla S \times \mathbf{B} \cdot \nabla \eta_3 + 3 \frac{i \eta_3 v_\parallel}{\varepsilon^2 B^2} \nabla S \times \mathbf{B} \cdot \nabla B^2 . \] (A.16)

(A.14) with (A.15) and (A.16) provide the third stability equation. From (A.10) and (A.11) an expression for \( \nabla \cdot \mathbf{v} \) is
\[ \nabla \cdot \mathbf{v} = \mathbf{B} \cdot \nabla \left( \frac{v_\parallel}{B} \right) - \frac{i}{\varepsilon} \phi_1 A + \frac{i}{\varepsilon} \nu_\perp \cdot \nabla S \] (A.17)

and to evaluate \( \nu_\perp \cdot \nabla S \) we need, to leading order \( O(\varepsilon^{-1}) \),
\[ \nabla S \cdot (\nabla \cdot \mathbf{W}) = - \frac{i}{\varepsilon} \frac{|\nabla S|^2}{3} \left( \frac{\omega}{3} - \frac{\eta_3 |\nabla S|^2}{\varepsilon^2 B} \phi_1 \right) . \] (A.18)

Therefore,
\[ \nu_\perp \cdot \nabla S = - \frac{i \mu \varepsilon}{B^2} \left( 1 + \frac{\eta_3 |\nabla S|^2}{\varepsilon^2 s \mu} \right) \left( p_1 - \frac{\omega}{3} + \frac{\eta_3 |\nabla S|^2}{\varepsilon^2 B} \phi_1 \right) . \] (A.19)

From the continuity equation (3.2) we have
\[ s \rho_1 + v_\parallel \frac{B_\perp}{B} \frac{\partial \rho}{\partial l} - \frac{i}{\varepsilon} \phi_1 \left( \frac{\partial \rho}{\partial A} + \frac{B_\perp}{B^2 B_\parallel} \nabla S \cdot \nabla A \frac{\partial \rho}{\partial l} \right) + \rho \nabla \cdot \mathbf{v} = 0 , \] (A.20)

where \( \nabla \cdot \mathbf{v} \) is obtained from (A.17) and (A.19).

Now turn to the energy equation (3.4). The main problem here is to evaluate the thermal conduction terms:
\[ - Q_1 = \left( \kappa_{\parallel 1} \frac{\mathbf{B} \cdot \nabla T}{B^2} + \kappa_{\parallel} \frac{\mathbf{B} \cdot \nabla T_1}{B^2} - 2 \kappa_{\parallel} \frac{\mathbf{B} \cdot \nabla T_1}{B^2} \mathbf{B} \cdot \mathbf{B}_1 + \kappa_{\parallel} \frac{B_1 \cdot \nabla T}{B^2} \right) \mathbf{B} + \]
\[ + \kappa_{\parallel} \frac{\mathbf{B} \cdot \nabla T}{B^2} \mathbf{B}_1 + \kappa_{\perp} \frac{i T}{\varepsilon} \nabla S + O(\varepsilon^2) . \] (A.21)

Here \( \kappa_{\parallel 1} \) is the change in the parallel thermal conductivity and takes the form
\[ \kappa_{\parallel 1} = \frac{\partial \kappa_{\parallel}}{\partial T} T_1 + \frac{\partial \kappa_{\parallel}}{\partial \rho} \rho_1 , \] (A.22)
where the $\partial \kappa_\parallel / \partial \Phi$ comes from the weak density dependence of the Coulomb logarithm (see Braginskii, 1965) and is normally neglected. The $O(1)$ contributions that arise from thermal conduction are therefore

\[
- \nabla \cdot Q = B \cdot \nabla \left( \kappa_\parallel \frac{B \cdot \nabla T}{B^2} + \kappa_\parallel \frac{B \cdot \nabla T_1}{B^2} + i \kappa_\parallel \frac{a_\parallel \nabla S \times B \cdot \nabla T}{\varepsilon B^2} + \kappa_\parallel \frac{B \cdot \nabla T}{B^2} \right) + \\
+ \frac{i}{\varepsilon B} a_\parallel (\nabla S \times B) \cdot \nabla \left( \frac{\kappa_\parallel B \cdot \nabla T}{B^2} \right) - \frac{\kappa_\perp |\nabla S|^2}{\varepsilon^2} T_1. \tag{A.23}
\]

$(B \cdot B_1)/B$ is to a linear approximation the change in magnetic field strength and can be obtained from (A.11) and (A.18). Thus the linearized energy equation is

\[
\frac{1}{\gamma - 1} \left\{ s p_1 + \frac{\partial p}{\partial l} v_\parallel \frac{B_p}{B} - \frac{1}{\varepsilon} \phi_1 \left( \frac{\partial p}{\partial A} + \frac{B_p}{B} B \cdot \nabla \cdot \nabla A \right) + \gamma p \nabla \cdot v \right\} = \\
= - \nabla \cdot Q - \frac{\partial \mathcal{L}}{\partial T} T_1 - \frac{\partial \mathcal{L}}{\partial p} p_1 - \frac{\partial \mathcal{L}}{\partial B} \frac{B \cdot B_1}{B}. \tag{A.24}
\]

Finally, the linearized gas law relates $p_1$, $\rho_1$, and $T_1$ through

\[
\frac{p_1}{p} = \frac{\rho_1}{\rho} + \frac{T_1}{T}. \tag{A.25}
\]

The stability variables are $a_\parallel$, $v_\parallel$, $\phi_1$, $p_1$, and $T_1$, with $j_1$, $\nabla \cdot v$, $\omega$, $\rho_1$ and $(B \cdot B_1)/B$ determined in terms of them. The six equations, that only involve derivatives along field lines, are then (A.2), (A.3), (A.14), (A.20), (A.23), and (A.24). Equation (A.25) can be used to eliminate $\rho_1$ in terms of $p_1$ and $T_1$. These equations are written in a suitable form for numerical computation in Section 3.

References