Differential asymptotic sound-speed inversions

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Summary. We present an asymptotic method for determining the internal solar sound speed from the differences between observed $p$-mode frequencies and those of a standard reference solar model. In order to test the method, it has been applied to the frequency differences between pairs of solar models, where in each case one takes the part of the reference model and the other the role of the Sun. The results of these numerical experiments indicate that by using this simple method one may in principle be able to determine the sound speed from the energy-generating core to the helium ionization zone.

1 Introduction

The observed solar 5-min oscillations are acoustic modes whose orders $n$, at least at low or moderate degrees, are high. Thus they can be approximated by plane sound waves, and consequently their behaviour, and hence their frequencies, are determined mainly by the variation of sound speed $c$ in the solar interior. It is therefore reasonable to attempt an inversion of the observed frequencies, to obtain the variation of $c$ with radius $r$.

The foundation for such inversions was established by Duvall (1982). From observational data for high-degree 5-min modes he noticed that, with a suitably chosen constant $\alpha$, the quantity $(n + \alpha)/\omega$ depended principally on frequency $\omega$ and degree $l$ only in the combination $w = \omega/L$, i.e.

$$\frac{(n + \alpha)\pi}{\omega} = F(\omega/L),$$

(1.1)

where $L = \sqrt{l(l+1)}$. Subsequently Gough (1984) used ray theory to relate $F(w)$ to $c(r)$ (cf. equation 2.2) and showed how the relation could be inverted; Christensen-Dalsgaard et al.

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(1985) applied this procedure to observed frequencies to infer the variation of sound speed throughout much of the Sun.

An attractive feature of this inversion is that the result is independent of any solar model; given the simplifying assumptions, the result depends only on the solar data. On the other hand, these assumptions are quite severe. Thus it is not obvious that ray theory provides an adequate approximation to the actual behaviour of the modes. In its simplest form it neglects the effects of stratification and the perturbation in the gravitational potential, the latter being particularly important for modes of low degree (e.g. Christensen-Dalsgaard 1984); moreover a more careful analysis indicates that the quantity $\alpha$ may in reality depend on frequency. Indeed, by applying the inversion procedure to computed frequencies of a solar model, Christensen-Dalsgaard et al. found that in the core and near the surface the accuracy of the inferred value of the sound speed was poor. The situation was improved, however, when the same procedure was applied to two theoretical models, for the difference between the inferred sound speeds was substantially more accurate. Evidently some of the errors introduced by the simplifications inherent in the method cancel out.

This procedure can be refined, in particular by choosing a frequency-dependent rather than a constant $\alpha$, by including effects of buoyancy and by taking the perturbation in the gravitational potential into account (Gough 1986a; Shibahashi 1988; Brodsky & Vorontsov 1987, 1988).

In the present paper we adopt the philosophy that, since differences in inferred sound speeds have been found to be more accurate than the inferred sound speeds themselves, we should invert for the sound-speed difference directly. This is achieved by linearizing equation (1.1) in terms of (assumed) small changes in $c$, $\omega$ and $\alpha$. Also we take the frequency dependence of $\alpha$ into account, and fit the linearized relation with cubic splines in $\omega$ and $\omega/L$. The latter fit can then be inverted to give the sound-speed difference as a function of position. To test the method, it has been applied to various pairs of theoretical models. We find that even for models that differ substantially, the sound-speed difference can be recovered to well within 1 per cent of the sound speed at most depths, even near the surface and in the core of the model, where the previous method failed.

2 Background

An application of short-wavelength asymptotic theory to the equation describing adiabatic acoustic-gravity waves propagating in a spherically symmetric star yields the approximate relation

$$\frac{(n + \varepsilon) \pi}{\omega} = \int_{r_1}^{r_2} \left[ 1 - \omega_s^2 - \frac{L^2 c^2}{\omega^2 r^2} \left( 1 - \frac{N^2}{\omega^2} \right) \right]^{1/2} \frac{dr}{c}$$

(e.g. Deubner & Gough 1984), where $\omega_s$ is essentially Lamb's acoustical cut-off frequency, $N$ is the buoyancy frequency and $\varepsilon$ is a constant that depends on conditions in the vicinity of the turning points $r_1$ and $r_2$, where the integrand vanishes. In deriving equation (2.1), some relatively minor aspects of spherical geometry and of the variation of the gravitational acceleration are neglected, as is the perturbation to the gravitational potential. We restrict attention to high-frequency $p$-modes, for which $\omega^2 \gg N^2$. If $c(r)$ and $\omega_s(r)$ are now approximated by their values for a plane polytropic envelope under constant gravity, and $N^2/\omega^2$ is neglected, equation (2.1) can be rewritten

$$\frac{(n + \varepsilon) \pi}{\omega} = \int_{r_1}^{r_2} \left( 1 - \frac{L^2 c^2}{\omega^2 r^2} \right)^{1/2} \frac{dr}{c}.$$
where $\alpha$ is a constant which depends on $\varepsilon$ and the polytropic index. This relation can also be obtained directly from ray theory (Gough 1986b) or from a simplified JWKB analysis (Christensen-Dalsgaard et al. 1985). The range of integration now extends from the approximate lower turning point $r_t$, where the square root in equation (2.2) vanishes, to the surface $r = R$ of the star. It should be noticed that equation (2.2) is of the same form as equation (1.1), which was first obtained empirically by Duvall (1982).

When equation (2.2) is derived from ray theory one finds that

$$L = I + \frac{1}{2}. \quad (2.3)$$

Except for the smallest values of $l$, however, the difference between this and the eigenvalue $\sqrt{l(l + 1)}$ of the associated Legendre equation is negligible. When $l$ is small, the value (2.3) appears to provide the better description (Brodsky & Vorontsov 1987, 1988), as it does for comparable asymptotic expansions of the solutions of Schrödinger’s equation (Kemble 1937). Thus we use the definition (2.3) here.

In reality the outermost part of the solar envelope is far from polytropic; in particular $\omega_c$ deviates substantially from the polytropic value in the thin superadiabatic boundary layer just beneath the photosphere. As a result, $\alpha$ cannot be considered to be constant. It is insensitive to the value of $l$, however, since in the superadiabatic layer $c^2$ is small, and $c^2 f^2/\omega^2 r^2 \ll 1$; furthermore, $|N^2/\omega^2| \ll 1$. Hence $\alpha$ should be regarded as a function of frequency alone, and is determined by conditions near the surface of the Sun.

Identifying equations (1.1) and (2.2) determines how the observable function $F(\omega/L)$ is related to $c(r)$:

$$F(\omega) = \left[ \ln \frac{R}{r} \right] \left( 1 - \frac{a^2}{w^2} \right)^{1/2} a^{-1} \, d\ln r, \quad (2.4)$$

where $a = c/r$. The equation is of the Abel type, and can be inverted analytically to obtain the sound speed implicitly, thus:

$$r = R \exp \left( -\frac{2}{\pi} \int_{a}^{w} (w^{-2} - a^{-2})^{1/2} \frac{dF}{dw} \, dw \right) \quad (2.5)$$

(Gough 1984). This relation was used by Christensen-Dalsgaard et al. (1985) to infer the sound speed in the solar interior.

As discussed in the introduction, the representation (2.5) is susceptible to systematic errors. These errors are related to the approximations made in deriving equation (2.2). It has been found, for example, that for the most deeply penetrating modes of low degree the perturbation to the gravitational potential has a substantial effect on the functions $F(\omega/L)$ obtained by fitting the relation (1.1) to discrete data; this could be the cause of the corruption of the inversion in the solar core. It appears that these errors cancel to some extent when differences are taken between inversions of different sets of frequencies. A disadvantage of this approach is that the results of the inversion depend on the reference model of the Sun.

The present paper is an examination of a differential inversion technique based on a linearization of the asymptotic formalism. By formally perturbing equation (2.2) and assuming that the perturbations $\delta \omega$, $\delta \alpha$, $\delta c$ in $\omega$, $\alpha$ and $c$ are small enough for the linearization to be valid, one obtains

$$S(\omega) \frac{\delta \omega}{\omega} = \int_{r_t}^{R} \left( 1 - \frac{c^2}{w^2 r} \right)^{-1/2} \frac{\delta c}{c} \, dr + \pi \frac{\delta \alpha}{\omega} \quad (2.6)$$
where

$$S(w) = \int_{r_0}^{R} \left( 1 - \frac{c^2}{w^2 r^3} \right)^{-1/2} \frac{dr}{c}. \quad (2.7)$$

Alternatively, noting that $2S(w)$ is the sound travel time along a ray between successive deflections at the surface, equation (2.6) can be derived from ray theory: since for a resonant wave the phase change along a ray path between deflections, taking due account of the phase jump at caustics, must be consistent with that of the coherent interference pattern on the surface, the perturbation $2\delta(\omega/c^- ds)$ to the phase change calculated along the unperturbed ray path ($s$ being the distance along the ray) and the perturbation $-2\pi\delta\alpha$ to the phase jump must sum to zero (cf. Gough 1989). In writing down equation (2.6) we have neglected a small term in $d\alpha/d\omega$.

Some properties of relation (2.6) are discussed by Christensen-Dalsgaard, Gough & Pérez Hernández (1988). Note in particular that the scaled frequency difference $S(w)\delta\omega/\omega$ can be expressed in the form

$$S(w)\frac{\delta\omega}{\omega} = H_1(w) + H_2(\omega) \quad \text{(2.8)}$$

where equation (2.6) defines the functions $H_1$ and $H_2$. Hence scaled frequency differences (between two models or between the Sun and a model) depend asymptotically on the interior sound-speed difference, through a function of $w$, and on differences in the surface layers, through a function of $\omega$ (Christensen-Dalsgaard 1986).

The variation of $S(w)$ with $w$ is illustrated in Fig. 1. $S(w)$ is plotted in units of

$$\tau_0 = \int_{0}^{R} \frac{dr}{c}. \quad (2.9)$$

Figure 1. The scaling function $S(w)/\tau_0$ (chain dashed line), and the location $r_t(w)/R$ of the lower turning point (solid line), as functions of $v/L$. 

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to which it tends for large $w$. Also shown is the dependence of $r_i/R$ on $w$. In this and all subsequent similar plots $w$ is replaced as abscissa by $v/L$, $v = \omega/2\pi$ being the cyclic frequency, which can be more simply related to the observations of the oscillations. It is evident that the formal turning points for radial modes of high frequency are extremely close to the centre. This provides some hope that inversion of such modes is sensitive to conditions in the solar core.

According to equations (2.6) and (2.7) $\delta\omega/\omega$ depends on a weighted average of $\delta c/c$ in the domain $c/w = r_i \leq r \leq R$ within which the waves propagate. Modes with smaller $w$ average over a shallower domain, and their frequencies are evidently more sensitive to perturbations in $c$, provided, of course, that the domain of propagation encompasses the region in which $c$ is perturbed. Since the range of $\omega$ is small, the sensitivity to $c$ is therefore generally greater for modes of high degree. An analogous result can be derived from a perturbation analysis of the full oscillation equations, without recourse to asymptotic theory (e.g. Christensen-Dalsgaard 1986); here the quantity $\tau_i^{-1} S(w)$ is replaced by the asymptotically equivalent ratio $E_l(v)/E_0(v)$, $E_l(v)$ being the energy of a mode with degree $l$ and frequency $v$. Plotting scaled frequency differences against frequency (Christensen-Dalsgaard 1987; cf. also Christensen-Dalsgaard & Gough 1984) or against $w$ exhibits the variation and the relative importance of the contributions from the solar interior and the surface layers, which is beneficial for an appreciation of the discrepancies between theoretical models and the Sun. As an example, Fig. 2 shows $\tau_i^{-1} S(w)\delta\omega/\omega$, the differences being between a chemically homogeneous model and a normal solar model. The corresponding unscaled plot, over a more limited range in $l$, was discussed by Christensen-Dalsgaard et al. (1988). The variation with $w$, corresponding to the term $H_1$ in equation (2.8), and the variation with $\omega$ for given $l$, corresponding to $H_2$, are clearly visible. The separation and analysis of these two contributions is discussed under Example (i) in Section 5 below.

![Figure 2](image-url)

**Figure 2.** Scaled relative differences $\tau_i^{-1} S(w)\delta\omega/\omega$ between the p-mode frequencies of a chemically homogeneous model and those of Model 1, plotted against $v/L$. Data corresponding to modes of like degree $l$ are connected by straight lines. All values of $l$ less than 200 are represented, and every tenth value thereafter up to $l = 1080$. Data are shown for all modes with frequencies in the range 1500–5000 $\mu$Hz, for those values of $l$ represented.
3 Spline-fitting method

In order to separate the two contributions $H_1(w)$ and $H_2(\omega)$ we fit to the data $D(w, \omega) = S(w)\hbar(\omega)$ the expression

$$H_1(w) + H_2(\omega)$$

by a least-squares method. Each of $H_1$ and $H_2$ is a cubic spline, the parameters of which are determined by the fitting procedure. We used $w$ and $\omega$ as independent variables. The splines are defined on two grids:

$$\log w_{\min} = x_1^{(1)} < x_2^{(1)} < \ldots < x_{n_1}^{(1)} = \log w_{\max}$$

$$\omega_{\min} = x_1^{(2)} < x_2^{(2)} < \ldots < x_{n_2}^{(2)} = \omega_{\max}$$

Here $n_1$ and $n_2$ are the numbers of knots in the log $w$- and $\omega$-splines respectively, and subscripts 'min' and 'max' denote the minimum and maximum values of the independent variables for which there are data in the given data set. Then each of $H_1$ and $H_2$ is defined by

$$H_k(x) = \sum_{j=1}^{n_k} A_j^{(k)}(x) + \sum_{j=1}^{n_k} B_j^{(k)}(x) + \sum_{j=1}^{n_k} C_j^{(k)}(x) + \sum_{j=1}^{n_k} D_j^{(k)}(x)$$

where the constants

$$\{ y_j^{(k)}, \ldots, y_n^{(k)} \} \quad (k = 1, 2)$$

are independent parameters, and the functions $A_j^{(k)} - D_j^{(k)}$ are defined by

$$A_j^{(k)} = \frac{x_j^{(k)} - x}{x_j^{(k)} - x_{j+1}^{(k)}} \quad C_j^{(k)} = \frac{1}{6}[(A_j^{(k)} - A_{j+1}^{(k)})^3 - A_j^{(k)}]$$

$$B_j^{(k)} = \frac{x - x_j^{(k)}}{x_j^{(k)} - x_{j+1}^{(k)}} \quad D_j^{(k)} = \frac{1}{6}[(B_j^{(k)} - B_{j+1}^{(k)})^3 - B_j^{(k)}]$$

The constants

$$\{ \eta_j^{(k)} : j = 1, \ldots, n_k - 1 \} \quad (k = 1, 2)$$

are determined by the requirement that $H_k$ be continuous with continuous first derivatives throughout $x_j^{(k)} < x < x_{j+1}^{(k)}$ (see Press et al. 1986). (Equations (3.3) and (3.5) ensure that the second derivatives are also continuous everywhere; in particular the second derivative of $H_k$ at $x = x_j^{(k)}$ is $\eta_j^{(k)}$.)

The values of the constants $y_j^{(k)}$ are found by minimizing

$$\chi^2 = \sum_\alpha \sigma_\alpha^{-2} [D(w_\alpha, \omega_\alpha) - H_1(w_\alpha) - H_2(\omega_\alpha)]^2$$

where the sum is over all modes in the dataset, and $\sigma_\alpha$ is the standard deviation of the error in data point $\alpha$. In the present work, since we have always employed frequencies from theoretical models to which no errors have been deliberately added, we have taken $\sigma_\alpha = 1$. Also we have imposed the conditions

$$\eta_1^{(k)} = \eta_{n_k}^{(k)} = 0$$

so that $H_1$ and $H_2$ are 'natural' splines.

The quantity $\chi^2$ was minimized using the singular value decomposition (SVD) method described by Press et al. (1986). Specifically, we used their SVDFIT routine, modified in a trivial
way to allow two independent variables and to enable equations (3.8) to be imposed as constraints. We draw attention to the fact that a constant may always be added to $\tilde{H}_2$, provided the same constant is subtracted from $\tilde{H}_1$. (The consequences of this indeterminacy on the sound-speed inversion are discussed in Section 4.) The SVD method handles the indeterminacy in a sensible way (see Press et al.). Alternatively one could constrain one of the parameters $y_j^{(k)}$ to remove this degree of freedom.

In all the examples discussed in Section 5 uniform grids in both $\log w$ and $\sigma$ were employed, with the sole exception that the interval between $x_{n-1}^{(l)}$ and $x_n^{(l)}$ was rather larger than the rest, since at large $\log w$ the density of modes is substantially lower than elsewhere (cf. Fig. 2) ($x_{n-1}^{(l)}$ was chosen, somewhat arbitrarily, to correspond to a mode of degree $l = 2$ and frequency 3 mHz). Some experiments using different numbers of spline knots are discussed in Section 5.

4 Inversion method

We now invert the frequency differences to infer $\delta c/c$. The function $H_1(w)$ is related to $\delta c/c$ by

$$H_1(w) = \int_{\ln r(w)}^{\ln R} \left( 1 - \frac{a^2}{w^2} \right)^{-1/2} \frac{\delta c}{c} a^{-1} d \ln r. \quad (4.1)$$

The right-hand side of equation (4.1) is the same functional of $w$ as that which arises in the asymptotic expression for the linear frequency splitting due to latitudinally-independent rotation at a rate $\Omega(r)$, with $\delta c/c$ instead of $\Omega$ (Gough 1984). Thus it can be inverted in the same way. Following the procedure presented in Appendix I, one obtains

$$\frac{\delta c}{c} = -\frac{2a}{\pi} \frac{d}{d \ln r} \int_w^{\infty} (a^2 - w^2)^{-1/2} H_1(w) dw, \quad (4.2)$$

where $a_s = a(R)$. This is not identical to the inversion formula derived by Gough (1984), but the two formulae are mathematically equivalent.

To apply the inversion formula (4.2) we use the estimate $\tilde{H}_1$ for $H_1$. Inevitably there is a gap between $w = w_{\min}$, determined by the most shallowly penetrating mode in the data set, and $w = a_s$. The treatment of $H_1$ in this gap is necessarily uncertain, and we have chosen simply to interpolate linearly (in $\log w$) between the value of $\tilde{H}_1$ found by the spline-fitting procedure at $w = w_{\min}$ and $\tilde{H}_1 = 0$ at $w = a_s$. The latter is the theoretical value of $H_1$ at the surface, but it must be borne in mind that $\tilde{H}_1$ contains an undetermined additive constant. Aside from the illustration in Fig. 3, we have in all cases used the function $\tilde{H}_1$ produced by the fitting procedure discussed above, without further adjustment. The numerical evaluation of the integral in equation (4.2) is described in Appendix II.

Within the framework of our choice of extrapolation near the surface one could add to our inversions a multiple $\lambda$ of the complementary function which corresponds to adding to the extrapolated $\tilde{H}_1$ the function

$$f(w) = \begin{cases} 
1 & \log w > \log w_{\min} \\
\log w - \log a_s & \log w_{\min} > \log w \geq \log a_s \\
\log w_{\min} - \log a_s & \log w_{\min} > \log w \geq \log a_s
\end{cases} \quad (4.3)$$

In our experience the values obtained for $\tau_0^{-1}H_1$ have differed from the actual $\tau_0^{-1}H_1$ by no more than $10^{-3}$ (see, for example, Figs 5 and 14), and we deduce that this is a representative
Figure 3. The dashed line shows the inversion of a constant function of value $10^{-3} \tau_0$, plotted against fractional radius $r/R$. The solid lines show the inversions of functions that have constant value $10^{-3} \tau_0$ for log $w > \log w_{cut}$ and vary continuously and linearly in log $w$ to zero at $w = c(R)/R$. The values of $w_{cut}$ represented are: (a) $2.89 \times 10^{-5}$ s$^{-1}$; (b) $4.30 \times 10^{-5}$ s$^{-1}$; and (c) $8.91 \times 10^{-5}$ s$^{-1}$. These correspond to the shallowest lower turning point radii, in units of $R$, of: (a) 0.994; (b) 0.989; and (c) 0.967.

value of $\lambda$ for the present calculations. Accordingly, in Fig. 3 we show the inversion of $10^{-3} \tau_0 f(w)$ for three different values of $w_{min}$. These correspond to modes with $l = 1080$, $\nu = 4694$ $\mu$Hz (the most shallowly penetrating mode in our set), $l = 500$, $\nu = 3428$ $\mu$Hz and $l = 150$, $\nu = 2135$ $\mu$Hz. Also shown for interest is a complementary function, obtained by inverting a constant function of value $10^{-3} \tau_0$ (cf. equation A1.7).

5 Results

The inversion method described above has been applied to the differences between $p$-mode frequencies of pairs of theoretical models. Except in Example (v), discussed at the end of this section, the reference model was in all cases Model 1 of Christensen-Dalsgaard (1982). The inversion method was implemented on the same grid as that used to compute the Model 1 frequencies, namely a 600-point mesh with points spaced roughly in proportion to $c^{-1}(r)$ (suitable for the computation of $p$-modes). In the experiments discussed below, four different models take the role of the Sun. Perturbation quantities are always defined in the sense: (proxy Sun) minus (reference model).

Except where explicitly stated otherwise, all the experiments used the same set of 3244 $p$-modes. This set comprises all modes with frequencies between 1500 and 5000 $\mu$Hz for each degree less than 200, and for every tenth value of $l$ in the range 200–1080, except for the following exclusions: since one might not expect the asymptotics to give a good approximation for modes of lowest order $n$, all modes with $n = 1$ were excluded. So also were low-frequency, low-degree modes; for these modes the effects of buoyancy and the cut-off frequency, whose deviations from appropriate plane-parallel polytropic values are formally neglected in the simplified asymptotic equation (2.2), are important. We have used a simple ad hoc criterion based on the size of the neglected terms in the integrand of equation (2.1) to determine which
modes are excluded. Above the base of the convection zone (at \( r = r_c \)) and excepting the surface layers, the buoyancy frequency is tiny and the effect of the cut-off frequency has mostly been taken into account in \( \alpha \). Thus we chose to retain modes according to the criterion

\[
\max_{r_c < r < r_c^0} \left| \frac{\omega^2}{\omega_c^2} - \frac{\alpha^2 N^2}{\omega^2} \right| < \delta
\]  

(computed using the reference model). By inspecting some examples to see which modes departed most from the simple representation (2.2), we chose \( \delta = 0.05 \). This had the effect of excluding the following modes: \( l = 0, \omega < 4480 \mu\text{Hz} \); \( l = 1, \omega < 3203 \mu\text{Hz} \); \( l = 2, \omega < 2430 \mu\text{Hz} \); \( l = 3, \omega < 2311 \mu\text{Hz} \); \( l = 4, \omega < 2111 \mu\text{Hz} \).

We present the results of numerical experiments involving five pairs of models:

**Example (i)**

The chemically homogeneous model of Christensen-Dalsgaard & Gough (1980) and Christensen-Dalsgaard et al. (1988) was used for the proxy Sun. Since the homogeneous model and Model 1 are quite dissimilar (see for example the graph of \( \delta c/c \) in Fig. 7 and the scaled frequency differences shown in Fig. 2) this example is quite a severe test of the method. The spline fit used 28 \( \omega \)- and 20 \( \omega \)-spline knots. Fig. 4 shows the scaled data after the fitted \( \bar{H}_2 \) has been subtracted. So as not to obscure the data, the fitted \( \bar{H}_1 \) is indicated with dots. The spread in the data for \( 10^2 \mu\text{Hz} < \nu/L < 10^3 \mu\text{Hz} \) is probably due to the effects of buoyancy and \( \omega_c \). Some of the spread at high degree arises because the orders of these modes are too low for the

![Graph showing scaled relative frequency differences](image)

**Figure 4.** Scaled relative frequency differences (homogeneous model minus Model 1) after the fitted function \( \bar{H}_2(\omega) \) has been subtracted, all scaled with \( \tau_0^{-1} \), for those modes in the set defined at the beginning of Section 5. Data corresponding to modes of like degree are connected by straight lines. The course of the scaled fitted function \( \tau_0^{-1} \bar{H}_1(\omega) \) is indicated by dots. (The dots have no further significance: they do not, for example, indicate the position of spline knots.) In the inset, data corresponding to the low-degree, low-frequency modes excluded from the mode set are also shown (though they have not been used in obtaining \( \bar{H}_1 \) and \( \bar{H}_2 \)). Note that subtracting \( \bar{H}_2 \) has succeeded in removing most of the variation with frequency (at fixed \( \omega \)) from the data shown in Fig. 2.
asymptotics to be accurate. The low-frequency, low-degree modes excluded from our mode set are included in the inset to Fig. 4, to display their discrepant behaviour.

The fitted $H_1$ is shown again in Fig. 5, together with the exact function $H_1$. The magnitude of $\tau_0^{-1}(H_1 - H_1)$ is nowhere more than about $10^{-3}$. The shortfall in the drop located in 60 $\mu$Hz $\leq \nu/L \leq 100$ $\mu$Hz appears to be caused by the neglected buoyancy and $\omega_c$ terms. The coincidence of $H_1$ and $\bar{H}_1$ at high $\nu$ is no doubt fortuitous. There is a scale shift at small $\nu$, which we do not yet understand, causing the bump at $\nu/L \approx 8$ $\mu$Hz to be displaced towards higher $\nu/L$ and deepening the trough to the right of this. The bump corresponds to the second helium ionization zone and arises from the difference in composition between the two models.

Fig. 6 shows the fitted function $H_2$, suitably scaled to be dimensionless and to correspond to $(3 \text{ mHz}/\nu)\delta\alpha$. Also shown is the same function obtained when the $\omega$-spline has only 10 knots. The curves are shifted relative to one another because of the slightly different arbitrary additive constants selected by the SVD method in the spline-fitting procedure. In the case with fewer splines, $H_2$ does not reveal the extent of the variation at low frequency. None the less, this has no significant effect on the fitted function $\bar{H}_1(\nu)$ and the sound-speed differences inferred therefrom.

The inferred function $\delta c/c$ (for the 20 $\omega$-spline case) is shown together with the exact function in Fig. 7. A remarkable feature of the inversion is the accuracy with which it reproduces the exact $\delta c/c$ over the entire range where it is valid (i.e. down to a radius $r \approx 0.013 R$). This is seen more clearly in Fig. 8(a), which shows the error in the sound speed thus inferred, as a fraction of the Model 1 sound speed.

Fig. 8(a) also illustrates the effects of excluding more modes from our mode set and of using more splines. Increasing the number of $\omega$-spline knots to 54 eliminates much of the wavy behaviour in the inferred sound speed in the convection zone and in the radiative region outside the energy-generating core, but does not have any significant effect on the overall accuracy of the inversion. (Using more splines actually introduces wavy structure in the core,

![Figure 5](image-url)
Asymptotic sound-speed inversion

Figure 6. The function $H_2$ scaled with $\pi^{-1}(3\text{ mHz} \times 2\pi)$, as a function of frequency $\nu$, obtained by fitting to the (homogeneous model minus Model 1) data using 20 (solid line) or 10 (dashed line) $\omega$-spline knots. This quantity is an estimate of $(3\text{ mHz}/\nu) \delta \alpha$.

Figure 7. The exact fractional sound-speed difference (dashed line) between the homogeneous model and Model 1, as a fraction of the sound speed of the latter, and the estimate (solid line) inferred using a 28 $\omega$-spline knot fit and the mode set defined at the beginning of Section 5. The functions are plotted against fractional radius $r/R$.

because in this region there are relatively few modes and the extra density of spline knots enables $H_1$ to follow some of the departures from a simple curve evident in Fig. 4.) Excluding the $l=0$ modes has little effect on the inversion, apart from limiting its validity to $r \geq 0.04R$; this indicates that including these modes has very little influence on the spline fit where $\nu/L \lesssim 3 \times 10^5 \mu\text{Hz}$. In particular, the magnitude of the error in the core is hardly affected. Similarly,
Figure 8. (a) The relative error in the sound speed inferred for the homogeneous model as a function of the Model 1 sound speed (i.e. $(\delta c/c)_{\text{inferred}}$ minus $(\delta c/c)_{\text{exact}}$) as a function of fractional radius $r/R$, obtained using the mode set defined at the beginning of Section 5 and using 28 $w$- and 20 $\omega$-spline knots (thick solid line), and the following variants: 54 $w$-spline knots (thin solid line); excluding $l=0$ modes (long dashed line); excluding modes with frequencies greater than 4 mHz (short dashed line). (b) The relative error in the sound speed inferred for the mixed core model as a fraction of the Model 1 sound speed, using the chosen mode set and 28 $w$- and 20 $\omega$-spline knots. (c) The relative error in the sound speed inferred for the homogeneous model using the mixed core model as the reference model, as a fraction of the sound speed in the latter. The mode set defined at the beginning of Section 5 was used, with 28 $w$- and 20 $\omega$-spline knots.

excluding all modes with frequencies greater than 4000 $\mu$Hz reduces the depth to which the sound speed may be inferred but has little effect on the errors.

It should be noticed that in the core the sound speed in the two models differs by 10 per cent; hence linearization in the perturbation method may introduce errors in the inversion. This is discussed further under Example (v) below.

Example (ii)

The proxy Sun was a model with a mixed core, described by Christensen-Dalsgaard (1986). This model is more like Model 1 outside the energy-generating core than is the homogenous model, though it is substantially different in the inner core. The fit again used 28 $w$- and 20
\( \omega \)-spline knots. The result of the inversion is shown in Fig. 9, and the error in the sound speed thus inferred can be seen more easily from Fig. 8(b). Because the reference model is more like the unknown one in this case, the inversion is much more accurate.

The fitted \( \bar{H}_2 \) for this example is shown in Fig. 10. Its shape is remarkably similar to the corresponding function in Fig. 6, although the magnitude is smaller by about a factor of 6. This function contains information about differences between the models in their outer layers. The details of the reflection of the waves at the upper turning point will be most dependent on the structure in the ionization zones for low frequency modes. Since each of these pairs of models

![Figure 9. Same as Fig. 7, but for the mixed core model instead of the homogeneous model.](image1)

![Figure 10. The function \( \bar{H}_2 \) scaled with \( \pi^{-1}(3 \text{ mHz} \times 2\pi) \), as a function of frequency \( \nu \), obtained by fitting to the (mixed core model minus Model 1) data using 20 \( \omega^- \) (and 28 \( \omega^- \)) spline knots. This quantity is an estimate of \( (3 \text{ mHz}/\nu) \delta \alpha \).](image2)
differs in composition one might expect this to affect the behaviour of low frequency modes at their upper turning point and thus contribute to $\delta \alpha$ for low frequencies. This may contribute to the more complicated behaviour of $H_2$ at low frequency, compared with its relatively smooth form at higher frequencies. Note that for two models with almost the same composition, discussed in Example (iv) below, the behaviour of $H_2$ at low frequencies is also smooth.

**Example (iii)**

Model A of Christensen-Dalsgaard, Gough & Morgan (1979) was used as the proxy Sun. The fit again used 28 $w$- and 20 $\omega$-spline knots. The results of the inversion are shown in Fig. 11. Except in the outer layers the error in the inferred sound speed is less than a third of 1 per cent where $r > 0.1 \, \text{R}$ and less than 1 per cent for $r < 0.1 \, \text{R}$. The improvement over the comparable inversion by Christensen-Dalsgaard et al. (1985) is particularly significant where $r < 0.3 \, \text{R}$.

**Example (iv)**

For the proxy Sun we have taken a model which is very similar to Model 1 in the interior but with a different treatment of the superadiabatic layer near the surface. It is described by Christensen-Dalsgaard (1986). The unscaled and scaled frequency differences are shown in Figs 12 and 13, respectively. Note that most of the variation with $w$ is removed when the data are scaled with $S(w)$. It is quite clear from Fig. 13 that most of the difference in frequencies comes from differences between the outer layers of the models, since $S\delta \omega / \omega$ is dominated by the contribution $H_2(w)$. This is rather similar to the case of the differences between observed solar frequencies and those of Model 1, though in that case there is a more significant contribution from $H_1(w)$. The shortening of the curves in Fig. 13 at high degree is an effect of mode selection, caused by there being no low frequency $p$-modes at high $l$.

The scaled data after subtraction of $H_2$ are shown in Fig. 14, the location of the spline fit $H_1(w)$ being indicated by dots as in Fig. 4; also shown is the exact $H_1$. The scale error at low $w$, discussed earlier for the homogeneous model, is more severe in this case. The fitted $H_2$ closely

![Figure 11](image-url)
approximates the behaviour of the individual curves in Fig. 13, which is particularly visible at high $v/L$. This is rather different from the earlier examples, reflecting the different structure of the outer layers of the present proxy model. The complicated behaviour of $\tilde{H}_2$ at low frequencies is absent, which is in accord with the hypothesis that this behaviour is due to differences in composition, since in that respect the present two models are almost identical.

The results of the inversion are shown in Fig. 15. The region $r < 0.92 R$ is not shown, since in this region $|\delta c/c|$ is no greater than at $r = 0.92 R$. The error in the inferred sound speed is likewise very small in this region except in the part corresponding to the final, long spline where the maximum error in the sound speed is 0.25 per cent. The inversion fails to reproduce accurately the major differences in the sound speed between the two models, which are in the outer 2 per cent of the star by radius. The inversion does produce a positive $\delta c/c$ near the surface but grossly underestimates its true magnitude. In the outer 0.5 per cent of the star the inverted $\delta c/c$ is almost zero because $\tilde{H}_1$ happens to be approximately zero at $w = w_{\text{min}}$ (cf. Fig. 14). The inverted $\delta c/c$ does not attain the extreme positive values of the true sound-speed
difference largely because, at small values of $\nu/L$, $\bar{H}_1$ is less steep than $H_1$. This seems to arise from the scale error mentioned above. The additive constant by which the fitting method has displaced $\bar{H}_1$ relative to $H_1$ also contributes to this shortfall. None the less, even in this example when the surface layers dominate the differences between the models, the inversion is very successful in the interior.
Figure 15. The exact fractional sound-speed difference (dashed line) between the model with modified superadiabatic layer and Model 1, as a fraction of the sound speed of the latter, in the outer part of the models. The estimate inferred using a 28 w-, 20 w-spline knot fit and the mode set defined at the beginning of Section 5 is shown with a solid line. The extreme values of the exact fractional sound speed difference, attained in the outer part of the models, are $1.78 \times 10^{-2}$ and $-3.46 \times 10^{-2}$.

**Example (v)**

In contrast to the previous examples, we here used as a reference the model with a mixed core (as was used in Example (ii) for the proxy Sun) and once more considered the chemically homogeneous model as the proxy Sun. The motivation for this was to ask whether, having ‘found’ from the first example that the central sound speed in the homogeneous model differs at the 10 per cent level from that in the original reference model, using a reference model with a similar central sound speed would enable us to infer more accurately the sound speed in the core. The sound speed in this pair of models differs by 4 per cent at $r = 0.7R$ and by 3 per cent in the inner core and in the outer layers. The error in the inferred sound speed (as a fraction of the sound speed of the present reference model) is depicted in Fig. 8(c). As one would expect, the errors in $r > 0.3R$ are very similar to those shown in Fig. 8(a). However, the error in the core has been greatly reduced, to much less than 0.5 per cent. No doubt there is now greater...
cancellation between the buoyancy and $\omega_\alpha$ terms of the two models, and the error arising from linearizing the perturbation quantities is correspondingly smaller.

6 Discussion

The differential method of asymptotic sound-speed inversion appears from these results to be capable in principle of reproducing the sound speed from the outer layers right down to the core with considerable accuracy. In all the examples we have considered, the sound speed has been recovered throughout most of the star with an error smaller than 0.5 per cent; and even in the core the accuracy was generally better than 1 per cent down to the radius of $r=0.013\, R$ at which the inversions stopped. We do not claim that such accuracy can always be achieved. Indeed in the cases considered the sound-speed differences varied quite smoothly with radius in the cores of the models, and this undoubtedly contributed to the success of the inversions. In fact experience with analogous asymptotic inversions for the solar internal rotation rate indicates that the resolution of such methods is limited in the core, due to the relatively small number of modes that contribute. Also the effect of including errors in the data must be investigated, though the spline-fitting method should be quite robust against non-systematic errors, except perhaps in the region where $r \leq 0.1\, R$, where the inversion depends on the frequencies of only a few modes.

Of course, the present inversion method sacrifices the model-independence of the method of Christensen-Dalsgaard et al. (1985). In return, however, it seems capable of obtaining substantially more accurate results from the highly simplified asymptotic description. It is true that taking into account the frequency dependence of the phase shift $\alpha$ (cf. Gough 1986a) in the direct inversion of the frequencies (as opposed to the inversion of the frequency differences) does improve the absolute determination of the sound speed considerably. However, Sekii & Shibahashi (1987), implementing the method described by Shibahashi (1988) to take into account the critical cut-off frequency (though not the buoyancy frequency), and Brodsky & Vorontsov (1987, 1988), using a somewhat different approach that also took into account the buoyancy frequency and the perturbation in the gravitational potential, remain with a relative error of 1–2 per cent in the vicinity of $r=0.1\, R$. Further out in the star it is not possible to estimate the accuracy of the inversions of Brodsky & Vorontsov from the results that they present; but a comparison of fig. 7 of Sekii & Shibahashi (1989) with our Fig. 8(b) indicates that our method appears to be a considerable improvement, relative to the model-independent inversions, over the whole range of $r$.

Linearization in the perturbed quantities does not appear to introduce much error, even though the difference between the sound speeds in the homogeneous model and Model 1 is as great as 10 per cent. This is probably rather greater than the difference in sound speed between Model 1 and the Sun. Therefore with accurate data one might anticipate an inversion at least as accurate as those presented here. However, there are additional departures from the simple representation (2.6) that first require further investigation, such as non-adiabatic effects, which are of course present in the Sun but not in our analysis. In this respect our apparent neglect of buoyancy and the perturbation $\Phi^\prime$ to the gravitational potential falls into a different category. Although they are not taken explicitly into account in equation (2.2), they have been included in the numerical computation of the eigenfrequencies that are analysed. Therefore our analysis, which can be regarded as a procedure for a sophisticated comparison of theoretical and observed frequencies, does implicitly include them. If the comparison model is close to the Sun, these small corrections largely cancel. This is particularly true of $\Phi^\prime$, which depends principally on the density structure; we have confirmed by numerical experimentation that the effects of $\Phi^\prime$ change little between similar models. The buoyancy term is a more sensitive
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quantity, since it depends on the relatively small departure of the density gradient from the adiabatic value, and we anticipate a more accurate inversion if we were to take it explicitly into account. Asymptotically this makes a contribution $\omega^{-2}H_3(w)$ to the scaled frequency, which has yet another functional dependence on $\omega$ and $w$ and so could in principle be separated from $H_1$ and $H_2$.

Finally we draw attention to the fact that our Example (v) indicates that the inversion is more accurate the closer the structure of the reference model is to that of the Sun. Therefore one can expect to obtain an even more faithful representation of the Sun by iteration. Subject to the equation of state being known, one can obtain a new solar model by integrating the linearized hydrostatic equations for the difference in pressure and density between the original reference model and its iterate. The equations are inhomogeneous, the inhomogeneity being dependent both on the differences in the abundances of those elements upon which the equation of state sensibly depends, and on the inferred sound-speed difference. We anticipate that in due course the first can be inferred by the procedure discussed by Däppen & Gough (1984, 1986); the second can be measured by the technique discussed in this paper.

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References

Appendix I. Inversion of equation (4.1)

Consider the equation

\[ H(w) = \int_{r_1}^{r_2} \left( 1 - \frac{a^2}{w^2} \right)^{-1/2} h(r) a^{-1} d\ln r, \tag{A1.1} \]

where \(a(r)\) is a monotonic decreasing function of \(r\) and \(a(r_1) = w\). Suppose \(H\) is known in the range \(w_1 < w < w_2\) with \(w_1 > a(R)\). Then it is possible to express \(h(r)\) in the range \(r_1 < r < r_2\) (where \(a(r_1) = w_1\)) in terms of the known function \(H(w)\) throughout the interval \(w_1 < w < a(r)\) and the unknown function \(h(r)\) throughout the region where \(r\) satisfies \(a_s < a(r) < w_1\). One way of inverting equation (A1.1) is to multiply both sides by \((a^2 - w^2)^{-1/2}\) and integrate with respect to \(w\) from \(w_1\) to \(w_2\). After interchanging the order of integration the double integral on the right-hand side becomes

\[ \left( \int_{w_1}^{w_2} \int_{r_1}^{r_2} da \int_{a_1}^{a_2} da' \int_{w_1}^{w_2} dw \right) (a^2 - w^2)^{-1/2} (w^2 - a'^2)^{-1/2} \frac{w}{a} \frac{d\ln r}{da} h(r'), \tag{A1.2} \]

where \(a' = a(r')\) is a dummy variable of integration. The integration with respect to \(w\) can now be carried out; it is simplified if one first makes the substitution

\[ w^2 = a^2 \sin^2 \theta + a'^2 \cos^2 \theta, \tag{A1.3} \]

from which it follows that

\[ \int (a^2 - w^2)^{-1/2} (w^2 - a'^2)^{-1/2} w dw = \theta = \sin^{-1} \left( \frac{w^2 - a'^2}{a^2 - a'^2} \right)^{1/2}. \tag{A1.4} \]

Substituting this into the expression (A1.2), equating the result with the left-hand side of the weighted integral of equation (A1.1) and differentiating with respect to \(a\) then yields

\[ h(r) = -\frac{2a}{\pi} \frac{d}{d\ln r} \int_{w_1}^{w_2} (a^2 - w^2)^{-1/2} H(w) \, dw + \frac{2}{\pi} \frac{da}{d\ln r} \int_{r_1}^{r_2} \frac{a^2}{a^2 - a'^2} \left( \frac{w_1^2 - a'^2}{a^2 - w_1^2} \right)^{1/2} h(r') a'^{-1} \, d\ln r'. \tag{A1.5} \]

This equation, with \(h = \delta c/c\) and \(H = H_1\), reduces to equation (4.2) when \(w_1 = a_s (r_1 = R)\). Note, moreover, that if one extends the definition of \(H(w)\) into \([a_s, w_1]\) in terms of the (unknown) function \(h(r)\) in \([r_1, R]\) according to equation (A1.1), it follows that the second integral in equation (A1.5) can be rewritten as

\[ -\frac{2a}{\pi} \frac{d}{d\ln r} \int_{w_1}^{w_2} (a^2 - w^2)^{-1/2} H(w) \, dw. \tag{A1.6} \]
The complementary function plotted as a dashed curve in Fig. 3 is a multiple of the function $\tilde{h}$ obtained by replacing $H$ in (A1.5) and (A1.6) by unity:

$$\tilde{h}(r) = -\frac{a_s}{\pi} \left(1 - \frac{a_s^2}{a^2}\right)^{-1/2} \frac{d\ln a}{d\ln r}. \quad (A1.7)$$

In this paper we have chosen to specify $H$ in $a_s < w < w_1$, which implies a choice of $\delta c/c$ in $r > r_1$. If for example one chose $H$ to be linear in $w$ in this region, matching to zero at $w = a_s$ and to the observed value $H(w_1)$ at $w = w_1$, the relative sound-speed difference above $r_1$ thus implied can easily be shown to be

$$\delta c/c = \frac{2H_1(w_1)a^2}{\pi w_1} \left(1 - \frac{a_s}{w_1}\right)^{-1} \left(1 - \frac{a_s^2}{a^2}\right)^{1/2} \frac{d\ln a}{d\ln r}. \quad (A1.8)$$

We chose to make $H$ linear in $\log w$: the implied $\delta c/c$ for our datasets is a multiple of the function plotted as curve (a) in Fig. 3, for $r > r_1 = 0.994 R$. An alternative procedure to that described in the main text is to assume a functional form for $\delta c/c$ in the outer layers, constraining it with the help of equation (A1.1) to ensure that the value of $H$, at $w = w_1$ has the observed value.

### Appendix II. The treatment of the integrable singularity

The integrals in equations (2.7) and (4.2) contain an integrable square-root singularity. Hence their evaluation requires a little care. To be general, we consider an integral of the form

$$I = \int_{x_1}^{x_2} f(x)[g(x)]^{-1/2} \, dx, \quad (A2.1)$$

over a single mesh interval $[x_1, x_2]$: if $g(x)$ has a zero in the interval, the integral is restricted to that part of the interval where $g$ is positive.

We approximate $f(x)$ and $g(x)$ by linear functions on $[x_1, x_2]$. Also we introduce

$$f_1 = f(x_1), \quad f_2 = f(x_2), \quad g_1 = g(x_1), \quad g_2 = g(x_2), \quad (A2.2)$$

as well as

$$\Delta x = x_2 - x_1, \quad \Delta f = f_2 - f_1, \quad \Delta g = g_2 - g_1.$$

Assuming initially that both $g_1$ and $g_2$ are positive, we then have

$$I \approx \Delta x \left[f_1 J_0 + \Delta f J_1\right], \quad (A2.3)$$

where

$$J_0 = \int_0^{\Delta t} [g_1 + t\Delta g]^{-1/2} \, dt, \quad J_1 = \int_0^{\Delta t} t[g_1 + t\Delta g]^{-1/2} \, dt. \quad (A2.4)$$

The evaluation of $J_0$ and $J_1$ is elementary. We write the results as

$$J_0 = g_1^{-1/2} \frac{2}{\epsilon} [(1 + \epsilon)^{1/2} - 1], \quad J_1 = g_1^{-1/2} \frac{4}{3\epsilon^2} \left[1 - \left(1 - \frac{\epsilon}{2}\right)(1 + \epsilon)^{1/2}\right], \quad (A2.5)$$
where \( \epsilon = \Delta g / g_1 \). It is evident that there are numerical difficulties in computing these expressions for small \( \epsilon \). Expanding in powers of \( \epsilon \) yields

\[
I_0 \approx g_1^{-1/2} \left( 1 - \frac{\epsilon}{4} + \frac{\epsilon^2}{8} \right), \quad I_1 \approx g_1^{-1/2} \left( 1 - \frac{\epsilon}{3} + \frac{3\epsilon^2}{16} \right). \tag{A2.6}
\]

In the computations we switched to equations (A2.6) for \( |\epsilon| < 0.01 \).

When \( g(x) \) has a zero on \( [x_1, x_2] \) we obtain, for \( g_1 > 0, g_2 \leq 0 \):

\[
I_0 = -\frac{2g_1^{1/2}}{\Delta g}, \quad I_1 = \frac{4g_1^{3/2}}{3\Delta g^3}, \tag{A2.7}
\]

whereas if \( g_1 < 0, g_2 > 0 \):

\[
I_0 = \frac{2g_2^{1/2}}{\Delta g}, \quad I_1 = \frac{2(g_2 - 3g_1)g_2^{1/2}}{3\Delta g}. \tag{A2.8}
\]