INTERFEROMETRIC IMAGING II:

Two-Dimensional Non-Redundant Arrays

J. B. ZIRKER

National Optical Astronomy Observatories*, National Solar Observatory/SP, Sunspot, NM 88349, U.S.A.

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Abstract. Non-redundant arrays offer a promising technique for producing diffraction-limited solar images. Pairs of two-dimensional non-redundant arrays yield sufficient information to recover the phases of all the spatial frequencies to which they respond. The algorithms to select and test such pairs are given and applied to a particular example.

1. Introduction

Images with subarcsecond resolution are in high demand for the investigation of several important topics in solar physics. For example, the fine-structure of solar prominences, the interaction of magnetic flux tubes and granulation, the evolution of granulation, and the build-up of magnetic energy at flare sites – all these topics require time-series of superb images for further progress.

Ground-based solar telescopes, with apertures of 50–100 cm, rarely achieve diffraction-limited resolution even at the best sites. Their image quality may be improved by a variety of techniques, each of which has its own advantages and disadvantages. Active optics are promising, but still developmental and expensive. Speckle interferometry, coupled with such image reconstruction algorithms as the Knox–Thomson (1974), speckle masking (Lohmann, Weigelt, and Wirnitzer, 1983) or hybrid mapping (Pearson and Readhead, 1984) are all reasonably successful, but involve massive post-observational computations and, in the case of speckle-masking, huge computer storage. Another approach is the use of non-redundant arrays.

A telescope can be converted to a non-redundant array by inserting a special mask in the pupil plane. Readhead et al. (1988) have emphasized the marked superiority of Non-Redundant Arrays (NRAs) over full apertures in imaging bright objects. This advantage arises from the suppression of redundant baselines and the consequent reduction of closure phase noise. NRAs have other advantages: they are cheap to build and their sparse data is economical to reduce. Unfortunately, their effective field of view is limited (so far) to the isoplanatic patch, but this disadvantage may not be disabling in some applications, and mosaicing techniques may also be feasible.

In previous papers (Zirker and Brown, 1986; Zirker, 1987) we proposed the simultaneous use of dual (or triple) non-redundant masks in solar astronomy, where

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photon noise is often negligible. We chose one-dimensional arrays to explain and to test the principles involved. However, two-dimensional arrays gather far more information per frame, if the signal-to-noise ratio is adequate. Thus, in this paper we examine pairs of two-dimensional arrays. We focus on the problem of recovering the phases of Fourier components of the object, ignoring for the moment the non-trivial difficulties of determining the amplitudes.

2. Phase Closure

NRAs eliminate the phase delays introduced by the Earth’s atmosphere by the ‘closure principle’ (Rogstad, 1968): Atmospheric phase delays cancel to first order in the sum of the phases of three spatial frequencies which correspond to three holes in a mask. Let \( P(i, mn) \) be the measured phase of the frequency that corresponds to the baseline \((m, n)\), in the \(i\)th frame:

\[
P(i, mn) = \psi(mn) + \varphi(m) - \varphi(n),
\]

where \(\psi(mn)\) is the object’s phase at the frequency \((mn)\) and the \(\varphi\)s are the phase delays at holes \(m\) and \(n\) in the mask. Then, the ‘closure phase’ is

\[
P(i, mn) + P(i, np) - P(i, pm) = \psi(mn) + \psi(np) - \psi(pm) + \[\varphi(m) - \varphi(n) + \varphi(n) - \varphi(p) + \varphi(m) + \varphi(p)\]
\]

and the atmospheric terms in the brackets cancel.

With a fully-filled aperture, many redundant baselines are present instantaneously in the pupil plane. Each baseline has its own phase delay \(\Delta\varphi\), and these do not cancel in forming the closure phase.

Each \(n\)-hole mask yields \(n(n - 1)(n - 2)/6\) closure equations (similar to (2)) in \(n(n - 1)/2\) unknowns (the object phase at each frequency). Thus the equations appear to outnumber the unknowns by the factor \((n - 2)/3\). However, only \((n - 1)(n - 2)/2\) equations are linearly independent (Zirker and Brown, 1986) so that, in fact, we have \((n - 1)\) too few equations to solve for all the unknowns. A pair of arrays, suitably chosen, may provide sufficient equations, however.

In choosing candidate pairs of arrays, we are guided by several principles. First, we want arrays that sample the Fourier plane with a nearly uniform distribution of spatial frequencies, in order to obtain an optimum representation of an arbitrary object. Second, the two arrays should sample the same spatial frequencies as far as possible, so as to maximize the number of available closure equations. Finally, the arrays should minimize the number of identical (e.g., dependent) closure equations.

Cornwell (1987) has recently published designs for arrays that produce fairly uniform coverage of the Fourier plane. Golay’s (1971) arrays cover the Fourier plane still more uniformly, however, and have the added advantage of sampling many of the same spatial frequencies out to some limit fixed by the maximum baseline.

Using these guidelines, we chose, as a possible pair, the 9-hole and 7-hole Golay arrays shown in Figure 1. Note that all the 7-hole frequencies coincide with 9-hole
Fig. 1. Golay's 7-hole and 9-hole non-redundant masks. The holes are labeled arbitrarily zero to \( n \). The autocorrelation functions of the arrays (or equivalently their spatial frequencies in Fourier space) are shown to the right.

frequencies within the central hexagonal core in the Fourier plane. This condition does not guarantee a sufficient number of independent closure equations, but it certainly enhances the prospects.

Table I shows the numbers of frequencies \( N(f) \), equations \( N(E) \) and linearly independent equations \( N(IE) \) for the 7- and 9-hole masks:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( N(f) )</th>
<th>( N(E) )</th>
<th>( N(IE) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>36</td>
<td>84</td>
<td>28</td>
</tr>
<tr>
<td>7</td>
<td>21</td>
<td>35</td>
<td>15</td>
</tr>
</tbody>
</table>

If, in the set of \( 28 + 15 = 43 \) equations provided by the two arrays, 36 are linearly independent, then we may solve for all unknown object phases.

3. Test

To examine whether this condition holds, we must first find the independent equations for each array. This is easily accomplished. We simply write down the triplets of holes
in the order:

\[(0, 1, 2), (0, 1, 3) \ldots (0, 1, n - 1) \quad n = 7 \text{ or } 9\]

\[(1, 2, 3), (1, 2, 4) \ldots (1, 2, n - 1)\]

\[\vdots\]

\[(n - 3, n - 2, n - 1)\].

Each array's set of triplet combinations is completely non-redundant, hence, the closure phase equations they represent are linearly independent. However, an equation in one array's set may be linearly dependent on equations in the other array's set. To find a set of mutually independent equations we note that each equation corresponds to a triangle in an array, and if a triangle in one array is congruent to another in the array (or in the complementary array), then the corresponding equations are linearly dependent (Zirker and Brown, 1986). The triangles may be flipped over and/or rotated in their own plane for this comparison.

This principle, though rigorous, is tedious to apply when the number of holes is large. Standard methods for testing systems of equations for linear dependence may be used instead. We used a matrix factoring routine in the IBM library ('DMFGR'), but the Schmidt orthogonalization algorithm (Margenau and Murphy, 1943) would work equally well.

To set up the system of closure phase equations, we must first rewrite the 7-hole mask's triplets in terms of the hole pairs of the 9-hole mask. For example, the 7-hole triplet \((0, 1, 2)\) involves the frequencies \((0, 1), (1, 2), \text{ and } (2, 0)\), as expressed in hole numbers of the 7-hole mask, or \((1, 5), (0, 7), \text{ and } (3, 2)\), when expressed in the 9-hole mask's numbers. Then the closure equation corresponding to this triplet is

\[
\psi(1, 5) - \psi(7, 0) - \psi(3, 2) = P(1, 5) + P(0, 7) - P(3, 2),
\]

where the phases, \(P\), are measured on the interferogram of the 7-hole mask. Now for any hole pair \((a, b)\) we have \(\psi(a, b) = -\psi(b, a)\), because the spatial frequency corresponding to the base line \(ab\) is a vector. Let us include in the equations only those 36 vector frequencies with positive \(u\) in the \(u, v\) frequency plane. This constraint would modify the preceding closure equation as follows:

\[
\psi(1, 5) - \psi(7, 0) - \psi(3, 2) = P(1, 5) - P(7, 0) - P(3, 2).
\]

Next we arrange the 36 9-hole mask frequencies in an arbitrary serial order and label each of them with an index \(j = 1, 2 \ldots 36\). The 43 closure equations (each with the form of Equation (2)) can then be represented in matrix form as

\[
M_{ij} \cdot \psi_j = P_i,
\]

where \(i = 1, 2 \ldots 43\), \(\psi_j\) is the column matrix of object phases (each corresponding to a vector frequency \(j\)) and \(P_i\) is the appropriate closure phase, e.g., the left-hand side of Equation (2). Each element \(M_{ij}\) equals 0 or \(\pm 1\). Table II shows the labeling scheme used.
in the present case; and Table III shows the first row of the matrix of coefficients of the system of 43 equations in 36 unknowns:

**TABLE II**
Labeling scheme

<table>
<thead>
<tr>
<th>Hole pair</th>
<th>Label of frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 1</td>
<td>1</td>
</tr>
<tr>
<td>0, 2</td>
<td>2</td>
</tr>
<tr>
<td>1, 2</td>
<td>3</td>
</tr>
<tr>
<td>0, 3</td>
<td>4</td>
</tr>
<tr>
<td>1, 3</td>
<td>5</td>
</tr>
<tr>
<td>2, 3</td>
<td>6</td>
</tr>
<tr>
<td>0, 4</td>
<td>7</td>
</tr>
<tr>
<td>etc.</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE III**

<table>
<thead>
<tr>
<th>$\psi_1$</th>
<th>$\psi_2$</th>
<th>$\psi_3$</th>
<th>$\psi_4$</th>
<th>$\psi_5$</th>
<th>...</th>
<th>$\psi_{36}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
</tbody>
</table>

4. Results

We find that 34 of the 43 available equations are linearly independent. This means that all but two of the 36 unknown phases may be determined from observations. The remaining two phases can be set equal to zero; this will introduce an arbitrary spatial shift in the final reconstructed image (see Appendix). The determinant of the $34 \times 34$ matrix equals $-39$, and the inverse matrix is easily obtained. This inverse matrix is a unique property of this pair of arrays, which allows 34 object phases to be recovered from the measured phases, by a simple matrix multiplication.

In practice, we need sufficient signal-to-noise ratio at all three frequencies in a triplet, to construct a closure equation like (2). Then we can sum each closure equation over all frames that contain a complete triplet (i.e., over the index $i$ in Equation (2)) to improve the precision of the result. Obviously, to solve the full system of equations, every closure phase must be measurable on a sufficient number of frames.

Suppose, however, that some observed phases are too noisy or are absent entirely – can we still solve for the remaining phases? The answer depends on how many (and which) phases must be rejected. For example, if the six highest spatial frequencies passed by the 9-hole mask [which form the outer ring in Fourier space (see Figure 1)] were undetectable, we would then have 30 unknown phases. After excluding all triplets that involve these 6 frequencies, we find only 27 independent equations, i.e., one too few for a unique solution.

The solution for the unknown object phases, $\psi$, is easily obtained by multiplying the column vector of observed closure phases (the left-hand side of Equation (2)) by the
inverse matrix for this array pair. Each closure phase will have an observational error. It is important that the process of matrix multiplication should not amplify these errors unduly. We tested this effect by generating random fractional closure errors $\varepsilon$, that lie in range $-1 < \varepsilon < 1$, in arbitrary units, and carrying out the matrix multiplication. The r.m.s. input error was 0.58, the r.m.s. output error was 1.22, or only a factor of 2 larger. Thus, error amplification does not appear to be serious.

5. Conclusion

We have demonstrated the feasibility of recovering uncontaminated Fourier object phases with a pair of two-dimensional non-redundant arrays. The particular pair of Golay arrays we examined would yield all but two unknown object phases, if each of the 36 observable phases could be measured on at least a few frames.

Algorithms for testing any candidate pair of arrays have been presented. They are simple and economical to apply.

Even larger Golay arrays may form complementary pairs (e.g., his 12-hole and 10-hole masks), and so offer even more spatial information.

As Readhead et al. (1988) and Baldwin et al. (1986), have demonstrated, single non-redundant arrays can be extremely useful in the interferometry of bright stars. Pairs of more complex arrays may offer real advantages for solar interferometry.

Appendix. Image Shift

If we set two arbitrary object phases equal to zero and solve for the rest, the reconstructed object will shift in absolute position by an arbitrary amount. This result follows from the shift theorem for Fourier transforms (Bracewell, 1965) as follows:

First, resolve each discrete vector frequency $\overline{\omega}$ into orthogonal components $u$ and $v$:

$$
\overline{\omega}_{nm} = u_n \hat{u} + v_m \hat{v}.
$$

We can write the object $O(x, y)$ in terms of a finite Fourier series:

$$
O(x, y) = \sum_{1}^{N} \sum_{1}^{M} A_{nm} \cos(xu_n + yv_m + \psi_{nm}).
$$

Here the phases $\psi_{nm}$ refer to the coordinate axes of $x, y$. Now replace the $\psi_{nm}$ by $\psi'_{nm}$ with

$$
\psi'_{nm} = \psi_{nm} + au_n + bv_m,
$$

and where $a, b$ are determined by the conditions

$$
\psi'_{N, M-1} = \psi_{N, M-1} + au_N + bv_{M-1} = 0,
$$

$$
\psi'_{N-1, M} = \psi_{N-1, M} + au_{N-1} + bv_M = 0.
$$
Then \( O(x, y) = \sum \sum A_{nm} \cos[(x - a)\mu_n + (y - b)\nu_m + \psi_{nm}] = O(x - a, y - b) \).

Thus, if we set two phases equal to zero, we shift the image, relative to the position it would have had with the two phases not equal to zero.

References