ON THE STABILITY OF MAGNETIZED ROTATING JETS: THE AXISYMMETRIC CASE

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ABSTRACT

We discuss the stability of a nonrelativistic, magnetized rotating flow to Kelvin-Helmholtz modes, using numerical methods for the analysis of the dispersion relation which allow complete and systematic mode classification. In the initial study presented here, we restrict attention to the axisymmetric case and work solely in the vortex sheet approximation. Major results include (1) interpretation of the reflected modes in terms of negative energy modes; (2) correction of previous results on the route to destabilization of reflected modes; (3) discovery of multiple maxima in the reflection mode growth rate as a function of Mach number; (4) discovery of a new unstable mode for magnetized, nonrotating beams which can be as important as the ordinary mode in a limited Mach number regime; and (5) the derivation of new unstable inertial modes for rotating jets. We also discuss some of the astrophysical implications of our study.

Subject headings: galaxies: jets — hydromagnetics

I. INTRODUCTION

The relatively recent, burgeoning interest in extragalactic and galactic jets has led to a revival of analytic and numerical work concerned with the stability properties of supersonic, beam-like flows. These studies have paid particular attention to destabilization by shear flow instabilities, especially of the Kelvin-Helmholtz type, in both magnetized and unmagnetized jets (Ferrari, Trussoni, and Zaninetti 1978; Hardee 1979; Ray 1981; Ferrari, Trussoni, and Zaninetti 1981; Fiedler and Jones 1984; Birkinshaw 1984; Payne and Cohn 1985). Because of the parallel efforts devoted to numerical simulation of such jets (Norman, Smarr, and Winkler 1983; Uchida and Shibata 1985; Woodward 1985), there has also been considerable interest in thorough exploration of the variety of ways in which these flows can be destabilized, at least in linear theory. One of the key recent results has been the recognition of the existence of so-called “reflected modes” (Ferrari, Trussoni, and Zaninetti 1981; Cohn 1984; Payne and Cohn 1985), which are a series of unstable solutions which arise for jet velocities larger than the sum of the internal and the external sound speeds (in the unmagnetized case) and are characteristic of astrophysical jets. In spite of these efforts, the systematic exploration of the possible routes to instability for these jets has been hampered by the seeming complexity of the dispersion relation to be solved. Such difficulties have also arisen in other domains in which Kelvin-Helmholtz-like instabilities figure, such as in magnetospheric physics; and we have taken advantage of recent work in this area (Pu and Kivelson 1983) to systematically explore the shear flow problem, as it pertains to astrophysical jets. The analysis presented here is based on the vortex sheet approximation, considers a compressible fluid, and focuses attention on axisymmetric disturbances; we will present the more general treatment of nonaxisymmetric disturbances in a later paper.

Because our analysis technique is sufficiently straightforward, we have been able to introduce another level of complexity into the jet problem: rotation. In virtually all models of the generation of jets in astrophysical systems, the matter which eventually makes up the jet derives from an accretion flow in which the matter is in orbit about a central object. This implies that the matter which enters the jet carries with it specific angular momentum, and it is of some interest to ask whether this makes any difference to the stability properties of jets. Thus, the additional complication of rotation seems unavoidable, and we have found it quite simple to include a study of its effects on the stability of jets.

Before beginning our analysis, it is important to note the very intimate connection between studies of the sort carried out here and investigations of wave propagation in cylindrical compressible magnetic flux tubes with finite areal cross section. The fundamental reason for this connection is simply that magnetic flux tubes and magnetized beams are really the same kind of physical object, the differences arising principally from the peculiar circumstances in which they occur. That is, magnetic flux tube studies have focused on the propagation and damping of internal body and surface waves which are excited by external flows (typically, the fluid turbulence associated with the outer atmospheres of stars); in contrast, previous beam studies have focused on the destabilization of internal and surface modes by the shear flow resulting from the existence of the beam itself within an external medium which is otherwise assumed to be static. Nevertheless, the results discussed here are also relevant to studies of magnetic flux tubes with internal flows since (unsurprisingly) the mathematical...
results for the two research problems bear a great deal of similarity (there also exist a number of consequential differences, which we shall point out as appropriate.) The reader unfamiliar with the problem of magnetic flux tube waves is referred to Edwin and Roberts (1983) and Cally (1985, 1986), who describe the most recent results relevant to the problem discussed here, and provide abundant references to earlier work in the field.

Our paper is structured as follows: we first present the equations to be solved and derive the dispersion relation for axisymmetric disturbances (§ II). In § III we focus on the solution of the dispersion relation from both qualitative and quantitative points of view. We summarize our results in § IV.

II. THE DISPERSION RELATION

We study the stability of a rotating, magnetized cylindrical flow through an ambient unmagnetized medium. We make use of a cylindrical coordinate system \((r, \theta, z)\). In the equilibrium configuration, the velocity is given by the vortex sheet approximation

\[
\mathbf{v}_0 = \begin{cases} (0, \Omega r, U) & r < a \\ (0, 0, 0) & r > a \end{cases}
\]

and the magnetic field is given by

\[
\mathbf{B}_0 = \begin{cases} (0, 0, B_0) & r < a \\ (0, 0, 0) & r > a \end{cases}
\]

where \(a\) is the radius of the cylinder. The relevant equations are the ideal MHD equations for a polytropic fluid

\[
\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot (\rho \mathbf{V}) = 0, \quad (2.1a)
\]

\[
\rho \left[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\nabla (p + \rho \mathbf{V} \cdot \mathbf{B}) + \frac{1}{8\pi} (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad (2.1b)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \mathbf{V} \times (\mathbf{V} \times \mathbf{B}), \quad (2.1c)
\]

\[
\frac{d}{dt} (pp^{-1}) = 0. \quad (2.1d)
\]

For equilibrium at the interface \(r = a\), we require equality of total pressures; that is,

\[
P_i = p_e,
\]

where \(P_i = p_i + B^2/8\pi\). Inside the cylinder, the total pressure gradient must balance the centrifugal force,

\[
\frac{dP_i}{dr} = \rho \Omega^2 r.
\]

Thus, in principle, the equilibrium quantities are not constant along \(r\). However, in the following we assume that

\[
\Omega^2 a^2 \ll c_{sa}^2, \quad (2.2)
\]

where \(c_{sa}\) is the internal sound speed; in this case, the equilibrium quantities can be considered constant. This approximation should be sensible for models of jets well outside the inner acceleration region, where the jet’s angular velocity has dropped as a consequence of jet expansion and conservation of angular momentum.

To study the stability of such a configuration, we carry out a temporal stability analysis. We note that there has been considerable debate regarding the appropriateness of such an analysis in the case of astrophysical jets, since one can reasonably argue that instabilities—if present—would take place in the spatial domain (for recent discussions on the details regarding temporal and spatial stability analysis see Birkinshaw 1984; Payne and Cohn 1985; Hardee 1984, 1986; see also Gaster 1962, 1968; Betchov and Criminale 1966); however, the ultimate question is really whether jet instabilities are absolute or convective in nature (see, e.g., Bers 1985); and the answer to this question is not as yet known (Bodo, Rosner, and Ferrari, 1988). In any case, the present analysis is not restricted to the astrophysical jet problem, but is rather aimed at understanding the full panoply of instabilities to which an axisymmetric, magnetized, rotating flow is subject. Hence, for the sake of simplicity, we do restrict attention to temporal analysis alone.

Thus, we linearize system (2.1), and consider perturbations of the form

\[
f(r) \exp \left[ i(-\omega t + kz + m\theta) \right]
\]

and obtain the following equation in the radial direction:

\[
\frac{1}{m} \frac{d}{dm} \left( m \frac{dp_i}{dm} \right) + \left( \frac{\tau_1^2 - m^2}{m^2} \right) p_1 = 0, \quad (2.3)
\]

where the indices \(i, e\) refer to the internal and external region, respectively, and the nondimensional radius is \(m = r/a\); the nondimensional radial wave numbers in the internal and external regions are defined, respectively, by

\[
\tau_i^2 \equiv \frac{\alpha^2 (\phi_i^2 - M_\lambda^2) - \phi_i^2}{\phi_i^2 (1 + M_\lambda^2) - M_\lambda^2}, \quad (2.4)
\]

\[
\tau_e^2 \equiv \frac{\alpha^2 (\phi_e^2 - \eta^2)}{\eta^2}, \quad (2.5)
\]

where \(\alpha \equiv k a\) is the nondimensional wave number along the beam axis, \(M_i = U/c_{si}\) is the Mach number, \(M_\lambda \equiv v_\lambda/c_{si}\) is the Alfvénic Mach number, \(R \equiv c_{si}/2\alpha a\), \(\eta \equiv c_{se}/c_{si}\) is the ratio of the external to the internal sound speeds, \(\phi \equiv \omega/kc_{si}\) is the nondimensional phase velocity measured in the reference system of the external fluid, and \(\phi_\pm \equiv \phi - M - m - m/(2\alpha R)\) (related to \(\phi\) by a simple Doppler shift formula) is the corresponding phase velocity in the frame of the internal fluid.

a) Boundary Conditions

Equation (2.3) must be solved in the internal and external regions subject to the appropriate boundary conditions: the solutions must be regular on the axis and must decay exponentially at infinity; this latter demand follows directly from the fact that we are considering modes which exponentially grow in time and does not depend on whether or not the external modes propagate radially (this is a key difference between the present analysis and that of Cally’s [1986] analysis of “leaky” and “nonleaky” flux tube waves). In addition, at infinity we require that there are only outgoing waves (Sommerfeld radiation condition), we do restrict attention to temporal analysis alone.

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\[
p_i \approx J_m(\tau_i m), \quad (2.6)
\]

where \(J_m\) is the Bessel function of the first kind of order \(m\), which is regular at \(r = 0\); the appropriate external solution is

\[
p_i \approx H^{(1)}_m(\tau_e m), \quad (2.7)
\]

where \(H^{(1)}_m\) is the Hankel function; note that \(p_i\) varies as exp \((\tau_e m)\) for large values of \(m\).
(b) Matching Conditions

The interface between the internal and external regions will be displaced by the amount

\[ \zeta = \zeta_0 \exp \left[ i (-\cot + k z + m \theta) \right], \]

and at the displaced interface we have to impose equality of the total pressure; in addition, we must impose the constraint that the displacement of the interface, as described by the equations in the two regions, is the same. In linearized form, these two matching conditions can be written as

\[ p_{1e} = p_{1i} + \frac{B_{0} B_{1i}}{4 \pi}, \quad (2.8) \]

\[ \nu_{1e} = \nu_{1i} + \nu_{0} \frac{v_{1e}}{v_{0}}. \quad (2.9) \]

Using conditions (2.8) and (2.9) and the given forms for the internal and external solutions, we obtain the following dispersion relation:

\[ \frac{\phi_\lambda^2 - M_\lambda^2}{\phi_\lambda^2/\alpha^2 R^2 - (\phi_\lambda^2 - M_\lambda^2)^2} \times \left[ \frac{J_m(\tau_\lambda)}{J_m(\tau_\phi)} - \frac{m}{\alpha R \tau_\phi \phi_\lambda^2 - M_\lambda^2} \right] = - \tau_\phi H_m^{(1)}(\tau_\phi) \phi_\lambda^2 H_\phi^{(1)}(\tau_\phi), \quad (2.10) \]

where

\[ \nu = \frac{p_e}{\rho_i}. \]

In the following, we will limit ourselves to the axisymmetric modes, i.e., modes with \( m = 0 \); the nonaxisymmetric modes will be subject of a following paper. In this axisymmetric case, equation (2.10) simplifies to read

\[ \frac{\phi_\lambda^2 - M_\lambda^2}{\phi_\lambda^2/\alpha^2 R^2 - (\phi_\lambda^2 - M_\lambda^2)^2} \times \frac{J_m(\tau_\phi)}{J_m(\tau_\phi, \phi)} \frac{m}{\alpha R \tau_\phi \phi_\lambda^2 - M_\lambda^2} \frac{J_m(\tau_\phi, \phi)}{J_m(\tau_\phi, \phi)} = - \tau_\phi H_m^{(1)}(\tau_\phi, \phi) \phi_\lambda^2 H_\phi^{(1)}(\tau_\phi, \phi), \quad (2.11) \]

where \( \tau_\phi(\phi, \phi) \) and \( \tau_\phi(\phi) \) are given by equations (2.4) and (2.5), respectively.

A qualitative analysis of equation (2.11) is most conveniently done by focusing on the neutrally stable solutions; this we do by considering the intersections of the curves generated by plotting the left-hand side and right-hand side of equation (2.11). Such an analysis is a useful (and in fact an essential) predecessor to a quantitative analysis of the numerically obtained solutions and is described in detail in the next section. However, before delving into these details, we note a basic feature of the solutions which follows immediately upon simple inspection of dispersion relation (2.11): the character of the solution depends crucially upon whether \( \phi_\lambda \) is complex or real. That is, the phase velocity \( \phi_\lambda \) is real for stable (or marginally stable) solutions; the radial wave number \( \tau_\phi \) is then either purely real or purely imaginary. Furthermore, because the behavior of the Bessel and Hankel functions, and hence the character of the solutions, depends upon whether \( \tau_\phi \) is real or not, we need to distinguish the following possible cases (Fig. 1):

1. If \( \phi_\lambda \) is complex, then (from 2.5) \( \tau_\phi \) is complex as well, and the external solution has the form

\[ p_e \approx \frac{1}{\sqrt{\omega}} \exp \left[ i \Re (\tau_\phi) \omega - \Im (\tau_\phi) \omega \right] \]

as \( \omega \to \infty \) (i.e., the external solution is oscillatory, but damped, at large \( \omega \)); in this case equation (2.11) can be satisfied, with (unstable) body modes in the tube interior.

2. If \( \phi_\lambda \) is real and \( \phi^2 > \eta^2 \), then (from eq. [2.5]) \( \tau_\phi \) is real; we thus have the following two cases.

Case 2a: \( \tau_\phi^2 > 0 \).—Since \( \tau_\phi \) is imaginary, the internal solution decays away from the boundary.

Case 2b: \( \tau_\phi^2 > 0 \).—Since \( \tau_\phi \) is real, the solution is a standing wave in the radial direction in the internal region.

In both cases, the external solution has the functional form

\[ p_e \approx \frac{1}{\sqrt{\omega}} \exp \left[ i \Re (\tau_\phi) \omega \right], \]

as \( \omega \to \infty \) (i.e., the solution is oscillatory and bounded at large radii). However, the dispersion relation (2.11) cannot be satisfied because the right-hand side is complex, while the left-hand side is purely real.

3. If \( \phi_\lambda \) is real and \( \phi^2 < \eta^2 \), then (again from eq. [2.5]) \( \tau_\phi \) is purely imaginary; hence we again have the following two possibilities.

Case 3a: \( \tau_\phi^2 < 0 \).—Since \( \tau_\phi \) is imaginary, the internal solution decays away from the boundary (i.e., is a surface mode).

Case 3b: \( \tau_\phi^2 = 0 \).—Since \( \tau_\phi \) is real, the solution is a standing wave (body mode) in the radial direction in the internal region.

In either case, the external solution has the form

\[ p_e \approx \frac{1}{\sqrt{\omega}} \exp (- | \tau_\phi | \omega) \]

as \( \omega \to \infty \), so that the external solution is exponentially damped at large radii; in these final cases, equation (2.11) can again be satisfied, with the internal surface or body modes coupling to external surface modes.

Finally, we note that in order to have an external solution decaying at infinity and simultaneously satisfying the Sommerfeld condition, we must require \( \Re (\tau_\phi) > 0 \) and \( \Im (\tau_\phi) \geq 0 \).

III. STABILITY ANALYSIS

Dispersion relations of the type (2.11) are known to lead to a rich variety of unstable modes, ranging from the surface (ordinary) modes to the body (reflected) modes. Because of this complexity, it is highly desirable to systematize the classification of the unstable modes. In the nonmagnetic case, it is well known that the surface mode is unstable for all values of \( M > 0 \), but that all other unstable modes in this nonmagnetic case appear by virtue of the destabilization of a stable mode as the Mach number is increased. In the magnetic case, all unstable modes (including the ordinary mode) arise via destabilization of stable modes as \( M \) increases from 0. For this reason, we first focus on a qualitative analysis of the mode structure of the dispersion relation, concentrating especially on the neutrally stable solutions \( \{ \Im (\omega) = 0 \} \) of equation (2.11) and following their routes to instability. This analysis will allow us to systematically capture all unstable modes, with the single exception of the nonmagnetic ordinary mode (which, as already stated, is always unstable for nonvanishing \( M \)).

The quantitative behavior of the unstable solutions, i.e., the variation of growth rate with wave number, Mach number, and (where appropriate) magnetic field strength and rotation rate, is then examined by solving the dispersion relation (2.11) numerically.

In order to identify the marginally stable modes corresponding to the distinct cases discussed immediately above, we seek
solutions to the dispersion relation (2.11) with $\phi$ real. These solutions can be readily found graphically by plotting both the right-hand side and the left-hand side of (2.11) as functions of the phase velocity (or non-dimensional frequency) $\phi$; recall that the left-hand side of (2.11) is a function of $\phi$ because $\phi_+$ is related to $\phi$ by a simple Doppler shift (at least for the simplest case of $m = 0$). The resulting intersections between these curves (Fig. 2) then correspond to the neutrally stable solutions of equations (2.11). In the following, we shall be particularly interested in those intersections which correspond to marginal stability.

As an aside, we note that the just-discussed curves show a qualitatively different behavior in the four relevant regimes (Fig. 2).

**a) Interior**

$\tau^2_+ < 0$.—The Bessel functions of first kind do not have zeros for imaginary values of the argument; we thus obtain a continuous monotonic curve for the interior branch, i.e., the left-hand side of equation (2.11), corresponding to cases 2a and 3a.

$\tau^2_+ > 0$.—The Bessel functions of first kind have a series of zeros for real values of the argument, and hence the curve corresponding to the left-hand side (interior) of equation (2.11) will be divided into many branches, each separated by vertical asymptotes from the others (cases 2b and 3b).

In either case (for $\tau$ real or purely imaginary), the Bessel function takes on real values.

**b) Exterior**

$\tau^2_- < 0$.—The Hankel function takes on real values if the argument is purely imaginary, and is plotted in Figure 2.

$\tau^2_- > 0$.—The Hankel function takes on complex values for real values of the argument; hence the right-hand side of equation (2.11) cannot be plotted in Figure 2, and the dispersion relation cannot have solutions.

Thus, unlike the single plane vortex-sheet case (Miles 1958; Pu and Kivelson 1983), in which a neutrally stable body mode appears above the upper Mach number cutoff, our dispersion relation does not have solutions with $\phi$ real if $\tau^2_+ > 0$; that is, we cannot have neutrally stable solutions which behave like body modes in the external region.

**c) The Simplest Case: the Nonmagnetic ($B = 0$), Nonrotating ($\Omega \equiv 0$) Jet**

We begin our analysis of the neutrally stable modes by starting with the simplest case, and ignore magnetic fields and rotation; for simplicity, we focus on a case with a homogeneous temperature distribution. In this case, the pressure confinement condition of the beam yields $v = 1$, and hence the dispersion relation (2.11) reduces to

$$
\frac{\tau_+}{\phi_+^2} \frac{J_0[\tau_+(\phi)]}{J_0[\tau_-(\phi)]} = \frac{\tau_-}{\phi_-^2} \frac{H_0^{(1)}[\tau_-(\phi)]}{H_0^{(1)}[\tau_+(\phi)]},
$$

(3.1)

where

$$
\tau^2_+ = \delta^2 (\phi^2 - 1), \quad \tau^2_- = \alpha^2 (\phi^2 - 1).
$$

Figure 1 shows the behavior of the square of the radial wave-number $\tau^2_+$ as a function of the proper frequencies $\phi_+$ and $\phi$ for the class of neutral modes; as already pointed out, the sign of $\tau^2_+$ determines the location of the domains of body modes and surface modes. Figure 2 shows the corresponding graphs of the left-hand side (solid line) and right-hand side (dashed line) of equation (3.1), plotted against frequency $\phi$, for $M = 3$. As just discussed, the intersections between the continuous and the dashed curves correspond to neutrally stable solutions; that is, each intersection yields either one or two stable modes. Which case obtains can be determined by carrying out a Taylor's series expansion in $\phi$ of the dispersion relation about any particular point of intersection. When the gradient of the
dispersion relation (3.1) with respect to \( \phi \) does not vanish at such an intersection, then there must exist a linear term in the Taylor's series expansion of the dispersion relation about the corresponding root; hence, the root is isolated. In contrast, when the gradient of equation (3.1) with respect to \( \phi \) vanishes, then the corresponding Taylor's series about this solution has no linear term, and hence the root is double. This latter case corresponds to the location of a marginally unstable root and is identifiable in graphs of the form Figure 2 by those intersections where the inner (left-hand side) and outer (right-hand side) branches of equation (3.1) are tangent.

Note that a change in the Mach number \( M \) only leads to a shift of the continuous curves relative to the dashed curve in Figure 2; hence one can easily follow the changes in the neutral solutions as a function of Mach number, and in particular it is possible to observe the emergence of marginally unstable solutions as \( M \) is varied. That is, there will be a series of values of the Mach number for which one of the continuous curves becomes tangent to the dashed line, and therefore two real roots (i.e., two intersections where the solid and dashed curves are not tangent) merge (the curves become tangent) and two complex conjugate roots appear, one of which corresponds to an unstable solution. We therefore expect that there is an unstable solution for each of the branches shown in Figure 2; this series of unstable modes is the series of reflected modes (Ferrari et al. 1981).

The route to destabilization of these unstable reflected modes can also be understood from a more intuitive, physical point of view, by considering the energetics of the perturbed configuration. If we consider modes with \( \text{Im}(\omega) \ll \text{Re}(\omega) \), the time-averaged energy per unit length of the perturbation inside the cylinder is given by

\[
E = 2\pi \int_0^\infty \langle \mathcal{E} \rangle r \, dr ,
\]

where \( \langle \mathcal{E} \rangle \) is the average over a period of the energy density, given by

\[
\mathcal{E} = \frac{1}{2} \rho_0 (v_r^2 + v_z^2) + \rho_1 U v_z + \frac{1}{2} c_s^2 \rho_1^2 ,
\]

with \( \rho_0 = \rho_i = \rho_e \). Note that we have considered only the second-order terms because the first-order terms average to zero. Inserting the expression for the eigenfunctions into equation (3.3), and performing the integration, we obtain for a stable perturbation

\[
E = \frac{\pi a^2}{2} \rho_0 \xi_n^2 (k_0 \omega)^2 \frac{\phi_+}{J_1^2(\tau_1)} \times \left\{ \frac{\phi_+}{J_1(\tau_1)} \left[ J_0(\tau_1 \omega) \right] ^2 \omega \, d\omega - \frac{\phi_+}{J_1(\tau_1)} \left[ J_1(\tau_1 \omega) \right] ^2 \omega \, d\omega \right\} \exp(2\gamma t) ,
\]

In the case of an unstable perturbation, we obtain instead of equation (3.4) the result

\[
E = \frac{\pi a^2}{2} \rho_0 \xi_n^2 (k_0 \omega)^2 \frac{\phi_+}{J_1^2(\tau_1)} \times \left\{ \frac{2a^2}{\tau_1^2} \text{Re} (\phi_+) \left[ J_0(\tau_1 \omega) \right] ^2 \omega \, d\omega \right\} \exp(2\gamma t) ,
\]

or

\[
E = \frac{\pi a^2}{2} \rho_0 \xi_n^2 (k_0 \omega)^2 \frac{\phi_+}{J_1^2(\tau_1)} \times \left\{ \frac{2a^2}{\tau_1^2} \text{Re} (\phi_+) - \frac{1}{\tau_1^2} \right\} \left[ \left[ J_0(\tau_1 \omega) \right] ^2 \omega \, d\omega \right] \exp(2\gamma t) ,
\]

where \( \gamma \equiv \text{Im}(\omega) \). As illustrated in Figure 3, for each of the modes considered, the energy tends to be negative for Mach
numbers for which that mode is expected to be unstable. What we mean by a negative energy perturbation is that the total energy of the system (zeroth order flow plus perturbation) is lowered by the presence of the perturbation. Such negative energy modes within the interior of the cylinder become unstable when they couple to radially propagating waves in the external region and can therefore extract energy from the cylinder. This reasoning accounts for the fact that these internal modes are stable for low Mach numbers (when they couple to evanescent [surface] modes in the external region that cannot extract energy from the cylinder); and that they become unstable for \( M \) larger than some critical value (which depends on the particular mode in question) when they couple to external propagating waves.

As already mentioned, this critical point of destabilization corresponds to the point in Figure 2 where one of the continuous branches becomes tangent to the dashed curve. At this point, the nondimensional phase velocity of the internal mode \( \phi \) (as seen in the external rest frame) becomes larger than \( \pi = 1 \), which is just the condition for having external radially propagating waves. The mathematical emergence of a marginally unstable solution is thus physically associated with the coupling of an internal body wave to an external propagating wave, and hence to radiation of energy from inside the cylinder to the external medium. This point can be made more quantitative by considering the time-averaged energy flow per unit length leaving the cylinder, given by

\[
F = 2na\langle p_1v_{tr} \rangle .
\]

Inserting the expressions for the eigenfunctions, we obtain

\[
F = \pi a p_0 c_0^2 \left[ \phi \right]^2 \text{Im} \left[ \frac{1}{\tau_0} \frac{W_0^{(r)}}{W_0^{(i)}} \right] \exp(2\gamma t) .
\]

Figures 3 and 4 show, respectively, the averaged total energy and the average energy flux of the perturbations inside the cylinder for the first of the reflected modes for three different wavelengths. We see that the energy associated with the perturbations is negative, as discussed, and that in general there is a good correspondence between the maximum growth rate (Fig. 5) and the minimum in the absolute value of energy (Fig. 3), as would be expected if we recall that the growth rate may be approximated by the ratio of the flux coming out of the cylinder to the energy content of the perturbation inside the cylinder.

To place this discussion in context, Figures 3d–5d show the energy, energy flux, and growth rate for the ordinary mode for a longitudinal wavenumber \( a = 0.5 \) (we do not consider higher wavenumbers because the growth rate is then of the order of the frequency, and therefore the averaging procedure defined above loses its meaning). We can immediately see that the energy associated with this perturbation is positive for low Mach numbers; in fact, the instability mechanism for the ordinary mode is different from that of the reflected modes: the ordinary mode is the only one that survives in the incompressible limit, and its mechanism is that of the incompressible Kelvin–Helmholtz instability. As the Mach number is increased, however, the figures show that the ordinary mode partakes some of the characteristics of the reflected modes.

We can also understand the difference between the plane vortex sheet case (in which we have an upper Mach number cutoff at which a stable body mode appears) and the jet case (in which it is not possible, as we already noted, to have a stable mode which behaves as a body mode in the external region) from the energetic point of view. In the first case (Duhau and Gratton 1975), the stable body mode consists of a negative energy wave propagating away from the vortex sheet in one medium and a positive energy wave in the other medium, thus assuring energy conservation. In the second case, in contrast, the external body wave can transfer energy outward from inside the jet, and therefore no oscillatory solution with time-independent amplitude (neutral solution) can be found.

The physical mechanism of destabilization of the reflected modes can also be understood in terms of overreflection (Berman and Ffowcs-Williams 1971; Payne and Cohn 1985). That is, if the reflection coefficient at the interface between the jet and the external medium \( |\mathfrak{r}| > 1 \), then repeated reflections at the jet boundary can cause the perturbations to grow. In fact, it is well known that overreflection can be related to the possible presence of negative energy waves (Acheson 1976): consider a reference frame in which the internal fluid is at rest. Then from the above, we know that the stable internal waves now correspond to positive energy waves, and the external modes to negative energy waves. If the reflection coefficient at the jet boundary for incident internal (positive energy) waves is larger than one, then the incident wave from the interior will draw energy from the transmitted negative energy wave, thus leading to instability without violating energy conservation. The critical value of the Mach number for instability is the limiting value for which it is possible to have a transmitted, external radially propagating negative energy wave, and therefore a reflection coefficient \( |\mathfrak{r}| > 1 \). In order to quantify these considerations, it is helpful to focus on the simpler case of the planar problem (Miles 1957), for which the reflection coefficient has the simple form

\[
\mathfrak{r} = \frac{\sin(2\delta) - \sin(2\epsilon)}{\sin(2\delta) + \sin(2\epsilon)} .
\]

where \( \delta \) is the angle of incidence, with \( \sin(\delta) = 1/\phi \) and \( \epsilon \) is the angle of transmission, defined as the angle between the wavevector for the transmitted wave and the normal to the vortex sheet plane, given by \( \sin(\epsilon) = 1/(\phi - M) \). Thus, \( |\mathfrak{r}| \) can be larger than one when \( \sin(\epsilon) \) is negative (i.e., when \( \epsilon \) is negative, so that the transmitted wave propagates in the opposite direction to the internal wave); thus, \( |\mathfrak{r}| > 1 \) is satisfied when \( \phi - M < 0 \), which is again roughly the condition for having negative energy waves. In addition, resonances are possible when

\[
\sin(2\delta) = - \sin(2\epsilon) .
\]

These resonances correspond to eigenmodes of the planar vortex sheet configuration and are given by

\[
M = \frac{1}{\sin(\delta)} , \quad M = \frac{1}{\sin(\delta)} + \frac{1}{\cos(\delta)} .
\]
Fig. 3.—Energy density $E[(\frac{1}{2} \pi \alpha^2 \rho_0 U^2 (\zeta_0/\alpha)^2)/2]$ vs. Mach number for several values of the longitudinal wavenumber $\alpha$: (a) the first reflected mode with $\alpha = 10$, (b) the first reflected mode with $\alpha = 5$, (c) the first reflected mode with $\alpha = 0.5$, and (d) the ordinary mode with $\alpha = 0.5$. 
Fig. 4.—Energy flux $F(\gamma)\int \rho d\mu / d$ vs. Mach number for several values of the longitudinal wavenumber $\alpha$: (a) the first reflected mode with $\alpha = 10$, (b) the first reflected mode with $\alpha = 5$, (c) the first reflected mode with $\alpha = 0.5$, and (d) the ordinary mode with $\alpha = 0.5$. 
First reflected mode with $a = 0.5$, and (d) the ordinary mode with $a = 0.5$.

Fig. 5: Growth rates of $|\text{Im}(\omega)/\omega|_\text{LSW}$ vs. Mach number for several values of the longitudinal wavenumber $z$: (a) the first reflected mode with $z = 5$, (b) the first reflected mode with $z = 10$, (c) the ordinary mode with $z = 0.5$, and (d) the ordinary mode with $z = 0.3$.
Fig. 6.- Reflection coefficient $|R_1|$ vs. Mach number for several values of the longitudinal wavenumber $\alpha$: (a) the first reflected mode with $\alpha = 10$, (b) the first reflected mode with $\alpha = 5$, (c) the first reflected mode with $\alpha = 1$, and (d) the ordinary mode with $\alpha = 0.5$. 

Reflection coefficient
Fig. 7.—Qualitative sketch of the behavior of the square of the radial wavenumber vs. the phase velocity \( \phi_+ \) for the case with the magnetic field, but no rotation. If we compare this result with Fig. 1, we note a new region where \( \tau_+^2 \) is positive, and therefore a new domain of body modes exists.

The mode is of the order of the radius of the cylinder, and therefore that large values of the longitudinal wavenumber \( x \) correspond to incidence angles \( \sim \pi/2 \); decreasing \( x \) is equivalent to decreasing the incidence angle (this is indeed how the dependence on \( x \) enters in Fig. 6). The two resonances given by equation (3.9) are well separated for \( \delta \approx \pi/2 \), but upon decreasing \( \delta \), they first approach and then separate again for \( \delta \approx 0 \). Indeed, from the figures we see that for \( x = 10 \) there are two distinct maxima in the reflection coefficient and in the growth rate (see, e.g., Ferrari, Trussoni, and Zaninetti 1981), while for \( x = 5 \) we have only one maximum for both the quantities, and for \( x = 0.5 \) we again see the presence of two distinct resonances in the reflection coefficient curve (the first maximum is not very prominent and does not show up in the growth rate curve). Finally, we note that the reflection coefficient for the ordinary mode, shown in Figure 6d, is only slightly larger than one; this fact points again to a different origin for this mode.

As an aside, it should be clear that since the points of marginal instability for the reflected modes correspond to \( \phi \approx \eta \) (which is just the condition for having radially propagating waves in the external medium), the claim of Payne and Cohn (1985) that it is the point \( \phi = 0 \) instead which is a point of marginal stability for these modes cannot be correct. It is straightforward to demonstrate this point explicitly. Consider a point of marginal instability for the reflected mode—it clearly must be a double root of the dispersion relation, and therefore we must have

\[
D(\phi) = \frac{\phi^2}{\tau_i J_\phi(\tau_i)} J_\phi(\tau_i) - \frac{\phi^2}{\tau_i} H^{(1)}_\phi(\tau_i) \left[ J_\phi(\tau_i) \right] = 0, \quad dD(\phi) \frac{d}{d\phi} = 0 .
\]

If we assume \( \phi = 0 \), then the first condition is modified to read

\[
M^2 J_\phi(\tau_i) \frac{d\phi}{\tau_i J_\phi(\tau_i)} = 0 ;
\]

similarly, with \( \phi = 0 \), the second relation can be rewritten (after eliminating all extraneous factors) in the form

\[
\frac{x^2 M^4}{\tau_i} = 0 .
\]

But \( \tau_i \) is finite, and neither the Mach number \( M \) nor the wavenumber \( x \) vanish at the point of marginal stability for the reflected modes; hence these results are inconsistent, and \( \phi = 0 \) cannot be a point of marginal stability.

\[d\) The Magnetized, Nonrotating Jet: \( B \neq 0, \Omega \equiv 0\]

In this case, the dispersion relation (2.11) reduces to

\[
\tau_i = \frac{1}{\phi^2 - M^2} \left[ J_\phi(\tau_i) \right] H^{(1)}_\phi(\tau_i) .
\]

with

\[
\tau_i^2 = \frac{z^2 \left( (\phi_+^2 - M^2) (\phi_+^2 - 1) \right)}{\eta^2 (\phi^2 - \eta^2)} .
\]

Figure 7 shows the qualitative behavior of \( \tau_i^2 \) as a function of the proper frequency \( \phi_+ \). If we compare this result with Figure 1, which displays the same plot, but for the unmagnetized case, we can see that the presence of the magnetic field significantly changes the behavior of \( \tau_i^2 \) by introducing a new domain in which \( \tau_i^2 > 0 \); that is, there must exist a new set of body modes. These modes have not been discussed in previous stability studies of magnetized jets. The positions of the points \( \phi_+ = a_1, z_1, \) and \( z_2 \) marking the new instability domain are given by

\[
a_1 = \left( \frac{M^2}{1 + M^2} \right)^{1/2} ,
\]

\[
z_1 = \begin{cases} M^2 & M^2 < 1 ; \\ M > 1 , \end{cases}
\]

\[
z_2 = \begin{cases} 1 & M^2 < 1 ; \\ M > 1 . \end{cases}
\]

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Thus, we obtain body modes in the intervals $\phi_+ > z_2$ and $a_1 < \phi_+ < z_1$. With the aid of equation (3.11), one can readily establish that these modes correspond to the familiar solutions for fast ($\phi_+ > z_2$) and slow ($a_1 < \phi_+ < z_2$) magnetosonic waves in the bulk fluid. Similarly, we find surface modes in the intervals $0 < \phi_+ < a_1$ and $z_1 < \phi_+ < z_2$.

Figure 8 displays the variation of the left-hand side (solid line) and the right-hand side (dashed line) of equation (3.10) as functions of the frequency $\phi$ for $M = 0$; because these functions are symmetric about $\phi = 0$, we show only the negative frequency domain. By comparing with Figure 7, one can identify the distinct behavior of the solutions in the regions with $\tau_1^2 > 0$ and $\tau_2^2 < 0$. Thus, Figure 8a shows the branches corresponding to the region of fast body modes for $\phi < -z_2$; Figure 8b is instead an enlargement of the region of slow body modes between $z_1$ and $a_1$, which is not resolved in Figure 8a. As before, the intersections between the continuous and the dashed line correspond to neutrally stable solutions for $M = 0$.

If we now vary $M$, the continuous curves in Figure 8 will be shifted relative to the dashed curve; this allows one to readily follow the changes in the neutral solutions as the Mach number varies, and in particular allows one to follow the destabilization of a stable mode as the two curves become tangent, and a double root—one of which is marginally unstable—emerges. Indeed, we expect that there is an unstable solution corresponding to every one of the branches shown in Figures 8a–8b, so that by increasing the Mach number, we can thereby follow the birth of all unstable modes. Thus, as the Mach number increases, the first mode to be destabilized is the ordinary mode (Ferrari, Trussoni, and Zaninetti 1981; Fiedler and Jones 1984; Payne and Cohn 1985), followed by the series of the previously neglected slow body modes (which we refer to as slow reflected modes), and finally the standard fast mode branches (which we refer to as the fast reflected modes). We note that in going from small to large wavenumbers, the separation between distinct branches corresponding to the fast modes decreases (since the argument of the Bessel functions contain $a$ as a factor and the position of each branch is determined by the zeros of the Bessel functions). Therefore, at a fixed Mach number, the number of unstable fast reflected modes increases with $a$ and, provided that $M > z_2 + a$, it is always possible to find a value of $a$ above which an increasing number of fast reflected modes become unstable.

Finally, we consider the variation of the growth rate of the unstable modes as a function of the Mach number (Fig. 9): the ordinary mode (solid curve), the first two of the slow reflected modes (dashed curves), and the first two of the fast reflected modes (dot-dashed curves). In the case given in Figure 9 (with $M_a = 0.8$), the growth rate of the new slow reflected modes is two orders of magnitude lower than the growth rate of the ordinary mode or of the (standard) fast reflected modes. However, as can be seen from Figures 10 and 11, the ordinary mode is separated into two branches and the role of the slow reflected modes becomes more important as the Alfvenic Mach number $M_a$ is increased; indeed, there exists a range of values of $M_a$ where the growth rate of these modes is of the order of that of the ordinary mode (as well as that of the fast reflected mode). This effect is well illustrated by Figure 11, which shows the growth rate plotted versus the Alfvenic Mach number for a fixed value of $M$; in this case, the growth rate of the slow mode increases with $M_a$ and, while negligible for small values of $M_a$, becomes comparable to that of the ordinary mode for larger values of $M_a$. Note further that the curve with largest growth rate represents the ordinary mode for small values of $M_a$, while for $M_a \approx 1$ there is an exchange of modes, and the curve with largest growth rate is associated with the first of the reflected modes. Finally, note that a further increase of $M_a$ leads to a stabilization of all modes, i.e., the field becomes sufficiently "rigid" to suppress all $m = 0$ unstable modes.

As expected, and as Figure 12 shows, rotation introduces yet another new region where $\tau_1^2 > 0$ at low frequencies; hence, yet another new series of unstable modes (which we refer to as inertial reflected modes) must come into play. From the above discussion, we can estimate the value of $M$ which leads to destabilization of these modes; this ought to be given by the Doppler shift necessary to superpose the branches of the inertial modes with the right branch of the curve corresponding to the right-hand side of the dispersion relation (3.14). These modes will therefore become unstable for $\phi > \eta$ and $\phi_+ \neq 0$ (Fig. 12), or $M > \eta$; that is, for flows which are slightly supersonic with respect to the sound speed in the external medium. The destabilization of the classical (acoustic) reflected modes will occur instead for $\phi > \eta$ and $\phi_+ < -z_2$, or $M > \eta + z_2$ (recall that there are two solutions for $z_2$, one positive and one negative; Fig. 12 only shows the positive solution, but destabilization occurs for the negative solution); from the expression for $z_2$, we see that rotation can cause a shift toward larger values of $M$ of the point at which we begin to find the reflected modes (especially for long wavelengths). Indeed, the effects of rotation are far more important for long wavelengths. This can be seen explicitly from the dispersion relation, in which the terms introduced by rotation become comparable to the other terms when $R/\omega M \approx 1$, or $\Omega / U \approx 1$, i.e., the effects of rotation begin to become noticeable when the difference in velocity introduced over a wavelength is comparable to the longitudinal velocity of the basic flow. We therefore conclude that for the slow rotation rates considered in this paper, the effects of rotation on the acoustic reflected modes are never important (apart from the increase of the Mach number at which these modes are destabilized) because these modes are unstable only for large values of $M$.

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Fig. 8.—(a) Behavior of the left-hand side (solid curve) and of the right-hand side (dashed curve) of the dispersion relation (3.10) as functions of the phase velocity for $M = 0$. Only the negative values of the phase velocity are shown because the functions are symmetric. In panel (b) we show a blow-up of the region between $\phi = -0.8$ and $\phi = -0.6$.

Fig. 9.—Growth rate $[\text{Im}(\omega)/\kappa c_s]$ vs. Mach number for the case $\alpha = 5$, $M_A = 0.8$, $\Omega = 0$. Solid curve represents the ordinary mode, dashed curves represent the first two slow reflected modes, and dot-dashed curves represent the first four fast reflected modes.

Fig. 10.—Growth rate $[\text{Im}(\omega)/\kappa c_s]$ vs. Mach number for the case $\alpha = 5$, $M_A = 2$, $\Omega = 0$. Solid curve represents the ordinary mode, dashed curve represents the first of the slow reflected modes, and dot-dashed curve represents the first of the fast reflected modes.
Figure 11.—Growth rate $[\text{Im} \omega]/k_0c_s$ vs. Alfvénic Mach number $M_A$ for the case $\alpha = 5, M = 3.5, \Omega = 0$. Dot-dashed curve (a) represents the ordinary mode in the region $M_A < 1$; for $M_A \approx 1$ there is an exchange of modes, so that curve (a) represents the first fast reflected mode for $M_A > 1$. In contrast, dot-dashed curve (b) represents the first fast reflected mode for $M_A < 1$, and the ordinary mode for $M_A > 1$ up to the cutoff; solid curve appearing for yet larger values of $M_A$ again represents the ordinary mode (there is thus a domain of $M_A$ for which the ordinary mode is stabilized, such that this mode becomes again unstable for yet larger Alfvén Mach numbers). Dashed curve represent the first slow reflected mode.

Figure 13 displays the growth rate $\text{Im} \phi$ versus Mach number $M$ for (a) the ordinary mode and (b) the most unstable of the inertial reflected modes in the case of $\Omega = 0.1$; for the purpose of comparison, we also display the growth rate for the ordinary mode (dashed curve) in the case of no rotation ($\Omega = 0$). The growth rate of the inertial mode is about one order of magnitude lower than that of the ordinary mode in the case of a rotating jet; furthermore, the ordinary mode in this case has a slightly lower growth rate than the corresponding mode for no rotation. Although the growth rate of the ordinary mode is thus not greatly affected by rotation, its spatial structure is in fact greatly modified, at least for low Mach numbers. If the jet is not rotating, the behavior of the eigenfunctions for the ordinary mode tends to be similar to that of the eigenfunctions for the incompressible case; that is, they decay away from the surface of discontinuity, with a decay length scale equal to the longitudinal wavelength. In contrast, if the jet is rotating, the corresponding eigenfunctions become more and more confined near the surface of discontinuity and furthermore are strongly oscillatory.

Figure 14 shows the variation of growth rate $\text{Im} \phi$ with longitudinal wavenumber $\alpha$ for the ordinary and the first inertial modes (the other inertial modes have lower growth rates and have their peak growth rate in the immediate vicinity of the peak growth rate for the first inertial mode). We again show the growth rate for the ordinary mode for $1/R = 0$ for comparison. We can see that the stabilizing effect of rotation on the ordinary mode becomes important only for $R/\alpha > 1$, as discussed above. It is useful to consider for a moment the physical reason for this stabilizing effect. For the small rotation rates considered here, the dominant effect of rotation is to couple the radial and azimuthal motions associated with possible perturbations of the jet fluid. This coupling (enforced by the Coriolis force) is such that outward radial motions produce an azimuthal velocity opposite in direction to that of the zeroth order rotation, while the opposite occurs for inward radial motions; as a result, both types of motion lead to a force which opposes the displacements themselves and consequently tend to be stabilizing. We note as well that this process is independent of $\alpha$, whereas the classic Kelvin-Helmholtz effect is proportional to $\alpha$; we therefore expect a larger stabilizing effect at lower values of $\alpha$, as is in fact observed.

Finally, we consider the effect of increasing the rotation rate on the growth rates of the ordinary and the inertial modes (Fig. 15). It is evident that stabilization for the ordinary mode is never very important. As for the inertial mode, its growth rate increases rapidly for small values of $1/R$, reaches a maximum, and then begins a slight decline. It must, however, be kept in mind that the results shown for the larger values of $1/R$ (i.e., larger rotation rates) in this figure lie at the limit of the range of validity of our approximations ($R^2 \gg 1$; see eq. [2.2]), so that the results in this limiting regime are only indicative of the expected behavior.

f) Rotating, Magnetized Jet: $B \neq 0, \Omega \neq 0$

In this final case, we have to solve the complete dispersion relation (2.11), with $\tau_2^2$ and $\tau_1^2$ given by equations (2.4) and (2.5), respectively. As before, the mode structure of the dispersion relation depends on the behavior of $\tau_2^2$ as a function of $\phi_+$. Upon examining equation (2.4), we can see that there are two vertical asymptotes and three zeros in the graph of $\tau_2^2$ versus $\phi_+$, whose positions are given by, respectively (see Fig. 16a),

$$a_{3,4} = \left\{ \begin{array}{l} \left[ M_A^2/(1 + M_A^2) \right]^{1/2} \\ M_A \end{array} \right\},$$

$$a_{5,6,7} = \begin{cases} 1 & \text{if } |1 + (1 + 4\alpha^2 R^2 M_A^2)^{1/2}|/2\alpha R > 1 \\ 1 & \text{otherwise} \end{cases}.$$

Note, however, that the relative positions of these five points
Fig. 12.—Qualitative sketch of the behavior of the square of the internal radial wavenumber vs. the phase velocity \( \phi_+ \) for the case with rotation but no magnetic field.

Fig. 13.—Growth rate \[ \text{Im}(\omega)/kc \] vs. Mach number for the case \( \alpha = 2 \times 10^{-2} \), \( M_A = 0 \), \( R = 10 \). Curve (a) represents the ordinary mode, and curve (b) represents the first inertial reflected mode. The ordinary mode for the case without rotation is also presented for comparison (dashed curve).

depend on the values of \( M_A \) and \( 1/\alpha^2 R^2 \). To illustrate this point, we plot in Figures 16a–16b \( \tau_i^2 \) versus \( \phi_+ \) for two specific cases; in either case, we have three regions in which \( \tau_i^2 > 0 \). Thus, the cylinder can support three different kinds of modes. By comparing with the magnetized, nonrotating problem, we locate the slow mode region near the asymptote \( \phi_+^2 = M_A^2/(1 + M_A^2) \), and the fast mode region after the largest of the zeros of \( \tau_i^2 \). Because the jet is also rotating, we find in addition a third region of body modes (intermediate modes), near the asymptote \( \phi_+ = M_A \) (these modes reduce to the inertial mode discussed above in the nonmagnetized, rotating case). As an aside, it should be pointed out that even in the simple magnetized, nonrotating case, there is a third mode supported by the cylinder, namely, the torsional Alfvén mode (see Spruit 1981); we have ignored this mode, however, because it does not introduce any coupling between internal and external region in the linear regime (no radial motions are associated with it in this limit, and therefore it is not destabilized by the flow). In contrast, when rotation is present, radial motions are introduced by the Coriolis force, and therefore this third type of body mode noted above must be taken into account—this mode does couple the internal and external regions, and does lead to destabilization by the flow.

Let us now progressively increase the Mach number, and consider the succession of instabilities which arise for the case illustrated in Figure 16b. The first series of modes to be destabilized will be the slow modes, which are destabilized for

\[
M > \eta + \frac{M_A^2}{(1 + M_A^2)}. \tag{3.19}
\]
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Fig. 14.—Growth rate $\text{Im}(\omega)/k\omega$ vs. the wavenumber for the case $M = 2.5$, $M_A = 0$, $R = 10$. Curve (a) represents the ordinary mode, and curve (b) represents the first inertial reflected mode. The ordinary mode for the case without rotation is also presented for comparison (dashed curve).

Fig. 15.—Growth rate $\text{Im}(\omega)/k\omega$ vs. the nondimensional rotation rate $R^{-1}$ for the case $M = 1.2$, $M_A = 0$. Curve (a) represents the ordinary mode and curve (b) represents the first inertial reflected mode.

Note that the value $M_A^2/(1 + M_A^2)$ always lies in the interval between 0 and 1, and that therefore these modes will always appear for Mach numbers not larger than $\eta + 1$. The growth rate of the slow modes in the nonrotating case was comparable to that of the other modes only for small wavelengths and for values of $M_A > 1$; the correction introduced by rotation will therefore be negligible for the slow modes, and the growth rate in the range of values of the parameters for which these modes are significant behaves as if there were no rotation.

The second series of modes to be destabilized will be the inertial modes, which become unstable for

$$M \gtrsim \eta + M_A.$$  \hfill (3.20)

For large magnetic field strengths, these modes are therefore unstable only for large values of $M$. Figure 17 shows the variation of the growth rate $\text{Im}(\phi)$ versus the Mach number $M$ for the ordinary mode (dashed line) and the most important of the inertial modes (solid line) for three values of the magnetic field ($M_A = 0, 0.15, 0.3$). We can see that the growth rate of the ordinary mode is, as expected, strongly reduced by the presence of the magnetic field. As for the inertial modes, their maximum growth rate is attained at larger Mach numbers as the magnetic field strength is increased. Whereas the maximum occurs just after destabilization of the mode for $M_A = 0$, the growth rate continues to increase after destabilization for $M_A > 0$, reaching its maximum value at larger values of $M$ as $M_A$ increases; in addition, the maximum growth rate continues to decline as $M_A$ increases.

Figure 18 displays the dependence of the growth rate $\text{Im}(\phi)$ on longitudinal wavenumber $\alpha$ for the same values of $M_A$ as in
Fig. 16.—Two illustrations of the qualitative behavior of the square of the internal radial wavenumber vs. the phase velocity for the case with both rotation and magnetic fields: (a) $\alpha R = 1.5$, $M = 0.8$; (b) $\alpha R = 0.1$, $M = 0.8$.

Fig. 17.—Growth rate $[\text{Im}(\omega)/k_c]$ vs. the Mach number for the case $\alpha = 2 \times 10^{-2}$, $R = 10$. Dashed curves represent the ordinary mode and solid curves the first intermediate reflected mode for (a) $M_A = 0$, (b) $M_A = 0.15$, and (c) $M_A = 0.3$.

Fig. 18.—Growth rate $[\text{Im}(\omega)/k_c]$ vs. the longitudinal wavenumber for the case $M = 2.5$, $R = 10$. Solid curves represent the ordinary mode, and dashed curves the first intermediate reflected mode for (a) $M_A = 0$, (b) $M_A = 0.15$, and (c) $M_A = 0.3$. 
Figure 17; we fix the Mach number at $M = 2.5$. It is evident that the growth rate of the ordinary mode is again strongly reduced by the presence of the magnetic field for small $x$ (large wavelengths in the beam direction) but is essentially unaffected by the magnetic field at high longitudinal wavenumbers. In contrast, the dominant intermediate reflected modes shift their point of maximum growth rate toward high $x$ as $M_A$ increased; and, as just observed above, the value of this maximum growth rate decreases as the magnetic field strength is increased. In any case, the growth rate of the dominant reflected mode is always negligible in comparison with the ordinary mode for $x > 0.1$. These various facets of the behavior of these modes are also illustrated by Figure 19, in which we plot the growth rate $\text{Im} \phi$ versus the Alfvénic Mach number $M_A$ for three different values of the longitudinal wavenumber $x$ (again for $M = 2.5$). Again we see that the decrease in growth rate of the ordinary mode occurs mostly at smaller wavenumbers and that the maximum growth rate for the inertial modes is reached at higher $M_A$ for larger wavenumbers; finally, it is evident that the growth of inertial modes is always negligible for $M_A > 0.5$.

In order to understand the behavior just described, it is useful to look at the resonances in the reflection coefficient, as discussed above. The reflection coefficient for the case at hand is given by

$$
\begin{align*}
\Lambda \sin(2\delta) - \sin(2\varepsilon)(1 - M_A^2/\phi_+^2) & \times \left[1 + 1/(aR\phi_\ast)^2(1 - M_A^2/\phi_+^2)\right] \\
\times \left[\Lambda \sin(2\delta) + \sin(2\varepsilon)(1 - M_A^2/\phi_+^2)\right] & \times \left[1 + 1/(aR\phi_\ast)^2(1 - M_A^2/\phi_+^2)\right]^{-1}
\end{align*}
$$

where

$$
\Lambda = \frac{1 + 1/2\gamma M_A^2}{\phi_+^2} \sin^2 \delta.
$$

The condition for resonance is given by

$$
(\phi_+ - M)^2 \sin^2 \delta = x^2,
$$

where

$$
x^2 = \frac{1}{2\chi^2} \frac{1}{\cos^2 \delta} \left[1 \pm (1 - 4\chi^2 \sin^2 \delta \cos^2 \delta)^{1/2}\right],
$$

and

$$
\chi^2 = \frac{\Lambda^2}{(1 - M_A^2/\phi_+^2)(1 + 1/[aR\phi_\ast]^2(1 - M_A^2/\phi_+^2))}.
$$

In the case at hand, we have the ordering $\tan^2 \delta \ll M_A^2 \ll 1$, so that the condition for resonance is given by

$$
M \approx M_A^2 \frac{M_A^2}{\sqrt{2} \sin \delta}.
$$

This equation show that (for fixed $\delta$, and therefore fixed longitudinal wavenumber $x$) the resonance point moves toward higher Mach numbers as the Alfvénic Mach number $M_A$ is increased; in contrast, if we fix $M$, we find that the resonance point shifts toward higher values of $x$ as $M_A$ increased and, finally, if we increase $x$, the resonance point shifts toward higher values of $M_A$. These results match exactly the behavior of the maximum growth rate displayed by the figures and discussed above.

The last series of modes to be destabilized are the fast modes, which become unstable for $M > \eta + \max(x_\eta, x_\zeta, x_\zeta)$; the precise Mach number clearly depends on the Alfvén Mach number $M_A$, the longitudinal wavenumber $x$, and the rotation frequency $R^{-1}$; as seen by considering equation (3.18b), in order for these modes to become unstable, the critical Mach number must increase as either the rotation rate or the magnetic field strength increases, or the longitudinal wavenumber decreases. Since these modes exist only for high $M$, or for large $x$, rotation does not significantly affect their growth rates, and hence they behave (in the unstable region) as if there were no rotation; for this reason, we have refrained from plotting the variation of growth rate versus Mach number, and so forth, for these modes.

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STABILITY OF MAGNETIZED ROTATING JETS

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Summarizing the results of this subsection, we find that at small wavenumbers the fast reflected modes are destabilized only at very large Mach numbers, the slow reflected modes have negligible growth rates, and therefore the only modes that are important are the ordinary and the intermediate reflected modes. Going to larger wavenumbers, the importance of the intermediate reflected modes decreases very rapidly, while the fast reflected modes are destabilized at lower Mach numbers (which in any case cannot be less than $M_{\lambda} + \eta$). For Mach numbers lower than $M_{\lambda} + \eta$, the modes of interest are the ordinary and the slow reflected modes (whose importance increases with $M_{\lambda}$), while, for Mach numbers larger than this value, the modes of interest are the ordinary and the fast reflected modes.

IV. SUMMARY

In this paper, we developed a systematic approach to studying the stability of a nonrelativistic, magnetized rotating jet to Kelvin-Helmholtz modes. We focused on axisymmetric disturbances and simplified the structure of the boundary layer by adopting the vortex sheet approximation; since we were able to derive a dispersion relation, this latter assumption allowed us to develop a clearer understanding of the nature of the various unstable modes, which is a prerequisite for further studies of shear velocity profiles.

In addition to a complete and systematic mode classification, our analysis also allowed us to (a) interpret the reflected modes first found by Ferrari, Trussoni, and Zanninetti (1981) for astrophysical jets in terms of negative energy modes; (b) discuss the counterparts to the "leaky" flux tube waves found by Cally (1985, 1986) in the case of MHD jets; (c) correct previous results by Payne and Cohn (1985) on the route to destabilization of reflected modes; (d) provide an explanation for the multiple maxima in the reflection mode growth rate seen when plotted as a function of Mach number; (e) discover several new unstable modes, including a slow mode which requires a magnetic field for its existence and an inertial mode which requires rotation (in the case of the slow mode, we show that its growth rate can be as important as the ordinary mode). This latter step is essential if we are to compare their discretizations, perforce deal with finite-thickness shear profiles.

The implications of our results for studies of astrophysical jets are straightforward: whereas magnetic fields have a major effect on the nature of the dominant instability of jets, rotation does not (at least as far as the instability growth rates for the axisymmetric case considered here are concerned). Thus, in agreement with previous results on Kelvin-Helmholtz instabilities, the maximum growth rate is found for modes with $ka \approx 1$ even when magnetic fields and rotation are taken into account. However, two attributes of the rotation-influenced instabilities may have interesting consequences even in the axisymmetric case: first, we have shown that the external modes become more and more confined to the shear flow interface region as the rotation rate increases; second, rotation does not allow for modes which dominate at low Mach number $M$ when the other (ordinary) modes are supposedly suppressed by the magnetic field.

Finally, we note that our results are quite relevant to the stability of magnetic flux tubes, encountered in the surface layers of stars such as the Sun. Such flux tubes are thought to have internal flows (see Spruit and Roberts 1983), and such flows in the presence of rotation have direct consequences for the stability properties of such tubes. This is a relatively unexplored area of study since, as mentioned above, the focus of previous work in this area has been on the wave propagation problem, rather than on the wave destabilization problem.

It is fairly obvious where the next steps in our study will take us: first, it is necessary to consider the nonaxisymmetric case, especially as far as the effects of rotation are concerned; having carried out the full linear analysis under the vortex sheet approximation, it is necessary to ascertain which features survive if one considers shear flow profiles instead (see Londrillo 1985). This latter step is essential if we are to compare our results with numerical simulations (which, by virtue of their discretizations, perforce deal with finite-thickness shear profiles).

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