FORMATION AND ERUPTION OF SOLAR PROMINENCES

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ABSTRACT

A model for the magnetic field associated with solar prominences is considered. It is shown that flux cancellation at the neutral line of a sheared magnetic arcade leads to the formation of helical field lines which are capable, in principle, of supporting prominence plasma. A numerical method for the computation of force-free, canceling magnetic structures is presented. Starting from an initial potential field we prescribe the motions of magnetic footpoints at the photosphere, with reconnection occurring only at the neutral line. As more and more flux cancels, magnetic flux is transferred from the arcade field to the helical field. Results for a particular model of the photospheric motions are presented. The magnetic structure is found to be stable: the arcade field keeps the helical field tied down at the photosphere. The axis of the helical field moves to larger and larger height, suggestive of prominence eruption. These results suggest that prominence eruptions may be triggered by flux cancellation.

Subject headings: hydromagnetics — Sun: corona — Sun: magnetic fields — Sun: prominences

I. INTRODUCTION

Solar prominences consist of relatively cool (~8000 K) plasma which is elevated above the solar chromosphere and embedded in the much hotter corona (Tandberg-Hanssen 1974; Hirayama 1985; Zirin 1988). When observed in Hz on the solar disk, prominences appear as dark, elongated structures (filaments) overlying the neutral lines that separate regions of opposite magnetic polarity in the solar photosphere. Prominences are found in both active and quiet regions, with some differences in morphology and typical size. Magnetic fields play a major role in supporting prominence plasma against gravity (Kippenhahn and Schlüter 1957). Occasionally prominences become unstable, leading to prominence eruption and coronal mass ejection (Kopp and Pneuman 1976; Moore et al 1980; Martin 1973) discuss the formation of filaments, and describes several characteristic cases: (1) formation within active regions, (2) formation in between active regions, and (3) formation in weak remnant fields. Prior to the formation of some filaments in cases (1) and (2), Hz fibrils are seen to form a path in which the individual fibrils are aligned approximately end to end. Accumulation of absorbing mass along this path is then seen to occur, eventually leading to a fully developed filament. This alignment of chromospheric fibrils along the neutral line suggests that the magnetic field associated with filaments is highly sheared (also see Foukal 1971). Direct evidence for the presence of sheared, nonpotential magnetic structures comes from vector field measurements at the photospheric level in active regions (e.g., Hagyard, Moore, and Emslie 1984). The fibril alignments and magnetic shear are believed to be due to large-scale shear flows in the vicinity of the magnetic neutral line (Martres, Soru-Escaut, and Rayrole 1971; Harvey and Harvey 1976; Athay, Jones and Zirin 1985a, b, 1986; Athay et al. 1986). Converging flows toward the neutral line may also be present (Rompolt and Bogdan 1986).

Magnetic field measurements using the Hanle effect provide information on the orientation of the vector field at the height of the prominence (Leroy 1979, 1985, 1988). Simultaneous measurements in two spectral lines indicate that the field is nearly horizontal (Athay et al. 1983; Querfeld et al. 1985; Bommier, Leroy, and Sahal-Brechot 1986). Leroy, Bommier, and Sahal-Brechot (1984) present a statistical analysis of a large number of measurements using the He D$_3$ line. They find that for quiescent prominences with maximum height larger than 30,000 km the component transverse to the filament axis generally points from the region of negative polarity in the photosphere to the region of positive polarity, i.e., opposite to the direction of the potential field (also see Bommier et al. 1985). For prominences with maximum height less than 30,000 km the transverse field is in the same sense as the potential field. The angle between the magnetic field and the prominence axis is $-25^\circ$ to $-30^\circ$ in the first case, and $+20^\circ$ in the second case. The large prominences of the polar crown are all of the nonpotential type (Leroy, Bommier, and Sahal-Brechot 1983).

Finally, erupting prominences often display a helical structure (e.g., Schmahl and Hildner 1977), supporting the idea that prominences are embedded in helical magnetic fields. Models of prominence magnetic structure have been discussed by a number of authors. Kippenhahn and Schlüter (1957) point out that a prominence can be stably supported only at places where the field lines are locally horizontal and curved upward ("toughs"). In the Kippenhahn-Schlüter model the field has a simple topology, with field lines threading the prominence in the same direction as the potential field. Kuperus and Tandberg-Hanssen (1967) considered the formation of a prominence in an open current sheet between two regions of opposite polarity. They proposed a model in which the current sheet breaks up into isolated filaments as a result of tearing-mode instability (also see Raadu and Kuperus 1973;
Raadu 1979), Kuperus and Raadu (1974) and van Tend and Kuperus (1978) argue that the current will eventually coalesce into a single filament, producing a closed magnetic structure with field lines circling around the filament in the same sense as the surrounding magnetic arcade. The horizontal field below the current filament is a possible site for stable support of prominence plasma. In the Kuperus-Raadu model the magnetic field threads the prominence from the region of negative polarity to the region of positive polarity, i.e., opposite to the direction of the potential field. The Kuperus-Raadu model is therefore consistent with vector-field measurements of quiescent prominences with maximum heights larger than 30,000 km, in particular those of the polar crown (Leroy, Bommier, and Sahal-Brechot 1983, 1984). The application of this model to two-ribbon flares has been discussed by Kaastra (1985) and Kuin and Martens (1986).

Pneuman (1983) proposed a somewhat different scenario for the formation of essentially the same "figure eight" type configuration. He argued that the radial outward distension of a bipolar region by gas-pressure gradients could lead to an inward collapse of the region toward the vertical plane containing the photospheric neutral line. This collapse presumably causes the field lines to reconnect. If the initial field is significantly sheared along the neutral line, the positive polarity part of one field line interacts with the negative polarity part of a second field line which is displaced along the neutral line relative to the first one. As a result the reconnection produces helical field lines in the region above the reconnection point. The prominence plasma presumably is located at the lowest points of the helical windings.

Recently it has been recognized that the disappearance of photospheric magnetic flux may play an important role in flare and filament activity. Martin, Livi, and Wang (1985) describe videomagnetograph observations of a decaying active region in the period 1984 August 3–8. They conclude that the 22 flares observed during this period were initiated at sites where photospheric magnetic fields were canceling (here "cancellation" is defined as "the apparent mutual loss of magnetic flux in closely spaced features of opposite polarity"). One of these flares is of the classical "two-ribbon" type, involving filament eruption (see Fig. 13 of Martin, Livi, and Wang 1985). Martin (1986) describes two observations in which for several hours before the formation of a filament small-scale fragments of opposite polarity flux were cancelling in the region below the (eventual) filament and at the ends of the filament. Hermans and Martin (1986) studied small-scale eruptive filaments in the quiet Sun, and found that the majority of these structures were related to cancelling magnetic features in videomagnetograms. Zwaan (1987) and Priest (1987) point out the important role of reconnection processes in the cancellation of photospheric magnetic features.

The observations by Martin and coworkers have led us to study the effect of flux cancellation on the structure of coronal magnetic fields. In this paper we propose that flux cancellation in a sheared magnetic field may drive a reconnection process which produces a configuration of the Kuperus-Raadu type and is therefore capable of supporting prominence plasma. The model is similar to that of Pneuman (1983) in that helical fields are produced by reconnection below the prominence, but it differs in the mechanisms which drive the reconnection. We argue that flux cancellation in a sheared magnetic field must involve a reconnection process which decouples the longitudinal and transverse field components: without such a decoupling magnetic loops cannot submerge below the photosphere because their curvature radii are too large. We also develop a numerical method for studying the equilibrium and stability of the resulting magnetic structure. Calculations for a particular set of boundary conditions at the photosphere are presented. The results suggest that prominence eruptions may be triggered by flux cancellation at the photosphere.

The paper is organized as follows. In § II we consider the submergence of flux in a sheared magnetic field. In § III we develop a two-dimensional model of an arcade in which flux cancellation is taking place. We assume that the field evolves through a series of force-free equilibria, i.e., gas-pressure gradients and the weight of the prominence plasma are neglected. After discussing a simple axisymmetric solution of the force-balance problem in § IV, we present a more general, numerical method in § V. The boundary conditions for a particular model of the flux distribution and velocity field at the photosphere are given in § VI. Numerical results are presented in § VII. Finally, in § VIII we discuss some of the approximations and limitations of the present model.

II. FLUX CANCELLATION IN SHEARED MAGNETIC FIELDS

Disappearance of magnetic flux from the solar photosphere may occur in a variety of ways (e.g., Zwaan 1987). In the simplest possible scenario a preexisting loop is pulled down below the photosphere as a result of converging flows below the solar surface (submergence). If the opposite polarity fields are not previously connected, such a connection must first be established by magnetic reconnection, either above or below the surface. In the former case the disappearance again involves the submergence of a magnetic loop through the photosphere, and for the purpose of this paper we will assume that this is the normal mode of flux disappearance on the Sun. The submergence of a magnetic loop through the solar atmosphere requires that the loop somehow overcomes its magnetic buoyancy (Parker 1955). Although converging flows may play an important role in the dynamics of the loops below the solar surface, flows in the atmosphere are much less important because the kinetic energy is small compared with the magnetic energy of the loop. Therefore, the only mechanism by which a flux loop can be pulled down through the atmosphere and submerge is magnetic tension and the forces associated with the downward curvature of the loop. To submerge, the downward force due to the curvature must be larger than the upward force due to magnetic buoyancy.

The curvature force (per unit volume) at the top of the loop is given by \( B^2/(4\pi R) \), where \( R \) is the local curvature radius and \( B \) is the field strength. For a loop in temperature equilibrium with its surroundings (\( T_l = T_p \)) the buoyant force per unit volume may be written as \( B^2/(8\pi H_p) \), where \( H_p = kT_p/m_g \) is the local pressure scale height (Parker 1955). Hence, for the curvature force to overcome magnetic buoyancy we require \( R < 2H_p \). The pressure scale height of the photosphere (\( H_p \approx 150 \text{ km} \)) is smaller than that of the layers above and below; hence, the photosphere acts as a barrier for the submergence of flux loops, requiring the smallest radius of curvature. For a semicircular loop the distance \( d \) between the footpoints at the base of the photosphere is twice the curvature radius; hence, for such a loop to submerge we require \( d < d_{\text{crit}} \), where \( d_{\text{crit}} = 4H_p \approx 600 \text{ km} \) is the critical distance at which curvature forces balance magnetic buoyancy. The actual value of \( d_{\text{crit}} \) is probably somewhat larger than \( 4H_p \) because the curvature radius of the field lines increases with decreasing height along the flux.
loop. Parker (1979, pp. 136–141) considers the equilibrium path of a flux tube in an isothermal, stratified atmosphere \((H_p = \text{constant})\) and shows that \(d_{\text{crit}} = 2\pi H_p\) in that case. Hence, for a loop to submerge the photospheric footpoints must approach each other within about 900 km.

Now consider the response of a closed coronal field to certain horizontal flows imposed at the solar photosphere. The evolution of the field is illustrated in Figure 1. At time \(t = 0\) the field consists of a set of loops connecting two regions of opposite magnetic polarity (Fig. 1a). We assume that the initial field is a potential field, so that the loops cross the neutral line more or less at right angles. We first apply a shear flow, i.e., the magnetic footpints on either side of the neutral line are displaced along the neutral line in opposite directions. This produces a shear, nonpotential magnetic structure in which the projections of the field lines onto the horizontal plane are more closely aligned with the neutral line (Fig. 1b). Next we apply a motion toward the neutral line, which brings the footpoints closer together and further enhances the magnetic shear (Fig. 1c). Shear displacements on the Sun typically are much larger than the critical distance for submergence, \(d_{\text{crit}} \approx 900\) km. For example, large-scale, horizontal velocities in active regions are of order \(0.1\) km s\(^{-1}\), producing displacements of order \(10^4\) km after only 1 day. For quiescent filaments outside active regions, the shear displacements are likely to be due to the solar differential rotation; the displacements depend on the distance to the neutral line from which the canceling field originates and could be \(\sim 10^5\) km or larger. Hence, the footpoints of a reconnected loop must be separated along the neutral line by a distance less than \(d_{\text{crit}} \approx 2\pi H_p\), where \(H_p\) is the photospheric pressure scale height. If the reconnection were to occur at a height much larger than \(H_p\), the footpoints of a reconnected loop would remain too far apart, unless the loop is very nearly perpendicular to the neutral line. The latter seems unlikely because the component of magnetic field along the neutral line is probably similar before and after reconnection. Therefore, we suggest that the reconnection takes place at a height of only a few times \(H_p\) above the base of the photosphere.

As more and more flux submerges, the magnetic flux in the helical field increases and the axis of the helical field moves to larger and larger height. Cool plasma may reside at the troughs of the helical field lines, forming a prominence. We do not consider here the question where the prominence plasma comes from, but we suggest there may exist siphon flows along the helical field lines which transport plasma to the prominence from the chromosphere at the far footpoints. Eventually the magnetic flux in the helical field becomes too large relative to that of the overlying arcade for the helical field to be held down at the solar surface. We suggest that the configuration then looses its equilibrium and the helical field erupts, taking the filament with it. In the next sections we consider a two-dimensional model of the magnetic structure on a length scale large compared to the photospheric pressure scale height, \(H_p\).
(i.e., we consider the limit $H_p \to 0$). Since the height of the reconnection point scales with $H_p$, the reconnection point in our model is located at the photospheric boundary and the details of the magnetic structure near the connection point are neglected. The purpose of these calculations is to study the equilibrium and global stability of the magnetic field.

### III. MODEL OF THE MAGNETIC STRUCTURE

We have seen that submergence of magnetic flux in a sheared field must involve a reconnection process which decouples the transverse field component from the longitudinal component. We do not consider here the details of the reconnection process, but simply assume that it can proceed at the rate imposed by the converging flow, without pile-up of flux near the neutral line. Although magnetic diffusion and reconnection are required locally for flux cancellation, further away from the neutral line magnetic diffusion is probably quite unimportant on account of the high electrical conductivity of the coronal plasma. Therefore, it seems reasonable to assume that diffusion and reconnection occur only at the neutral line, and that the rest of the magnetic structure evolves as an ideal MHD fluid, i.e., the field is “frozen” into the plasma. As we will see below, this assumption greatly simplifies the analysis of the equilibrium and stability of the magnetic structure: except for reconnection at the neutral line, magnetic field lines retain their identity, and therefore the magnetic shear present at one time can be related to the photospheric motions that produced that shear.

Consider, then, a magnetic configuration with two topologically distinct flux systems, the helical field and the surrounding arcade field. We introduce a Cartesian coordinate system $(x, y, z)$, with the $y$-axis along the neutral line and $z$ measuring the height above the photosphere. The magnetic field $\mathbf{B}$ is independent of the $y$ coordinate and can be written in the form

$$ \mathbf{B}(x, z, t) = \left( \frac{\partial A}{\partial z}, B_y, -\frac{\partial A}{\partial x} \right), $$

where $A(x, z, t)$ is the magnetic potential, which is constant along field lines. The induction equation in ideal MHD has the following form

$$ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), $$

where $\mathbf{v}(x, z, t)$ is the velocity. From the $x$- and $z$-components of equation (2) one finds

$$ \frac{\partial}{\partial x} \left( \frac{dA}{dt} \right) = \frac{\partial}{\partial z} \left( \frac{dA}{dt} \right) = 0, $$

where $dA/dt$ is the comoving time derivative. The general solution of equation (3) is $dA/dt = f(t)$, but we may choose the gauge of $A(x, z, t)$ such that $dA/dt = 0$, i.e., $A(x, z, t)$ is a constant of motion. We assume that $A$ vanishes at large distance from the neutral line. The amount of flux present in the initial potential field ($t = 0$) is then given by the value of the magnetic potential at the neutral line, $A_{\text{max}} = A(0, 0, 0)$. Hence, $A(x, z, t)$ lies in the range $0 \leq A \leq A_{\text{max}}$.

Now we let some of this flux cancel, but not without first giving it some shear displacement along the $y$-axis. Then, at some later time $t$ the magnetic flux in the arcade, $A_0(t)$, is again given by the value of the potential at the neutral line, $A_0(t) = A(0, 0, t)$, while the flux in the helical field is given by $A_{\text{max}} - A_0(t)$. Hence, all field lines present initially are still present at later times; the field lines have been transferred from one flux system to the other. This statement is valid only if the field lines are given a shear displacement larger than some minimum value $d_{\text{crit}}$; otherwise, the arcade field would simply submerge without leaving helical field lines (on the Sun $d_{\text{crit}} \approx 900$ km; see §II).

We assume that the flux $A_0(t)$ of the arcade decreases monotonically with time. Let $t_0(A)$ be the inverse of the function $A_0(t)$. Then $t_0$ is the time at which field line $A$ reconnects at the neutral line.

Now consider the force balance of the configuration. As a first approximation we neglect gas-pressure and gravity effects, i.e., we assume that the field is force free (Lorentz force vanishes). In a force-free field the electric currents flow along the magnetic field lines,

$$ \mathbf{V} \times \mathbf{B} = \mathbf{z} \mathbf{B}. $$

Inserting expression (1), we find that $B_y$ must be a function of the magnetic potential, $B_y = B_y(A, t)$, and that the scalar $\alpha$ is given by:

$$ \alpha(A, t) = -\frac{\partial B_y}{\partial A}. $$

The potential $A(x, z, t)$ satisfies the following partial differential equation:

$$ \nabla^2 A + B_y \frac{\partial B_y}{\partial A} = 0. $$

Together with two boundary conditions, equation (6) defines a nonlinear boundary-value problem for the magnetic potential $A(x, z, t)$ (for a review of this problem, see Priest 1982, pp. 143–149 and 369–377). One of the boundary conditions is the flux distribution at the photosphere, $A(x, 0, t)$. The other condition can be chosen in several different ways. The simplest one is to prescribe the functional form of $B_y(A, t)$ (e.g., Low and Nakagawa 1975; Low 1977a; Jockers 1978; Birn and Schindler 1981). However, a physically more relevant problem is to find solutions for prescribed positions of the photospheric footpoints (Sturrock and Woodbury 1967; Barnes and Sturrock 1972; Yang, Sturrock, and Antiochos 1986; Zwingmann 1987; Klimchuk, Sturrock and Yang 1988). The function $B_y(A, t)$ is then not a priori known, but must be determined in conjunction with $A(x, z, t)$ using the boundary conditions.

To specify the boundary conditions we now consider in more detail the motions of magnetic footpoints at the photospheric plane $(z = 0)$. The converging flow toward the neutral line is described by the velocity component $v_y(x, 0, t)$, or equivalently, by the mapping function $x(x_0, t)$, where $x_0$ is the position of a photospheric footpoint at time $t = 0$. If the initial distribution $A(x_0, 0, 0)$ of the magnetic potential is given, we can identify a value of $A$ with each initial coordinate $x_0$, and write its later position $x$ as a function of $A$ and $t$. Thus we obtain two functions $x = X^+(A, t)$ and $x = X^−(A, t)$, corresponding to the footpoints with positive and negative polarity for each field line. Similarly, the shear flow along the neutral line can be described by the velocity component $v_y(x, 0, t)$, which may be integrated with respect to time to yield the relative displacement $Y(A, t)$ of the two footpoints:

$$ Y(A, t) = \int_0^t v_y [X^+(A, t), 0, t] dt - \int_0^t v_y [X^−(A, t), 0, t] dt. $$
The functions \(X^{(\pm)}(A, t)\) and \(Y(A, t)\) define the footpoint positions for field lines in the arcade region \([0 \leq A \leq A_0(t)]\). For field lines in the helical-field region, there are no photospheric footpoints, but we can define a similar quantity \(Y_{\text{helix}}(A, t)\), defined as the length along the \(y\)-axis of one full helical winding. We introduce the distance \(s\) along the projection of a field line onto the \((x, z)\) plane. The displacement \(y(s, A, t)\) along a field line is determined by the relation \(dy/B_y = ds/|\nabla A|\), where \(|\nabla A|\) is the magnitude of the \((x, z)\) component of the magnetic field. Integrating this relation along a field line, we obtain

\[
Y(A, t) = -B_y(A, t) \int_{s_0}^{s_{\text{max}}} \frac{ds}{|\nabla A|} \quad \text{for } 0 \leq A \leq A_0(t), \quad (8a)
\]

\[
Y_{\text{helix}}(A, t) = -B_y(A, t) \int_{s_0}^{s_{\text{max}}} \frac{ds}{|\nabla A|} \quad \text{for } A_0(t) \leq A \leq A_{\text{max}}, \quad (8b)
\]

where \(s^{+}\) and \(s^{-}\) denote the positive and negative polarity footpoints, and the integral in equation \((8b)\) is over a closed contour in the \((x, z)\) plane. Relations \((8a)\) and \((8b)\) can be used to express \(B_y(A, t)\) in terms of \(Y(A, t)\) or \(Y_{\text{helix}}(A, t)\).

Once a field line has entered the helical-field region, the length \(Y_{\text{helix}}\) of a helical winding is a constant of motion. To see this, consider the flux \(\Delta \Phi_y = \int B_y \, dx \, dz\) through an area in the \((x, z)\) plane between two helical field lines \(A\) and \(A + \Delta A\), with \(\Delta A\) small. Assuming \(v_z = v_y(A, t)\), the \(y\)-component of equation \((2)\) yields

\[
\frac{\partial B_y}{\partial t} + \frac{\partial}{\partial x} (B_x v_y) + \frac{\partial}{\partial z} (B_z v_y) = 0. \quad (9)
\]

Integrating this expression over the comoving area between the two field lines, one finds that \(\Delta \Phi_y\) is independent of time. The distance between the projections of the field lines \(dn = \Delta A/|\nabla A|\); hence, \(\Delta \Phi_y\) can also be written as

\[
\Delta \Phi_y = \int B_y \, dn \, ds = -Y_{\text{helix}} \Delta A. \quad (10)
\]

Since \(\Delta \Phi_y\) and \(\Delta A\) are constants of motion, \(Y_{\text{helix}}\) is also a constant of motion. This result is a consequence of our assumption that the magnetic and velocity fields are invariant with respect to \(y\): there is no stretching of the helical windings in the direction along the filament \((\partial v_y/\partial y) = 0\).

During the transformation of field line \(A\) from the arcade region to the helical-field region \([\text{at time } t = t_0(A)]\), the coordinate \(y(s, A, t)\) changes with time in a continuous fashion. Hence, \(Y_{\text{helix}}(A)\) is equal to the shear displacement of the photospheric footpoints at the time of reconnection:

\[
Y_{\text{helix}}(A) = Y[A, t_0(A)]. \quad (11)
\]

Therefore, the function \(Y_{\text{helix}}(A)\) can be computed directly from the known displacements \(X^{(\pm)}(A, t)\) and \(Y(A, t)\) at the photospheric boundary. Thus, \(Y_{\text{helix}}(A)\) provides a "boundary condition" on the field lines in the helical-field region, even though these field lines are no longer connected to the photospheric boundary. In \(\S\ V\) we will describe how this boundary condition is implemented in a numerical code.

One of the main characteristics of our model is that the helical field spirals around a horizontal axis which runs parallel to the neutral line at some height \(h(t)\) above the photosphere. Because of our assumption that the field was sheared before any flux was allowed to cancel, none of the field lines that were present initially could submerge without producing helical field lines (see \(\S\ II\)). Therefore, all field lines present initially are still present at later times, and the magnetic potential at the helical-field axis is given by \(A_{\text{max}}\); the total flux present in the initial potential field. The height \(h(t)\) of the axis depends on the force balance of the magnetic structure and cannot be computed \(a\ priori\). We assume that in the neighborhood of the axis the projections of the field lines onto the \((x, z)\) plane are approximately concentric circles, i.e., the helical field in this region has axial symmetry as well as translational symmetry. In the Appendix we discuss the asymptotic behavior of the field near the helical-field axis. It is shown that the field may be regular or singular at the axis, depending on the nature of the shear flow at the photosphere. The axisymmetric model to be discussed in the next section is an example of a field with regular behavior, while our numerical solutions (\(\S\ VI\)) use boundary conditions that lead to singular behavior.

IV. AXISYMMETRIC MODELS

In this section we discuss a general method for constructing force-free fields with helical field embedded in a surrounding arcade. To make the problem tractable we assume that the field has both translational and axial symmetry, with the symmetry axis located at \(x = 0, z = h\). Let \((r, \phi, z)\) be a cylindrical coordinate system aligned with the symmetry axis. Then the arcade field is located in \(r \geq h\), because field lines in this region intersect the photospheric boundary plane \((z = 0)\). The helical field is located in the cylinder \(r < h\). Instead of specifying the shear displacement \(Y(A, t)\) at the photosphere as a function of \(A\) and \(t\), we will specify it here as \(Y'(x, h)\), where \(x\) is the distance to the neutral line, and \(h\) is the height of the helical-field axis (which plays the role of time). For field lines in the arcade region we rewrite \(Y'(x, h)\) as a function of \(r\) and \(h\):

\[
Y'(x, h) = Y'(\sqrt{x^2 - h^2}, h) \quad r \geq h. \quad (12)
\]

For field lines in the helical-field region we again introduce the length of one helical winding, \(Y_{\text{helix}}(r, h)\), which is a constant of motion (see \(\S\ III\)). For simplicity we assume that \(Y_{\text{helix}}(r, h)\) is independent of \(h\), i.e., we assume that there is no radial compression or expansion of the helical field. Thus \(Y_{\text{helix}} = Y_{\text{helix}}(r)\), and this function is determined by the condition that the \(y\)-displacements are continuous at the time of reconnection:

\[
Y_{\text{helix}}(r) = Y(r, r). \quad (13)
\]

Now define a quantity \(\lambda(r, h)\) by

\[
\lambda(r, h) = \frac{dy}{d\phi} = r B_y(r, h) / B_\phi(r, h). \quad (14)
\]

Integration over \(\phi\) yields a relation between \(\lambda(r, h)\) on the one hand and \(Y(r, h)\) or \(Y_{\text{helix}}(r)\) on the other hand:

\[
Y(r, h) = 2[\pi - \arccos (h/r)] \lambda(r, h) \quad \text{for } r \geq h, \quad (15a)
\]

\[
Y_{\text{helix}}(r) = 2\pi \lambda(r, h) \quad \text{for } r < h. \quad (15b)
\]

The force balance for the axisymmetric case is discussed in the Appendix. For the purpose of this section we consider \(\lambda\) to be a function of \(r\); hence, equation \((A3)\) is interpreted as a first-order differential equation for \(\partial A/\partial r\). Integrating with respect to \(r\) yields the following general solution:

\[
\frac{\partial A}{\partial r} = -\frac{B_0}{\sqrt{\lambda^2(r, h) + r^2}} \exp \left[ -\int_r^{r_{\text{max}}} \frac{r \, dr}{\lambda^2(r, h) + r^2} \right]. \quad (16)
\]

where \(B_0[= B_y(r = 0)]\) is the field strength on the axis. This expression is valid provided \(\lambda(r)\) does not approach zero too
rapidly for \( r \to 0 \); otherwise the integral on the right-hand side of equation (16) diverges; if we describe the asymptotic behavior for \( r \to 0 \) by \( \lambda(r) \sim r^m \), then we avoid divergence by requiring \( m < 1 \).

Equations (15) and (16) may be used to construct cylindrical symmetric force-free fields for arbitrary displacement \( Y'(x, h) \) at the photosphere. The normal component of the field at the photosphere, \( B_z(x) \), follows from this analysis and cannot be arbitrarily prescribed. The method is complementary to that of Priest and Milne (1980), who describe a procedure for constructing axisymmetric force-free fields with prescribed \( B_z(x) \).

As an example of the present method we consider the case

\[
Y'(x, h) = 2Ay = \text{constant}. \tag{17}
\]

From equations (15a) and (15b) we derive an expression for \( \lambda(r, h) \); inserting this expression into equation (16) yields \( \partial A/\partial r \) as a function of \( r \). We integrate this function numerically (for \( Ay = 1, R_0 = 1 \)), thus obtaining \( A(r, h) \). Figure 2 presents the result of this calculation. Each panel in Figure 2 shows a perspective view of the field lines, with \( h \) increasing from panel (a) to panel (d); this sequence represents a sequence in time. The same five field lines (values of \( A \)) are plotted in each panel. Note that the photospheric footpoints march toward the neutral line and become connected. This example illustrates how the cancellation of magnetic flux in a sheared arcade leads to the formation of helical field lines, and an increase in height of the helical structure.

V. NUMERICAL METHOD

We now return to the general problem outlined in § III, namely, to find a force-free field with prescribed positions of the photospheric magnetic footpoints. Two closely related numerical methods for solving this problem have been described in the literature. The first method is to reformulate the problem in terms of Euler potentials, \( B = \mathbf{v} \times \mathbf{V} \), and to prescribe the boundary conditions for \( u \) and \( v \) at the photosphere; since the Euler potentials are constants of motion, this uniquely specifies how the footpoints are connected. A number of authors have used this technique to study the evolution of a sheared magnetic arcade without flux cancellation (Sturrock and Woodbury 1967; Barnes and Sturrock 1972; Yang, Sturrock, and Antiochos 1986; Zwingmann 1987; Klimchuk, Sturrock, and Yang 1988). Van Ballegooijen (1988) applied the same method in three dimensions to study the fine structure of solar coronal loops. The second method is to use a Lagrangian formulation in which the grid is tied to the magnetic field lines (e.g., Sakurai 1979; Craig and Sneyd 1986). In the latter case the variables of the problem are the shapes of the magnetic field lines, and the boundary conditions are the positions of the photospheric footpoints. In the present paper we use the Lagrangian method, since it is more easily adaptable to the case with detached, helical field lines.

To describe the shapes of field lines, we introduce the functions

\[
x = x(s, A, t), \quad y = y(s, A, t), \quad z = z(s, A, t), \tag{18}
\]

where \( s \) measures the distance along the projection of a field line onto the \((x, z)\) plane. These functions are subject to a number of boundary conditions (see § III). In the arcade region \( [0 < A < A_0(t)] \) we require

\[
x(s^{+}, A, t) = X^{+}(A, t), \quad x(s^{-}, A, t) = X^{-}(A, t), \tag{19a}
\]

\[
y(s^{+}, A, t) = + Y(A, t)/2, \quad y(s^{-}, A, t) = - Y(A, t)/2, \tag{19b}
\]

\[
z(s^{+}, A, t) = 0, \quad z(s^{-}, A, t) = 0, \tag{19c}
\]

where \( s^{(\pm)} \) are the coordinates of the photospheric footpoints.

Fig. 2.—Time sequence of an axisymmetric force-free field in which magnetic flux is canceling at the neutral line.
For the helical field lines \([A_0(t) < A < A_{\text{max}}]\) we require

\[
y(s^{(')}, A, t) = + Y_{\text{hel}}(A)/2,
\]

\[
y(s^{('-)}, A, t) = - Y_{\text{hel}}(A)/2,
\]

(20)

where \(s^{(')}\) represents an arbitrary point on the projection of the field line onto the \((x, z)\) plane, and \(s^{('-)}\) represents the same point after one full turn along this (closed) contour. The magnetic field \(B(s, A, t)\) can be expressed in terms of the functions \(x, y,\) and \(z\) as follows:

\[
B = \frac{1}{J} \begin{bmatrix}
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}
\end{bmatrix},
\]

(21)

where \(J(s, A, t)\) is the Jacobian of the transformation \((s, A) \rightarrow (x, z),\)

\[
J = \frac{1}{|V A|} = \frac{\partial x}{\partial s} \frac{\partial z}{\partial A} - \frac{\partial x}{\partial A} \frac{\partial z}{\partial s}.
\]

(22)

The magnetic energy per unit length along the filament is defined by:

\[
W \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{B^2}{8\pi} \, dx \, dz.
\]

(23)

Inserting expression (21) and using \(dx \, dz = J \, ds \, dA,\) we obtain

\[
W = \int_{0}^{A_{\text{max}}} \int_{0}^{A_{\text{max}}} \left( \frac{\partial x}{\partial s} \frac{\partial z}{\partial A} - \frac{\partial x}{\partial A} \frac{\partial z}{\partial s} \right)^{-1}
\]

\[
\times \left[ \left( \frac{\partial x}{\partial s} \right)^2 + \left( \frac{\partial y}{\partial s} \right)^2 + \left( \frac{\partial z}{\partial s} \right)^2 \right] \, ds \, dA.
\]

(24)

We now make use of the fact that the problem of finding a stable force-free field (with prescribed footpoint positions) is equivalent to the variational problem of finding the state of minimum magnetic energy (e.g., Sakurai 1979). In the present case not all field lines are anchored in the photosphere, so we must introduce the additional constraint (20), which is a consequence of our assumption of ideal MHD (see § III). Therefore, to find a force-free field we must vary the functions \(x(s), A(t),\) \(y(s, A, t),\) and \(z(s, A, t)\) subject to constraints (19) and (20) and search for the minimum of the energy \(W.\)

We cast this problem in finite-difference form by introducing a grid in the \((s, A)\)-plane. Each grid point \((i, j)\) represents a point at distance \(s = s_{ij}\) along field line \(A = A_j,\) where \(A_j\) is a decreasing function of \(j(1, \ldots, M). One of the field lines is chosen to coincide with the critical field line \(A_{0}(t)\) which forms the interface between the arcade region and the helical-field region. To overcome numerical problems in the region where the interface touches the photosphere, the number of points along a field line, \(2N_j + 1,\) is a function of \(j; N_j\) decreases with \(j\) for the field lines just outside the interface. Note that expression (21) is invariant under the transformation \(s \rightarrow s(s);\) therefore, we may change the parametrization along the field lines such that

\[
s_{ij} = i - N_j - 1, \quad i = 1, \ldots, 2N_j + 1.
\]

(25)

The "end points" of a field line are then located at fixed points on the edge of the grid:

\[
s^{(')} = -N_j, \quad s^{('-)} = +N_j.
\]

(26)

At each grid point we define variables \(x_{ij}, y_{ij},\) and \(z_{ij},\) which describe the position of the point in space. Replacing the partial derivatives in equation (24) by their finite-difference analogs, we obtain an expression for the magnetic energy in terms of the variables \(x_{ij}, y_{ij},\) and \(z_{ij}\) at all interior grid points, \(W(x_{ij}, y_{ij}, z_{ij}).\) By minimizing this function subject to constraints (19) and (20), we obtain a finite-difference approximation of the force-free field at time \(t.\)

The minimization is accomplished using the conjugate gradient method. This is an iterative procedure for finding the minimum of a function of a large number of variables. In each iteration step the minimum along a straight line in parameter space is determined: \(x_{ij}, y_{ij},\) and \(z_{ij}\) are considered linear functions of some variable \(p\) and the minimum of \(W(p)\) is determined. The coefficients of \(p\) in the above linear relations are chosen according to the Polak-Ribiere variant of the conjugate gradient method (see Press et al. 1986). Since expression (24) contains the Jacobian (22) in the denominator, the quantity \(W(p)\) becomes singular when \(J\) goes through zero for any one of the computational cells. Therefore, before each line minimization we first determine an appropriate interval \([0, p_{\text{max}}]\) in which to search for the minimum.

The boundary conditions (19) and (20) are implemented as follows. For arcade field lines the values of \(x_{ij}, y_{ij},\) and \(z_{ij}\) at the end points \(i = 1\) and \(i = 2N_j + 1\) are fixed, i.e., they are not allowed to vary during the iteration process. For helical field lines we only fix \(y_{ij}\) at the "end points" and let \(x_{ij}\) and \(z_{ij}\) be variable. To make the projection of a helical field line close on itself, we make \(x_{ij} = x_{2N_j+1,j}\) and \(z_{ij} = z_{2N_j+1,j}.\)

Since the magnetic field vanishes far away from the neutral line, we introduce an outer boundary defined by the shape \([x(s), z(s)文物保护 of the initial potential field. Hence, the outer boundary is a flux surface, \(A = A_{\text{outer}},\) and its location in space is the same at all times. The contribution to the magnetic energy from the region beyond the boundary \((0 < A < A_{\text{outer}})\) is neglected. The parameters \(x_{ij}\) and \(z_{ij}\) at the outer boundary are fixed, while \(y_{ij}\) is allowed to vary.

To avoid the singularity that occurs at the axis of the helical field (see the Appendix), we also introduce an inner boundary \(A = A_{\text{cylinder}}\) defined as a slender cylinder concentric with the axis. Thus we exclude the helical-field axis from our computational domain. The boundary conditions at \(A = A_{\text{cylinder}}\) are similar to those at the outer boundary; during the iterative search for a force-free solution, the parameters \(x_{ij}\) and \(z_{ij}\) at the cylinder are fixed while \(y_{ij}\) is allowed to vary. The radius of the cylinder, \(r_c = r(A_{\text{cylinder}}),\) is determined by numerically integrating equation (A3) of the Appendix, using equation (A5) as the boundary condition. Since \(y_{\text{hel}}(A)\) is independent of time, \(r_c\) is the same at all times. However, the location of the cylinder is not constant: it is determined by the condition that the net force on the cylinder should vanish (because the weight of the cylinder is neglected). To enforce this condition, we treat the \(x-\) and \(z-\)coordinates of the cylinder axis as free parameters: for each time \(t\) we compute a series of force-free fields with different locations \((x_c, z_c).\)

The solution with the lowest value of the magnetic energy is the true equilibrium solution in which the net force on the cylinder vanishes.

VI. INITIAL AND BOUNDARY CONDITIONS

For the initial distribution of the vertical component of magnetic field at the photosphere we assume:

\[
B_0(x, 0, 0) = \frac{2x}{(1 + x^2)^2},
\]

(27)

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which reaches its extreme values at $x = \pm 1/\sqrt{3}$. Hence, the initial separation of magnetic polarities is used as the measure of length, and the magnetic flux is normalized such that $A_{\text{max}} = \int_0^\infty \mathcal{B}_1(x, 0, 0)dx = 1$. The initial field is assumed to be a potential field ($V^2A = 0$), which yields the following expression for the magnetic potential:

$$A(x, z, 0) = \frac{1 + z}{(1 + z^2 + x^2)}.$$  

(28)

Rewriting this expression as

$$\left(\frac{1}{2A} - 1 - z\right)^2 + x^2 = \left(\frac{1}{2A}\right)^2,$$

(29)

we see that the field lines are circles with radii $1/(2A)$, all of which pass through the point $x = 0, z = -1$. Hence, the initial field can be thought of as the potential field of a line of dipoles located at unit depth below the photosphere.

For the outer boundary of the computational domain we take a circular field line $A = A_{\text{min}}$ of the initial potential field. Let $|x| = x_m$ be the intersection of this field line with the photosphere ($z = 0$); then $A_{\text{min}} = (1 + x_m^2)^{-1}$. In the present calculation we used $x_m = 5$, so that $A_{\text{min}} = 0.0385$.

We assume the following flow pattern at the photosphere. The velocity component toward the neutral line is given by

$$v_{\text{v}}(x, 0, t) = \frac{1}{x} + \frac{x}{x_m},$$

(30)

independent of time $t$. The first term diverges for $x \to 0$, which is necessary to force magnetic flux to cancel at the neutral line ($\nu_r \mathcal{B}_r$ must be finite at $x = 0$). The second term is introduced so that $v_{\text{v}}(\pm x_m, 0, t = 0)$, consistent with our assumption that the outer boundary is fixed. Solving $dx/dt = v_{\text{v}}(x, 0, t)$, we find the positions $x = X^{(2)}(x_0, t)$ of the magnetic footpoints:

$$x^2 = x_m^2 - (x_m^2 - x^2) e^{2\pi s/(2y_0^2)}.$$  

(31)

Here $x_0$ is the initial footpoint position, which is related to the magnetic potential via $A = (1 + x_0^2)^{-1}$. A footpoint reaches the neutral line ($x = 0$) at a time $t_0$ given by

$$t_0(x_0) = \frac{1}{2} x_m^2 \ln \left(\frac{x_m^2}{x_m^2 - x_0^2}\right),$$

(32)

at which time the associated field line undergoes reconnection. For the velocity component along the neutral line we assume

$$v_{\text{v}}(x, 0, t) = v_0 \frac{|x|}{x_m^2 + x^2},$$

(33)

where $v_0$ and $c$ are free parameters. Inserting equation (31) and integrating with respect to time, we obtain for the relative displacement of the footpoints (see eq. [7]):

$$\begin{align*}
Y(x_0, t) &= 2v_0 \frac{x_0^2}{c^2 + x_0^2} \\
&\times \left\{ t - \frac{1}{2} c^2 \ln \left[ \frac{c^2 + x_0^2}{c^2 + x_m^2 - (x_m^2 - x_0^2) e^{2\pi s/(2y_0^2)}} \right] \right\}.  
\end{align*}$$

(34)

This expression applies for $0 \leq t \leq t_0(x_0)$, when the field line characterized by $x_0$ still has a footpoint at the photosphere. For $t > t_0(x_0)$ the field line is detached from the photosphere, and the length of a helical winding is given by

$$Y_{\text{helix}}(x_0) = Y(x_0, t_0(x_0)) = v_0 \frac{x_m^2}{c^2 + x_m^2} \left[ x_m^2 \ln \left(\frac{x_m^2}{x_m^2 - x_0^2}\right) - c^2 \ln \left(\frac{c^2 + x_0^2}{c^2}\right) \right].$$

(35)

Since the magnetic potential $A$ is a constant of motion, its distribution on the photosphere at time $t$ is given by

$$A(x, 0, t) = A(x_0, 0, 0) = \frac{1}{1 + x_0^2} \left[ 1 + x_m^2 - (x_m^2 - x^2) e^{-2\pi s/(2y_0^2)} \right]^{-1}.$$  

(36)

Hence, the flux in the arcade field is

$$A_0(t) \equiv A(0, 0, t) = [1 + x_m^2(1 - e^{-2\pi s/(2y_0^2)})]^{-1},$$

(37)

For $t \ll x_m^2/(2c^2)$ this may be approximated by

$$A_0(t) \approx \frac{1}{1 + 2c^2 t}.$$  

(38)

Hence, the time $t$ is given in units of the time scale of flux cancellation.

For the magnetic potential at the inner, cylindrical boundary we take $A_{\text{cylinder}} = 0.95$, i.e., we exclude 5% of the magnetic flux initially present in the photosphere. The radius of the cylinder is $r_c = 0.0193$. The assumption of a cylindrical inner boundary is reasonable only if the radius of the helical flux tube is somewhat larger than $r_c$. Therefore, we only consider models with $t \geq 0.1 [A_0(t) \leq 0.83$; see eq. (38)].

VII. RESULTS

The numerical method described in § V is quite general and can be used with an arbitrary distortion of flux at the photosphere. However, in the present case the boundary conditions at the photosphere are symmetric with respect to the plane $x = 0$, and we assume that the coronal field also exhibits this symmetry. Therefore, for the present paper we have used a simpler version of the method in which only the region $x \geq 0$ is considered. By symmetry the axis of the helical field is located at $x = 0$, directly above the neutral line; hence, to find the equilibrium position of the axis, we only need to vary its height, $z_a = h$. For each time $t$ we perform a series of calculations with different height $h$. The true equilibrium field is the solution with the lowest magnetic energy. In the modified version of the code there are $N_j + 1$ points along field lines; the point $i = N_j + 1$ corresponds to the apex of a field line, which is located at the symmetry plane ($x = 0$). The boundary conditions at the apex are $x_{ij} = 0$ and $y_{ij} = 0$, but the height $z_{ij}$ is allowed to vary so that a minimum energy state may be obtained.

In this paper we present results for the case $v_0 = 2$ and $c = 0.1$, corresponding to a rather abrupt change in $v_{\text{v}}$ near the neutral line (see eq. [33]). We use $N_j = 13$ for helical field lines and $N_j = 10-12$ for arcade field lines, depending on time $t$ and height $h$. The number of field lines varies from 40 to 50. We test the convergence of the solutions by computing the minimum and maximum values of $B_z$, along a field line; in a force-free field $B_z(4)$ is independent of $s$, so the minimum and maximum values should be equal. We find that at least 1000 iterations are required to reach a converged solution. The magnetic energy is estimated to be accurate to a few tenths of a%.
The search for the equilibrium height $h_{\text{eq}}(t)$ is illustrated in Figure 3, where we compare three solutions for time $t = 0.3$. When $h$ is too large ($h = 0.60$), the arcade field is stretched out, causing an increase of the magnetic energy. When $h$ is too small ($h = 0.18$), the helical field is pressed against the photosphere, again causing an increase of magnetic energy. The lowest energy state is obtained for $h = 0.33$.

The equilibrium solutions for $t = 0.2$, $0.4$ and $0.6$ are shown in Figure 4. Note that the equilibrium height of the axis increases with time as more and more flux enters the helical-field region. At early times the helical field is pressed against the photosphere by the overlying arcade, but at $t = 0.6$ this effect has decreased substantially, causing the helical field to pull away from the photosphere.

In Figure 5 we show the minimum and maximum values of $B_y(s, A, t)$ along field lines, plotted as function of $A$ for three different times. In a force free field $B_y(s, A, t)$ is independent of $s$; hence the differences between minimum and maximum values are indicative of residual deviations from a force-free field. To bring out the differences more clearly we plot $B_y/A^3$ versus $A$. Note that there are substantial deviations from a force-free state in the outer regions of the model, $A < 0.2$, where the rate of convergence of our iteration scheme is very slow. The effects of these errors on the magnetic energy $W$ are small, however, because the field strength in the outer region is small. Note also that $B_y(A, t)$ increases with time between $t = 0.2$ and $t = 0.4$ as a result of the velocity shear at the photosphere. However, $B_y(A, t)$ decreases with time between $t = 0.4$ and $t = 0.6$. This effect is due to the positive divergence of the velocity field at the photosphere, $\partial v_x/\partial x = x^{-2} + x_m^{-2}$, which causes an overall weakening of the magnetic field.
A three-dimensional view of the equilibrium field for $t = 0.6$ is shown in Figure 6. Field lines in the arcade region are plotted twice to show the shear displacement of the photospheric footpoints ($Y \approx 2.4$). The helical field line with two windings is the critical field line which separates the helical from the arcade field ($A = A_0$); this field line touches the photospheric plane at the neutral line (dashed line). The helical field is wound around an axis which lies at a height $h = 1.2$. Note that, unlike the semianalytic example of Figure 2, the central part of the helical field is very tightly wound ($Y_{\text{centr}} \rightarrow 0$ for $A \rightarrow A_{\text{max}}$). This is a consequence of our assumption that $v_x(x)$ is a continuous function of $x$ at the photosphere (see the Appendix).

Figure 7 shows the magnetic energy as function of height $h$ of the helical-field axis, for different times $t$. For early times the curves have a minimum, indicating there exists a stable equilibrium in which the net force on the inner cylindrical boundary vanishes. The equilibrium height, $h_{\text{eq}}(t)$, increases with time until at $t = 0.75$ the minimum becomes very shallow. Figure 8 shows the equilibrium energy as function of time $t$, together with the energy of the corresponding potential field. The latter is defined as the current-free field with the same photospheric flux distribution (eq. [36]). After a slight initial decrease (not resolved in the present calculations) the magnetic energy of the force-free field increases with time for $t > 0.1$, whereas the potential-field energy decreases (primarily because of the decrease of photospheric magnetic flux). At $t = 0.75$ the force-free field contains about eight times as much energy as the potential field.

For $t > 0.75$ the curves in Figure 7 decline monotonically with $h$, indicating that an equilibrium does not exist within the computed height range ($h \leq 3$). Problems with numerical
resolution and slow convergence prevent us from exploring the region \( h > 3 \) in any detail. However, the energy cannot decrease indefinitely with increasing height; in our model the field is contained within a finite volume, so when \( h \) approaches \( z_{\text{max}} = x_m^2 = 25 \) the field is compressed against the outer boundary of the volume and the magnetic energy will increase with increasing \( h \). Therefore, for \( t > 0.75 \) the energy curves must have a minimum somewhere in the range \( 3 < h < 25 \). We cannot say at present whether the curves have one or more minima in the height range \( 0 < h < 25 \). If there are two minima, there exists a possibility for a catastrophic transition from one equilibrium state to the other. The shallowness of the energy curve for \( t = 0.75 \) suggests, however, that there is only a single minimum which simply shifts to larger and larger height as time progresses. Therefore, the shift to larger height indicated in Figure 4 is a manifestation of the quasi-static evolution of the field, and does not have the character of an instability.

During the later phases of evolution the helical field has the tendency to pull away from the photosphere, as shown for example by the teardrop-shaped cross section of the helical-field region for \( t = 0.6 \) (Fig. 4c). It is possible that the helical field will actually separate from the photosphere, producing a detached helical flux tube located above an infinitesimally thin current sheet which stands vertically above the photospheric neutral line. We do not allow for the formation of such a current sheet in the present version of the code: the lowest point of the helical-field region is assumed to be located at

**Figure 7**—Magnetic energy \( W \) as function of height \( h \) of the axis of the helical field. The curves are labeled with the time \( t \).

**Figure 8**—Magnetic energy as function of time \( t \) (circles), and the corresponding potential field energy (dashed curve).
z = 0. Therefore, some of the solutions with large h may be incorrect, and the energies for these solutions are too high. Unfortunately, we do not know for which values of t and h current-sheet formation is energetically favorable; thus, we do not know precisely which solutions are affected, nor do we know how much they are affected. Further improvements in the code are required before we can answer these questions.

VIII. DISCUSSION

The results of the previous sections may be summarized as follows. We propose that flux cancellation in a sheared magnetic field leads to the formation of helical field lines which are capable, in principle, of supporting prominence plasma. We develop a model of prominence magnetic structure based on the following assumptions: (1) the magnetic field has an ignorable coordinate, y; (2) reconnection occurs only at the photospheric neutral line (x = z = 0), and everywhere else the field evolves according to ideal MHD; (3) the time scale of evolution is long compared with the Alfvén time; (4) gas pressure gradients and gravity have a negligible effect on the magnetic structure. Starting from a potential magnetic field at time t = 0, we simulate the response of the field to certain motions of the photospheric footpoints. We find that the field is stable against perturbations which keep the field invariant with respect to the coordinate y along the filament.

In the present calculation the field is assumed to be contained within a finite volume, which prevents the field from opening up. However, even if the entire half-space z > 0 were available, the helical field would not expand indefinitely. Aly (1984) has shown that in a two-dimensional open field the magnetic energy per unit length along the neutral line is infinite (the energy within radius r from the neutral line diverges logarithmically with r). If the helical field expands indefinitely, the arcade field is completely opened up, which requires an infinite amount of energy. Hence, the open field always has a larger magnetic energy than the closed field, and for large but finite values of h the magnetic energy increases logarithmically with increasing h. We conclude that a true eruption of the helical field cannot occur in a two-dimensional model if the field is assumed to evolve according to ideal MHD. Of course, on the Sun magnetic structures have a finite size L_y in the direction along the filament, so when the axis of the helical field reaches a height h of the order of L_y, three-dimensional effects become important. In active regions L_y is typically of the order of the distance between the opposite polarities; therefore, h = 3 in our model corresponds to a height on the order of the length of the filament. Hence, for h > 3 the assumption of a y-invariant magnetic structure is no longer valid. When three-dimensional effects are taken into account the arcade field may be pushed aside in the y-direction, allowing the helical field to slip through and expand to larger heights. Therefore, we suggest that the cancellation of flux at the photosphere would eventually lead to filament eruption.

By assuming invariance with respect to the y coordinate we also suppress kink modes of the helical field. In fact, it seems likely that the helical field in our model is kink unstable because each helical field line makes a large number of turns (see Fig. 6), while linear perturbation theory predicts instability when the field lines make more than about one turn (e.g., Sh秀丽rov 1957; Kruskal et al. 1958; Hood and Priest 1979). Therefore, the ideal MHD model developed in this paper should not be taken too literally. Kink instabilities in sheared arcades with and without embedded helical fields have been studied by a number of authors (e.g., Low 1977b; Hood and Priest 1980; Birn and Schindler 1981; Hood 1983, 1984; Migliuolo and Cargill 1983; Hood and Anzer 1987; An, Suess, and Wu 1988). We suggest that kink instabilities do not necessarily lead to filament eruption: in the nonlinear development of these instabilities reconnection processes may occur which effectively reduce the number of twists per unit length along the filament, without destroying the overall magnetic structure. It is unclear how this relaxation process will affect the eruptive instability. On the one hand, the field may open up more easily when three-dimensional perturbations are allowed. On the other hand, part of the magnetic energy is released during the relaxation of the three-dimensional kink instability, so less energy is available for driving the eruptive instability which opens up the magnetic field. Hence, premature energy release via kink instabilities may actually enhance the overall stability of the magnetic structure and delay the time of prominence eruption.

Further progress in modeling the eruption of solar prominences requires three-dimensional models which take the finite length of the magnetic structures into account. Although it is possible to extend the present Lagrangean method to the three-dimensional case, one disadvantage of the method is that resistive effects cannot easily be included. Therefore, if resistive energy release plays an important role, a more suitable approach would be to use an Eulerian grid (e.g., Mikic, Barnes, and Schnack 1988). Further work with both Lagrangean and Eulerian codes is needed to determine the relative merits.

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APPENDIX

ASYMPTOTIC BEHAVIOR NEAR THE AXIS OF THE HELICAL FIELD

The projections of the helical fieldlines onto the (x, z) plane are closed curves which circle around a certain point P. In the neighborhood of this point, the curves are nearly concentric circles. Therefore, we can write the magnetic potential as \( A(r, t) \), where r measures the distance to P in the (x, z) plane. Using the fact that \( A(r, t) \) is a monotonically decreasing function of r, we obtain from equation \( (8b) \)

\[
B_y(A, t) = \frac{\lambda(A)}{r} \frac{\partial A}{\partial r},
\]  

\( (A1) \)
where $\lambda(A)$ is defined by

$$\lambda(A) \equiv \frac{Y_{\text{helix}}(A)}{2\pi}. \quad (A2)$$

Inserting this into equation (6), we find the following ordinary differential equation for $A(r, t)$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A}{\partial r} \right) = \frac{\lambda(A)}{r} \frac{\partial A}{\partial r} = 0. \quad (A3)$$

Since $Y_{\text{helix}}(A)$ and hence $\lambda(A)$ can be derived from the conditions on the photosphere, equation (A3) is a nonlinear, second-order differential equation for $A(r, t)$. The boundary condition at the axis is $A(0, t) = A_{\text{max}}$.

To determine the asymptotic behavior of $A(r, t)$, we need to consider the shear displacement $Y(A, t)$ of the very first field lines that are reconnected at the neutral line. If the shear velocity $v_y(x, 0, t)$ is a continuous function of $x$ at $x = 0$, the relative displacement $Y(A, t)$ of the footpoints must vanish for field lines close to the neutral line (see eq. [7]). Hence, the first field lines to be pushed to the neutral line have only a small shear displacement $Y(A, t)$, and the first helical windings produced from these field lines have only a small length $Y_{\text{helix}}(A)$ along the neutral line (see eq. [11]). Therefore, in the case of a continuous shear flow at the photosphere, $Y_{\text{helix}}(A)$ and $\lambda(A)$ vanish in the limit $A \to A_{\text{max}}$. Let us describe the asymptotic behavior of $\lambda(A)$ by a power law,

$$\lambda(A) = C(A_{\text{max}} - A)^n, \quad (A4)$$

where $C$ and $n$ are assumed known ($n > 0$). Assuming that $A(r, t)$ is also given by a power law, we write

$$A_{\text{max}} - A(r, t) = Q r^k, \quad (A5)$$

where $Q$ and $k$ are to be determined. Inserting equations (A4) and (A5) into equation (A3), we obtain

$$k^2 Q r^{k-2} + k(k - 2 + nk) C^2 Q^{2n+1} A_{\text{max}}^{n+2} = 0. \quad (A6)$$

We can distinguish two separate cases. First, if the exponent of $r$ in the first term is larger than the exponent in the second term, $k - 2 > k - 4 + 2nk$, then in the limit $r \to 0$ the coefficient of the second term must vanish, $k - 2 + nk = 0$. This yields $k = 2/(n+1)$, which satisfies the above inequality for the exponents provided $n < 1$. The field near the axis is then given by

$$B_\phi \approx \frac{2Q}{n+1} r^{(1-n)(1+n)}, \quad B_r \approx -\frac{2CQ}{n+1} = \text{const} \quad (0 < n < 1). \quad (A7)$$

Second, if the two exponents in equation (A6) are equal, we require $k = 1/n$ and $C^2 Q^{2n} = 1/(n-1)$. This solution applies when $n > 1$. The field is then given by

$$B_\phi \approx \frac{Q}{n} r^{(1-n)/n}, \quad B_r \approx \frac{Q}{n^{1/n} n-1} r^{(1-n)/n} \quad (n > 1). \quad (A8)$$

Expression (A5) does not apply to the special case $n = 1$; in that case the solution of equation (A3) may be written as

$$A_{\text{max}} - A(r, t) = C^{-1} f(r), \quad (A9)$$

where $f(x)$ is a slowly varying function which satisfies the following differential equation:

$$(1 + f^2) \frac{d^2 f}{dx^2} + 2(1 + f^2) \frac{df}{dx} + \left[ 1 + \left( \frac{df}{dx} \right)^2 \right] f = 0. \quad (A10)$$

Note that expression (A7) is regular at the origin, whereas expressions (A8) and (A9) are singular at $r = 0$. The singularity for $n \geq 1$ is a consequence of the fact that the helical field lines are very tightly wound, causing strong curvature forces which compress the field toward the axis. The current density $j_x(r)$ also diverges, but the total current within radius $r$ is finite. Note also that in the singular case the value of $Q$ follows from the asymptotic analysis,

$$Q = \left( \frac{1}{C^{1/n}} \right)^{1/n}, \quad (A11)$$

whereas in the regular case $Q$ is determined by conditions at large $r$.

For the boundary conditions used in our numerical calculation the asymptotic relation is $Y_{\text{helix}}(A) = v_0/(2c^2)(1 - A)^2$, so that

$$n = 2 \quad \text{(see § VI)}. \quad \text{Hence the field is singular at the origin,}$$

$$B_\phi(r) \approx -B_r(r) \approx C \frac{\pi}{v_0 r}. \quad (A12)$$

REFERENCES


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