THE INFLUENCE OF A CHROMOSPHERIC MAGNETIC FIELD ON THE SOLAR \textit{p}- AND \textit{f}-MODES

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ABSTRACT

The influence of a chromospheric magnetic field on \textit{p}- and \textit{f}-mode frequencies is evaluated theoretically for a simple model of the solar plasma, consisting of a polytrope in the solar interior, above which is an isothermal atmosphere. The atmosphere is permeated by a horizontal magnetic field. Frequency changes and shifts in phase factors due to the presence of a magnetic atmosphere are calculated analytically in the long-wavelength limit, and numerically for arbitrary wavelength. The results reveal that at low to moderate degree \( l \) an increase in chromospheric magnetic field leads to a frequency increase for the \( n = 1 \) \textit{p}-mode, whereas the overtones \((n = 2, 3, \ldots)\) suffer a frequency decrease. At high \( l \), all the \textit{p}-modes suffer a frequency decrease. The change in behavior for the \( n = 1 \) \textit{p}-mode takes place at about \( l = 500 \), the precise value depending upon the field strength. The \textit{f}-mode suffers a frequency increase for all \( l \). These effects are roughly proportional to the square of the magnetic field strength in the atmosphere and are largest at high degree \( l \).

The magnetoacoustic cutoff frequency for the chromosphere depends strongly on the degree of the mode and the magnetic field strength in the atmosphere. This allows the chromosphere to act as a "window" for \textit{p}- and \textit{f}-modes, in that at fixed \( l \) a mode may evanesce in the atmosphere at one field strength but propagate at another field strength. In this way active regions may act as sinks for oscillations. The effect may be relevant to an explanation of the observed absorption of \textit{p}- and \textit{f}-modes in the vicinity of sunspots.

It is pointed out that oscillation data sets gathered at different times may reflect changes in the chromospheric magnetic field, both as that field evolves on the long time scale of the solar cycle and as it changes more suddenly and locally in association with the birth and death of active regions.

\textit{Subject headings:} Sun: magnetic fields — Sun: oscillations

I. INTRODUCTION

The solar \textit{p}-modes are acoustic waves trapped in the solar interior within a cavity, formed at its lower level by the increasing sound speed in the convection zone and below, and at the upper level by the reflective properties of the photosphere and chromosphere. The modes manifest themselves as spatially and temporally coherent oscillations in the vertical velocity of the solar surface and are generally observed by means of their Doppler shifts. A power spectrum of the oscillations reveals that power is concentrated around the 3.3 mHz (5 minute) wave band and lies on approximately parabolic ridges that designate the radial overtones of the modes (see the reviews by Deubner and Gough 1984; Christensen-Dalsgaard \textit{et al.} 1985; Leibacher \textit{et al.} 1985).

The parabolic ridges exhibited by \textit{p}-modes are largely a reflection of the structure of the solar interior, and as such they provide important information about the run of sound speed within the convection zone and below (Christensen-Dalsgaard \textit{et al.} 1985). In this the solar atmosphere plays only a minor role, essentially providing a correction to the mode frequency. Nonetheless, such corrections are likely to prove to be of increasing importance as our knowledge of solar oscillations improves and as measurements achieve ever greater accuracy. It is important, then, to assess the influence of the solar atmosphere on global oscillations.

Now the Sun's surface layers are permeated by magnetic field. In the low photosphere and below, the magnetic field resides in the form of flux tubes, either compressed together to form sunspots or remaining isolated from one another in intense tubes (\textit{e.g.}, Parker, 1979; Zwaan, 1981, 1987). Field strengths at the photospheric level are in the kilogauss range, typically 3 kG in sunspots and 1.5 kG in isolated tubes. In the chromospheric layers the flux-tube sources of magnetic field have expanded out to fill the atmosphere, creating a magnetic canopy (Giovannelli 1980) of broadly horizontal field. The field strengths within such canopies and the heights at which they form depend upon whether the structures are in \textit{active regions} (associated with sunspots) or \textit{quiet regions} (far from sunspots), and also upon the internal structure of a flux tube (Spruit 1981; Pneuman, Solanki, and Stenflo 1986). Quiet regions are characterized by field strengths of the order of 10 G, active regions by field strengths of 100 G or so (Spruit 1981; Anzer and Galloway 1983).

The chromosphere, then, is dominated by the presence of magnetic field and as such must have a bearing on the nature of \textit{p}- and \textit{f}-modes. Moreover, since the chromospheric magnetic field will vary from location to location and from time to time, as active regions are born and decay, such variations are likely to be reflected in the frequency of oscillations. Long-term variations with the solar cycle are also to be expected. It is of interest, then, to consider the influence of magnetic fields on solar oscillations, and to assess the dependence of frequencies on magnetic field.

In this paper we examine theoretically the effect of a horizontal magnetic field on \textit{p}- and \textit{f}-mode frequencies. We ignore the influences of vertical tubes in the convection zone (Bogdan and Zweibel 1985; Zweibel and Bogdan 1986), of a toroidal field at the base of the convection zone (Roberts and Campbell 1986; Campbell 1987; Campbell and Roberts 1988), and of a...
global field (Dziembowski and Goode 1984; Gough and Taylor 1984), concentrating instead on the effect of the chromospheric field. Indeed, it is in this part of the solar atmosphere, where the plasma $\beta$ may be small (elsewhere it is of order unity or large), that the influence of a magnetic field on oscillation frequencies is likely to be most marked, especially for high-degree modes. Our aim is to display the role of the chromospheric magnetic field in determining frequency shifts.

II. DISPERSION RELATIONS

a) Oscillations in the Absence of a Magnetic Field

To model the solar interior, we consider a plane stratified polytrope of constant index $m$. In such a medium, linear isentropic disturbances of the form

$$u = (u_x(z), 0, u_z(z)) \exp i(\omega t - kz) ,$$

for frequency $\omega$ and horizontal wavenumber $k$, satisfy the equations (Lamb 1932)

$$\frac{d^2u_x}{dz^2} - \frac{gk^2}{\omega^2} u_x = - \omega^2 c_s^2 \frac{d\Delta}{dz} - g(\omega^2 - k^2 c_s^2) \Delta ,$$ (2)

$$\frac{du_x}{dz} = - \frac{gk^2}{\omega^2} u_x - \left( \frac{k^2 c_s^2}{\omega^2} - 1 \right) \Delta .$$ (3)

Here $g$ is the gravitational acceleration, $\gamma$ is the ratio of specific heats, $c_s(z)$ is the sound speed at depth $z$, and

$$\Delta = \text{div} \, u = \left( \frac{\partial u_x}{\partial x}, 0, \frac{\partial u_z}{\partial z} \right) .$$ (4)

Elimination of $u_x$ yields the second-order ordinary differential equation (Lamb 1932)

$$\frac{d^2\Delta}{dz^2} + \left( \frac{c_s^2(z)}{c_s^2} + \frac{\gamma g}{c_s^2} \right) \frac{d\Delta}{dz} + \left\{ \frac{\omega^2 - k^2 c_s^2}{c_s^2} - \frac{gk^2}{\omega^2} \right\} \Delta = 0 ,$$ (5)

where the prime denotes the derivative with respect to depth $z$.

In the special case of a sound speed given by

$$c_s^2(z) = c_s^2 + c_z^2 z , \quad z \geq 0 ,$$ (6)

for constants $c_s^2$ and $c_z^2$, equation (5) has the solution (cf. Lamb 1932)

$$\Delta(z) = e^{-kz + zo}[CM(-a, m + 2, 2kz + 2kz_0)$$

$$+ Du(-a, m + 2, 2kz + 2kz_0)] ,$$ (7)

where $z_0 = c_s^2/c_z^2$, $M$ and $U$ are confluent hypergeometric functions (Abramowitz and Stegun 1965), and $C$ and $D$ are arbitrary constants. The parameter $a$ is given by

$$2a = \frac{m + 1}{\gamma} \frac{\omega^2}{gk} + k q \left[ \frac{\gamma - 1}{\omega^2 c_s^2} - 1 \right] - (m + 2) ,$$ (8)

and

$$m = \frac{\gamma g}{c_s^2} - 1$$ (9)

is the polytropic index. In the case of an adiabatically stratified atmosphere, to be considered here, $c_s^2 = (\gamma - 1)g$, and so

$$2a = m(\frac{\omega^2}{gk}) - (m + 2) , \quad \gamma = 1 + \frac{1}{m} ,$$ (10)

$$z_0 = (1 + m)H_0 ,$$

where $H_0 = c_s^2/(\gamma g)$ is the pressure scale height at $z = 0$.

The simplest case to consider is that for which the atmosphere is closed at $z = 0$, obtained by setting $c_z^2 = 0$ in equation (6). Then the confluent hypergeometric function $U$ is singular at $z = 0$, and we are left with the $M$ function. For this to remain bounded as $z \to \infty$, a consideration of the asymptotic behavior of $M$ reveals that the parameter $a$ must be zero or an integer $n$; that is, for an adiabatically stratified atmosphere (Spiegel and Unno 1962; Christensen-Dalsgaard 1980).

$$\frac{m \omega^2}{gk} - (m + 2) = 2(n - 1) , \quad n = 1, 2, 3, \ldots .$$ (11)

Introducing $\Omega^2 = \omega^2/gk$, we may rewrite equation (11) in the form

$$\Omega^2 = \Omega^2_n \equiv 1 + \frac{2n}{m} , \quad n = 1, 2, 3, \ldots .$$ (12)

These are the $p$-modes of order $n$.

We should note too the $f$-mode. Equation (5) possesses the solution $\Delta = 0$. Returning to equations (2) and (3), we see that with $\Delta = 0$ a solution of the form

$$u_x = e^{-kz}$$ (13)

arises with

$$\omega^2 = gk$$ (14)

This is the $f$-mode. It is independent of the thermal stratification.

The above approach ignores the role of the "chromospheric" atmosphere ($z < 0$) in modifying, through its propagation characteristics, the frequencies of $p$- and $f$-modes. It is necessary, then, to retain $c_z^2 \neq 0$, and to match the behavior of modes in the "convection zone" ($z > 0$) with their behavior in the atmosphere ($z < 0$). Since the solar chromosphere is dominated by magnetic field, it is also necessary to include magnetooacoustic effects, to which we now turn.

b) The Influence of a Magnetic Field

In the presence of a stratified horizontal magnetic field $B(z)x$, it proves convenient to work directly in terms of $u_x$. Adopting the equations of ideal magnetohydrodynamics and supposing an equilibrium gas pressure $p_0(z)$ and density $\rho_0(z)$ stratified according to

$$\rho_0(z) = \frac{p_0(z) + B^2(x)}{2\mu_0} = \rho_0(z)g ,$$ (15)

we may obtain the following second-order ordinary differential equation as a description of linear perturbations (Goedbloed 1971; Adam 1977; Roberts 1985):

$$\frac{d}{dz} \left[ \rho_0(c_s^2 + c_z^2)(\omega^2 - k^2 c_s^2) du_x \right]$$

$$= \left[ \frac{\rho_0 g^2 k^2}{\omega^2 - k^2 c_s^2} - \rho_0(\omega^2 - k^2 c_z^2) - gk^2 \left( \frac{\rho_0 c_s^2}{\omega^2 - k^2 c_z^2} \right) \right] u_x ,$$ (16)
Here $c_s(z)$ and $v_A(z) = B(z)/[\mu_0 \rho_0(z)]^{1/2}$ are the sound and Alfvén speeds within the field, and $c_r(z)$, defined by

$$c_r = \frac{c_s v_A}{(c_s^2 + v_A^2)^{1/2}},$$

(17)
is the magnetohydrodynamic (MHD) subsonic, sub-Alfvénic cusp speed (e.g., Roberts 1981, 1985).

To model the magnetic chromosphere ($z < 0$), we assume it to be isothermal with a temperature equal to that at the top of the field-free medium, i.e., in $z < 0$, we take $c_s(z) = c_o$. Also, we suppose that the Alfvén speed $v_A(z)$ is a constant. This case is amenable to an exact treatment (see Yu 1965; Thomas 1983), though we should note that the atmosphere is subject to instability if the field strength is too large. Thus, the sound speed is taken to be continuous across the magnetic interface at $z = 0$.

There is, however, a density jump across $z = 0$, since pressure balance dictates that the total (gas plus magnetic) pressure (as $z \to 0^-$) equal that in the field-free region below (as $z \to 0^+$):

$$p_0(0^-) + \frac{B_0^2}{2 \mu_0} = p_0(0^+),$$

(18)

where $B_0$ denotes the magnetic field strength at the magnetic interface ($z = 0$). Coupled with the ideal gas law, we then obtain

$$\frac{\rho_0(0_+)}{\rho_0(0_-)} = 1 + \frac{1}{2} \gamma \left(\frac{v_A^2}{c_s^2}\right) = 1 + \frac{\gamma}{2\beta},$$

(19)

where $\beta = c_s^2/v_A^2$ is the ratio of the squares of the sound and Alfvén speeds in the magnetic atmosphere.

With the assumptions of constant sound and Alfvén speeds, equation (16) reduces to

$$\frac{d^2 u_z}{dz^2} + \frac{1}{H_y} \frac{du_z}{dz} + Au_z = 0,$$

(20)

where

$$A = \frac{(\Gamma - 1)k^2 - \omega^2 - k^2c_0^2}{c_0^2 + v_A^2} (\omega^2 - k^2c_s^2),$$

(21)

and

$$\Gamma = \frac{2\beta\gamma}{2\beta + \gamma}, \quad H_y^{-1} = \frac{\rho_0}{\rho_0} = \frac{\Gamma g}{c_0^2}$$

(22)

are the magnetically modified adiabatic exponent and pressure scale height, respectively. (In the absence of a magnetic field, $\Gamma = \gamma$ and $H_y = H_0 = \rho_0/p_0 g$.)

Equation (20) possesses solutions of the form

$$\exp\left\{\pm \frac{2k}{2H_y} \left[ -1 + (1 - 4AH_y^2)^{1/2} \right] \right\}, \quad z < 0.$$  (23)

We suppose that $4AH_y^2 < 1$ and choose the plus sign in equation (23), so that $e^{2k/H_y} u_z$ tends to zero as $z \to -\infty$. This corresponds to disturbances in the atmosphere ($z < 0$) being evanescent. We return to this matter in § IV.

The solution (23) is to be matched to that pertaining in the field-free region ($z > 0$). We require that $u_z$ be continuous across $z = 0$. Also, the total pressure perturbation must be continuous across $z = 0$; this is equivalent to continuity of

$$\rho_0(z)(c_s^2 + v_A^2 \omega^2 - k^2c_s^2) \frac{du_z}{dz} + \left(\frac{gk^2 \rho_0 c_s^2}{\omega^2 - k^2c_s^2}\right) u_z$$

across $z = 0$. Finally, we require that the kinetic energy density of the flow in $z > 0$ be finite at infinity, and this implies that $C = 0$ in equation (7). Thus

$$\Delta = De^{-kz - k\omega^2} U(-a, m + 2, 2kz + 2kz_0), \quad z > 0,$$

(24)

with $u_z$ in $z > 0$ given by equation (2).

Application of the continuity conditions across $z = 0$ to solutions (23) and (24) finally yields the dispersion relation

$$2k \omega^2 c_0^2 \frac{U(-a, m + 2, 2kz_0)}{U(-a, m + 2, 2kz_0)} + g \omega^2 - k^2c_0^2 - gk^2c_s^2$$

$$= \frac{(\omega^2 - k^2c_0^2)(\omega^2 - k^2c_s^2) + \frac{1}{2}\lambda k^2}{gk^2 c_0^2 + (c_0^2 + v_A^2) (\omega^2 - k^2c_s^2)\lambda},$$

(25)

where

$$\lambda = \frac{1}{2H_y} \left[ -1 + (1 - 4AH_y^2)^{1/2} \right],$$

(26)

and $U(-a, m + 2, 2kz_0)$ denotes the derivative (with respect to $z$) of the confluent hypergeometric function $U(-a, m + 2, z)$, evaluated at $z = 2kz_0$.

Dispersion relation (25) provides a complicated description of the modes of oscillation in an adiabatically stratified field-free medium on top of which resides an isothermal atmosphere with horizontal magnetic field (of constant Alfvén speed). As such, it includes magnetoacoustic surface waves which may propagate along the surface $z = 0$, as well as $p$- and $f$-modes. In the next section we examine the behavior of the $p$- and $f$-modes, determining the manner in which the chromospheric magnetic field influences the frequencies of the modes. We consider analytically the dispersion relation in the long-wavelength limit (corresponding to $kz_0 = 0$), and then go on to present numerical solutions of equation (25). The magnetoacoustic surface waves will not be pursued here (see Campbell 1987 for a more extensive discussion) save to note that they are closely confined to the magnetic interface and for $k < 4$ $Mm^{-1}$ have frequencies comparable to those of $p$-modes, although their horizontal wave speeds are significantly less. This places them below the $p$- and $f$-modes in the standard $\omega$-$k$ diagram.

III. MAGNETIC EFFECTS

It is convenient to rewrite dispersion relation (25), introducing $\Omega^2 = \omega^2/(gk)$:

$$2\Omega^2 \frac{U'}{U} + (m + 1) \frac{\Omega^2}{kz_0} - (1 + \Omega^2)$$

$$= \frac{(m + 1)(1 - \Omega^4)(m\Omega^2 - kz_0)}{\Gamma X},$$

(25')

where

$$X = mkz_0 + \lambda kz_0 \left[ 1 + \frac{1}{\beta} \left( m\Omega^2 - \frac{kz_0}{1 + \beta} \right) \right].$$

(27)

To pursue equation (25') further, we eliminate the $U$-functions in preference to $M$-functions. This then permits us to seek solutions for $kz_0$ small. There are two types of solution of equation (25') that are of interest here, the $p$-modes and the $f$-mode. We consider them separately.

a) $p$-Modes

In the limit $kz_0 \to 0$, equation (25') possesses solutions with $\Omega^2 = \Omega_n^2 \equiv 1 + 2n/m$, for $n = 1, 2, 3, \ldots$. These are the $p$-
modes, noted earlier (eq. [12]). To obtain the correction due to a magnetic atmosphere, we set
\[ \Omega^2 = \Omega_{20}^2 + \delta, \quad \delta \ll \Omega_{20}^2, \] (28)
and expand the dispersion relation for small \( \delta \). After some algebra (for details see the Appendix) we obtain the result
\[ \delta = - \frac{(m + 1)^{m+1} \Gamma(1 + m + n)}{m(1 + m) \Gamma(2 + m) \Gamma(n)(\Omega_{20}^4 - 1)} \times \left[ a_8 \Omega_{n0}^2 + a_4 \Omega_n^4 + a_0 - \left( \frac{m}{m + 2} \right) \Omega_n^2 (\Omega_n^4 - 1) \right] (2kH_0)^{m+2}, \] (29)
where \( \Gamma(n) \) is the gamma function. The coefficients \( a_8, a_4, \) and \( a_0 \) are given by
\[ a_8 = \frac{\gamma}{4\beta(\beta + 1)} \left( 1 + \frac{2\beta^2}{\gamma} \right), \quad a_4 = -1 - \frac{\gamma}{\beta} + 2(\Gamma - 1)a_8, \]
\[ a_0 = 1 + \frac{(\beta - 2\beta \Gamma - \Gamma^2)\gamma}{(\beta + 1)^2}. \] (30)

In obtaining equation (29) we have supposed that \( m \) is not an integer.

It is apparent from the form of the coefficients \( a_8, a_4, \) and \( a_0 \) that the expansion leading to equation (29) ceases to be valid in the limit \( \beta \to 0 \), corresponding to an extremely large magnetic field \( \beta \), and \( \Omega_n = 2n/m \). For \( \beta \) not too small, relation (29) provides a useful guide to the behavior of the p-modes in the presence of a magnetic atmosphere. We note that the correction to \( \Omega \) depends upon \( k = L/R_0 \), and therefore the correction to frequency \( \omega \) depends upon \( k(m + 2) \).

In the limit of zero magnetic field, corresponding to \( \beta \to \infty \) and a nonmagnetic atmosphere, we see that the coefficients \( a_8, a_4, \) and \( a_0 \) reduce to
\[ a_8 = \frac{1}{\gamma}, \quad a_4 = 1 - \frac{2}{\gamma}, \quad a_0 = \frac{1}{\gamma} - 1 (\beta \gg 1). \]
Thus
\[ \delta = - \frac{(m + 1)^{m+1} \Gamma(1 + m + n)}{m(1 + m) \Gamma(2 + m) \Gamma(n)(m + 2)} \times \left( \frac{1}{\gamma} \left( \Omega_n^2 - \Omega_{n0}^2 \right) + 2\Omega_{n0}^2 \right) (2kH_0)^{m+2}. \] (31)

This is the correction due to the presence of a field-free isothermal atmosphere.

The zero-field correction given by equation (31) may be compared with the result given in Belvedere, Gough, and Paterno (1983), who discuss the influence of nonlinearities on p-modes in a nonmagnetic atmosphere. With nonlinearities set to zero, the discussion in Belvedere et al. covers the nonmagnetic limit of our treatment. Comparing equation (31) with their equation (3.11), applied to linear oscillations, we see that the two analyses are essentially in agreement, save for the factor \( (m + 2) \) in the denominator of equation (31), which is absent in their result; there is also a typographical error in their definition of \( K \) in equation (3.13), which, in the linear approximation, should read (D. O. Gough 1987, private communication):
\[ K = 1 - \left[ \gamma^{-1}(s_n^2 - s_{n0}^2) + 2s_{n0}^2 \right] H \kappa. \]
\( (H \equiv H_0, s_n \equiv \Omega_n \) in our notation.

Returning to the magnetic case, we consider the reduction of expression (29) in the circumstances of a strong magnetic field, corresponding to \( \beta \) small. When the Alfvén speed dominates over the sound speed \( (v_A \gg c_0) \), the coefficients \( a_8, a_4, \) and \( a_0 \) reduce to
\[ a_8 = \frac{\gamma}{4\beta}, \quad a_4 = -\frac{3\gamma}{2\beta}, \quad a_0 = \frac{\gamma}{4\beta} (\beta \ll 1). \] (32)

Thus
\[ \delta = - \frac{(m + 1)^{m+1} \Gamma(1 + m + n)(\Omega_n^8 - 6\Omega_{n0}^4 + 1)}{m(1 + m) \Gamma(2 + m) \Gamma(n)(\Omega_n^4 - 1)} \times \left( \frac{\gamma}{4\beta} \right) (2kH_0)^{m+2}. \] (33)

We note that \( \delta \) is proportional to the square of the magnetic field strength at the base of the chromosphere.

It is convenient to present our results in terms of cyclic frequency \( \nu = \omega/(2\pi) \) and spherical degree \( l \). To do this, we set \( k = L/R_0 \), where \( R_0 \) is the solar radius \((= 6.963 \times 10^5 \) km) and \( l^2 = ll + 1 \). Then, in the presence of a strong magnetic field, we have
\[ \frac{\nu}{(gh)^{1/2}} = \frac{\Omega_n^8 - 6\Omega_{n0}^4 + 1}{16\pi m(1 + m) \Gamma(2 + m) \Gamma(n)(\Omega_n^4 - 1)} \times \left( \frac{\nu_A^2}{c_0^2} \left( \frac{2LH_0}{R_0} \right)^{m+2} \right). \] (34)

with \( \Omega_n^4 = 2n/m \).

Evidently, then, in the presence of a strong magnetic field p-modes suffer a frequency shift that is proportional to the square of the chromospheric field strength. It is interesting to note that the sign of this frequency shift, for small \( kH_0 \), depends upon the sign of \( (\Omega_n^8 - 6\Omega_{n0}^4 + 1) \). For polytropic index \( m \) of the order 3/2 (corresponding to \( \gamma = 5/3 \) in an adiabatically stratified gas) we see that \( (\Omega_n^8 - 6\Omega_{n0}^4 + 1) \) is negative for \( n = 1 \), positive for \( n \geq 2 \). Thus, in the presence of a strong magnetic field, the \( n = 1 \) p-mode suffers a frequency increase, whereas the overtones \( (n = 2, 3, \ldots) \) suffer a decrease in frequency. There is, then, an anisotropy in the frequency shift for p-modes in the presence of a magnetically dominated chromosphere. This is in contrast with the nonmagnetic case, where all p-modes \( (n = 1, 2, 3, \ldots) \) suffer a frequency decrease in the presence of a field-free atmosphere.

We may also observe from equation (34) that frequency shifts are proportional to \( L^{m+2/2} \); for \( m = 3/2 \), this gives a fourth-power dependence upon \( L \). High degree \( l \) p-modes, then, suffer a frequency shift that is proportional to \( L^4 \) and proportional to the square of the magnetic field strength \( (\beta \ll 1) \).

The division between strong and weak field may broadly be marked by \( \beta = 1 \), when the sound and Alfvén speeds are equal. For conditions typical of the temperature minimum, we may take
\[ \beta = \left( \frac{180}{B_0} \right)^2, \] (35)
where \( B_0 \) is the field at the base of the magnetic atmosphere, measured in gauss.

b) f-Mode

We turn now to a consideration of the f-mode. In the absence of magnetic field the f-mode is a surface mode, in that its amplitude declines from the top of the atmosphere (see eq.
have \( a = m/2 \).

In the absence of an atmosphere (cf. eq. [11]), we would simply have \( a = m/2 \). In the presence of a strong magnetic atmosphere we see from equation (33) that

\[
\alpha = \frac{m}{2} - \frac{\gamma(m + 1)^{m+1}1(1 + m + n)(\Omega^2 - 6\Omega^4 + 1)}{8\Gamma(1 + m)\Gamma(2 + m)\Gamma(\gamma)\Omega^2(\Omega^2 - 1)} \times \frac{\mu_a^2}{c_0^2} (2kH_0)^{m+2}. \tag{40}
\]

The anisotropy pointed out for \( p \)-modes is reflected in the phase factor \( \alpha \): for \( kH_0 \ll 1 \), \( \alpha = m/2 \) is positive for \( n = 1 \), negative for \( n = 2, 3, \ldots \). In the absence of a magnetic field, \( \alpha = m/2 \) is negative for all \( n \).

We discuss the phase factor \( \alpha \), and frequency shifts in general, in more detail in § V.

**IV. THE WINDOW EFFECT**

The approximate solutions presented in § III serve to provide qualitative guidance to the expected behavior of \( p \)- and \( f \)-modes in the presence of a magnetic field. They are not, however, a substitute for numerical solutions of the full dispersion relation. These we present in § V. But first it is necessary to bring out another role of the magnetic atmosphere in modifying oscillation frequencies: the magnetoacoustic cutoff.

Returning to equations (20) and (23), we see that the form of the solution depends upon the sign of \( \mathcal{D}(\omega, k) \equiv (1 - 4AH^2) \). If \( \mathcal{D} \) is positive, the mode is evanescent, with the energy density associated with the wave declining in height. If \( \mathcal{D} \) is negative, however, then the mode will propagate in the magnetic atmosphere and no trapping arises; consequently, the atmosphere acts as a sink for \( p \)- and \( f \)-modes generated within the convection zone (and below).

The magnitude of \( \mathcal{D}^{1/2} \) provides information about the way the magnetic atmosphere is disturbed by the motions. For small \( k \), \( \mathcal{D} = 1 \) and the total (kinetic plus magnetic) energy density declines on a scale close to \( H_B \), and this increases with field strength. Similarly, the stronger the magnetic field, the more the velocity \( u_z \) penetrates into the magnetic atmosphere.

The division \( \mathcal{D} = 0 \) provides the cutoff frequency between propagation and evanescence in the solar atmosphere. It is a quadratic equation in \( \omega^2 \):

\[
(\omega^2 - k^2 c_0^2)(\omega^2 - k^2 v_A^2) + (1 - 1)k^2 q^2
= \frac{(c_0^2 + v_A^2)}{4H_B^2} (\omega^2 - k^2 c_f^2). \tag{41}
\]

For frequencies of the order expected in \( p \)-modes only the larger of the two roots of equation (41) is of interest, and this corresponds to the fast magnetoacoustic wave.

Expanding equation (41) for small \( k \), we obtain

\[
\omega^2 = \omega_{\text{mag}}^2 \equiv \frac{c_0^2 (c_0^2 + v_A^2)}{(c_0^2 + \gamma v_A^2)^2} \omega_A^2, \tag{42}
\]

where \( \omega_A = \gamma g/2c_0 \) is the acoustic cutoff in a field-free isothermal atmosphere (Lamb 1932). Relation (42) provides the magnetoacoustic cutoff frequency \( \omega_{\text{mag}} \) for small \( k \); corrections are of order \( k^2 \). Note that \( \omega_{\text{mag}} \) decreases with increasing field strength (Yu 1965; Thomas 1983).

In Figure 1 we display the behavior of the magnetoacoustic cutoff (cyclic) frequency \( \nu_{\text{mag}}(\equiv \omega_{\text{mag}}/2\pi) \) as a function of field strength for various degrees \( l \), as determined by solution of equation (41). Conditions typical of the temperature minimum are used to fix the sound speed: we take \( c_0 = 6.76 \text{ km s}^{-1} \). It may be noted that \( v_{\text{mag}} \) is not a monotonic function of field.
Fig. 1.—The magnetoacoustic cyclic frequency $v_{\text{mag}}$ as a function of base field strength $B_0$ (in gauss) for various degrees $l$. At $B_0 = 0$, $v_{\text{mag}} \approx \omega_0/2\pi \approx 5.38 \text{ mHz}$ (at the temperature minimum). Frequencies above $v_{\text{mag}}$ result in propagation in the chromosphere, while frequencies below $v_{\text{mag}}$ are evanescent.

strength, but reaches a minimum value before increasing indefinitely at very high $B_0$. It is clear that a mode of fixed degree $l$ may be evanescent in the chromosphere at one field strength but propagate at another field strength.

It is apparent, then, that the magnetoacoustic nature of the chromosphere may lead to a "window" effect, whereby a mode of fixed $l$ may change its form if the magnetic field evolves in time. For example, we see for $l = 100$ that a magnetic field that evolves from (say) $10$ to $400$ G may result in an evanescent mode becoming propagating, with wave energy that was trapped within the interior and atmosphere at the lower field strength now leaking into the atmosphere at the higher field strength (since waves are no longer trapped in the atmosphere). Such an evolution may occur as quiet regions evolve into active regions or as we approach closer to the center of an active region from the surrounding quiet region. Active regions, then, may appear to act as "sinks" for $p$-modes.

In this context we should note the recent results of Braun, Duvall, and LaBonte (1987, 1988), which demonstrate that sunspots absorb up to half the power in $p$-modes. Our model, of course, is for a horizontal magnetic field and so does not apply to the sunspot itself, though it is applicable to the surroundings of a spot where the magnetic field has fanned out and become largely horizontal. We note that the absorption of $p$-modes is not limited to the umbra of a spot but extends to the penumbra and beyond (Braun, Duvall, and LaBonte 1988).

Where the field has become horizontal, the model demonstrates the ability of active regions to act as sinks for $p$-modes simply because of the changing nature of the magnetoacoustic cutoff frequency. As we move horizontally from the quiet region surrounding a spot toward the center of the activity, the magnetic field strength increases and the magnetoacoustic cutoff frequency decreases substantially for all except extremely high degree modes (Fig. 1). This effect is substantial for all save the extremely high degree modes in any region which embraces a change in field strength from very low to high (say $\sim 500$ G) values.

V. NUMERICAL SOLUTION OF THE DISPERSION RELATION

We turn now to a numerical solution of the general dispersion relation (25), solved subject to the constraint $\mathcal{D} > 0$. Figure 2 shows the behavior of the $f$-mode and the first 10 $p$-modes. We have taken the sound speed in the isothermal atmosphere to correspond to that at the temperature minimum, yielding $c_0 = 6.76 \text{ km s}^{-1}$ for adiabatic index $\gamma = 5/3$. The scale height $H_0$ is then $100$ km, and the polytropic index $m$ is $3/2$. A chromospheric field strength of $100$ G is assumed. Note the magnetoacoustic cutoff (dotted line); above this line, modes propagate in the chromosphere and therefore leak energy into that region from below.

It is of interest to examine how the $p$- and $f$-modes are influenced by an evolving magnetic field. We are thinking of

Fig. 2.—Cyclic frequency versus wavenumber for $p$- and $f$-modes in the presence of a magnetic chromosphere with field strength $100$ G at its base. Note the magnetoacoustic frequency cutoff (shown dotted) near $4 \text{ mHz}$. Note that $k = 1.0 \text{ Mm}^{-1}$ corresponds to $l = 700$. 

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what happens when, for example, a quiet region gives way to an active region or when an active region decays. Also, since the solar magnetic field undergoes an activity cycle, data collected at different stages of that cycle may reflect, in part, that cycle, though more local occurrences (birth and death of active regions) may predominate at various times. In Figures 3–5 we present the behavior of the frequency \( v \) of the \( n = 1, 2, \) and 3 \( p \)-modes as a function of field strength \( B_0 \) for degrees \( l = 100 \) and 500. Note the window effect. For magnetic field strengths within the window the mode is no longer trapped by the magnetic chromosphere. At fixed \( l \) the width of the window increases with the order \( n \) of the mode.

It is apparent from the above figures that frequency shifts due to a change in magnetic field strength depend strongly upon both the order and the degree of the mode. For example, the \( n = 1 p \)-mode with \( l = 100 \) undergoes a frequency shift of only 0.7 \( \mu \text{Hz} \) in a magnetic field evolving from zero to 3 kG. By contrast, the \( n = 3 p \)-mode with degree \( l = 500 \) changes by 160 \( \mu \text{Hz} \). These two values illustrate extremes, and generally frequency changes are intermediate between them.

It is of interest to consider the frequency difference \( \Delta v \), defined by

\[
\Delta v = v(B_0) - v(0) ,
\]

where \( v(B_0) \) is the frequency of a mode associated with a base chromospheric field \( B_0 \), and \( v(0) \) is the corresponding frequency of the field-free case. The behavior of \( \Delta v \) as a function of \( k \) is illustrated in Figure 6 for the \( n = 1 p \)-mode, and in Figure 7 for the \( n = 2 \) mode. As found in § III, for small wavenumber \( k \) an increase in chromospheric field strength leads to an increase in the frequency of the \( n = 1 \) mode, but a decrease in that of the \( n = 2 \) mode. But at large values of \( k \), corresponding to modes of degree \( l \geq 500 \) (depending upon the field strength), the frequency shift in the \( n = 1 p \)-mode is opposite in sign to the trend at small \( k \). Notice that the magnitude of the frequency shift (for \( n = 1 \) and 2) is strongly dependent upon the field strength \( B_0 \).

Turning now to the phase factor \( \alpha \) discussed in § IIIc, we consider the \( n = 1 p \)-mode and display in Figure 8a the variation of \( (\alpha - m/2) \) with horizontal wavenumber \( k \) for various field strengths \( B_0 \), and in Figure 8b, the variation of \( (\alpha - m/2) \) with field strength for various values of \( l \). A weak field leads to a negative shift, while a strong field gives a positive shift.

Finally, we consider the \( f \)-mode. Figure 9 displays the behavior of the frequency \( v \) of the \( f \)-mode as a function of magnetic field strength for \( l = 100 \) and \( l = 500 \). Notice that the frequency \( v \) of the \( f \)-mode in a magnetic atmosphere lies above its field-free value, just as indicated by the expansion (36). In Figure 10 we show the behavior of the frequency difference \( \Delta v = v(B_0) - v(0) \) as a function of \( k \). Notice the strong dependence upon the field strength \( B_0 \).

VI. DISCUSSION

The analysis presented above demonstrates the role of a magnetic atmosphere in modifying the frequencies of \( p \) - and \( f \)-modes. These effects are generally not large, though within currently attainable measurement accuracies, but they are distinctive. The \( n = 1 p \)-mode, for example, in the presence of a strong magnetic atmosphere suffers a frequency shift at low \( l \) that is opposite in sign to that experienced at high \( l \) and that experienced by its overtones \((n = 2, 3, \ldots)\). The \( f \)-mode suffers a frequency increase above its value in the absence of a magnetic field; this is particularly distinctive because in the absence of a magnetic field the \( f \)-mode propagates with a frequency that is independent of the thermal stratification. Thus, any departure from that frequency is a reflection of magnetic effects.

It may be useful at this stage to offer some physical explanation (albeit partial) of the various results for frequency shifts. There would seem to be a number of factors at play. It is usual to think of \( p \)-modes in terms of cavity physics. The lower extent of the cavity is fixed by the sound speed in the nonmagnetic region and is therefore independent of the magnetic field strength in the atmosphere. Frequency shifts, then, are purely a response of the modes to conditions in the atmosphere. In the atmosphere there are competing effects. On the one hand, the characteristic speed of propagation is the fast magnetoacoustic speed, \((c_s^2 + v_A^2)^{1/2}\), and this increases with field strength. Thus the transit time in the magnetic field is reduced, which in turn tends to increase the frequency. On the other hand, the presence of the magnetic field adds support to the atmospheric gas (the density scale height \( H_p \) increases with increasing field strength \( B_0 \)) and so adds inertia to the system, and this tends to reduce the frequency. The effect is larger for those modes that penetrate deepest into the magnetic atmosphere. Both the \( f \)- and the \( p \)-modes penetrate an appreciable distance into the magnetic atmosphere, and this increases with field strength. The \( f \)-mode, however, penetrates less into the atmosphere than the \( p \)-modes and accordingly gains more from the enhanced elasticity of the magnetic field; accordingly, its frequency is increased. The \( p \)-modes, with the exception of the \( n = 1 \) mode, are reduced in frequency. However, why the \( n = 1 \) mode at small \( k \) is close to a balance between the two competing effects is not clear to us.

In addition to displaying the frequency wavenumber diagrams for various field strengths, we have examined frequency shifts, thus making clear the changes induced by the magnetic atmosphere. In so doing we are thinking of the fact that the solar magnetic field is not constant, but evolves on a long time scale over the solar activity cycle and, additionally, may change more suddenly and more dramatically as local active regions are born and decay. Such active regions influence the \( p \)- and \( f \)-modes, and local magnetic activity may temporarily outweigh the longer term trend associated with the solar cycle. It follows, then, that data sets gathered at different times may display systematic differences, reflecting changes in the solar atmosphere that have taken place between the two data collection times. Modes of high degree \( l \) are the most strongly influenced, as § III makes clear. But local magnetic activity may well complicate any comparison between data sets and at times dominate the general trend associated with the solar cycle.

As noted in § V, in performing the computations of the \( p \) - and \( f \)-modes we have taken the isothermal atmosphere to be at a temperature consistent with the temperature minimum, giving a sound speed of \( c_0 = 6.76 \text{ km s}^{-1} \). The magnetic field at this level is then taken to be 10 G to represent a quiet region and 100 G to represent an active region. Of course, some variations in these values is to be expected in reality. But we should note that the assumption of constant Alfvén speed, made for simplicity in modeling the magnetic atmosphere, tends to underestimate the influence of the magnetic field on the modes. This is because with \( v_A \) assumed constant in an isothermal atmosphere the magnetic field strength \( B(z) \) declines as \( \rho(z)^{1/2} \), giving a scale of \( 2H_0 \); with \( H_0 = 100 \text{ km} \), this yields a magnetic scale of only 200 km. In reality, the magnetic field declines far slower than this, and so its influence is likely to be greater than the constant Alfvén speed model predicts. Accordingly, while
Fig. 3a

Fig. 3a: Variation of the frequency of the $n = 1$ $p$-mode as a function of magnetic field $B_0$, for (a) $l = 100$ and (b) $l = 500$. Note the window effect in (b).
FIG. 6. Variation of the frequency difference $\Delta v = v(B) - v(0)$ for the $n = 1$ p-mode as a function of wavenumber $k$, for (a) a weak field of $B_0 = 10$ G and (b) a strong field of $B_0 = 100$ G.
Fig. 7a — Variation of $\Delta v$ with $k$ for the $n = 2$ p-mode for (a) $B_o = 10$ G and (b) $B_o = 100$ G. Note the mode cutoff at $k = 0.03$ Mm$^{-1}$ in (a) and at $k = 0.05$ Mm$^{-1}$ in (b).
Fig. 8a. Shift in the phase factor $\alpha$ of the $n = 1$ p-mode as a function of (a) horizontal wavenumber and (b) magnetic field strength. For illustration we have taken $m = 3/2$. 

Fig. 8b.
Fig. 9.—Frequency $\nu$ of the $f$-mode as a function of magnetic field strength for (a) $l = 100$ and (b) $l = 500$. Notice that an increase in field strength results in an increase in the frequency of the $f$-mode.
Fig. 10a—Frequency increase $\Delta v = v(B_0) - v(0)$ of the $f$-mode as a function of wavenumber $k$ from 0 to 1 $\text{Mm}^{-1}$ (corresponding to $l = 0$ to $l = 100$) for (a) a magnetic field $B_0 = 10\text{ G}$ and (b) $B_0 = 100\text{ G}$.
$B_0 = 10 \, \text{G}$ and $B_0 = 100 \, \text{G}$ may be representative of the mean fields in quiet and active regions, we expect that somewhat larger values are in fact more appropriate as a guide to actual frequency shifts.

In addition to the direct effect of a magnetic field on $p$- and $f$-mode frequencies, we should note that magnetic activity implies mechanical heating and consequently a hotter atmosphere. This is presumably manifest in a larger scale height $H_0$; and so at fixed degree $l$ we may expect frequency shifts from this effect alone. Thus, actual frequency shifts may be larger than the estimates provided by the calculations performed here. We have not, however, attempted to allow for this effect but have instead kept $H_0$ constant throughout.

Concentrating on frequency shifts, the results presented in § V may be summarized as showing that such shifts are negligible in a quiet region but become significant in an active region when modes with high $l$ are considered. For the Sun considered as a whole, the quiet region results are probably the most appropriate. But for modes measured with high spatial resolution, considering only a section of the solar disk, it is clearly important to know whether that region is an active one or not. Indeed, this raises the question of observationally comparing modes from active regions with those from quiet regions. In this respect the $f$-mode may be of particular interest, since its frequency shift is a direct measure of the presence of a magnetic atmosphere; in the absence of a magnetic field, the $f$-mode's frequency $\omega$ (in a planar atmosphere) is $(gk)^{1/2}$, independent of the thermal stratification (be it isothermal, superadiabatic, or otherwise). In the presence of a magnetic field, the $f$-mode suffers a frequency increase which is largest at large $l$ and broadly proportional to the square of the magnetic field strength.

The linking of the $f$-mode to magnetic effects leads us to speculate on a possible means for its generation. The question of the generation of $p$- and $f$-modes is problematic (see Libbrecht 1988 for a review), with the $f$-mode being particularly puzzling in that the mode is almost compressionless and so cannot be directly excited by the $\kappa$-mechanism (which depends upon the overstability of sound waves). Also, transfer of energy by mode coupling drains energy from the $f$-mode (see discussion in Libbrecht 1988). The connection of the $f$-mode to the magnetic canopy overlying the photosphere suggests that the $f$-mode could arise simply in response to buffeting of that canopy by granules and exploding granules and the sound waves they generate.

Now the Sun's magnetic field is not especially strong compared with that expected in the atmospheres of some stars. So it is interesting to speculate that the effects we have described—broadly proportional to the square of the field strength—will be more apparent in those stars with stronger fields and more pronounced activity cycles. However, since the effects are only significant at high $l$, it is unlikely that they will be detectable.

It is appropriate at this stage to note the recent analyses of oscillation data for intermediate- and high-degree modes. Considering $p$-mode data sets for late 1981 and mid-1984, Rhodes et al. (1988) conclude that the two data sets are in agreement to within a level of $0.02 \, \text{kHz}$ for modes of degree $l$ in the range $6 \leq l \leq 89$. By contrast, Duvall et al. (1988), considering $p$-mode data sets with $l$ in the range $4 \leq l \leq 99$ for the years 1981 and 1985, report systematic frequency differences ranging from less than $0.1 \, \text{kHz}$ at low $l$ to of order $0.6 \, \text{kHz}$ at $l = 99$.

As our analysis makes clear, changes in the chromospheric magnetic field strength can be expected to cause changes in the $p$-mode frequencies; systematic trends with degree $l$ are to be expected, with high $l$ showing the largest changes. For low and intermediate $l (< 100)$, however, these changes are negligible, a result in agreement with the conclusion reached by Rhodes et al. (1988) but contrary to that arrived at by Duvall et al. (1988).

We note too the data presented by Libbrecht and Kaufman (1988), who considered high-degree modes ($30 \leq l \leq 1320$). Of particular interest here is the fact that Libbrecht and Kaufman recorded the $f$-mode, and they remark that its measured frequency $v$ was in excess of the $v$ value by as much as $13.4 \, \text{kHz}$ (systematic errors perhaps accounting for $10 \, \text{kHz}$). As we have demonstrated here, the presence of a chromospheric magnetic field leads to an increase in the frequency of the $f$-mode, and an increase of the reported magnitude is entirely consistent with our results for large $l$ ($\geq 1000$, say).

Finally, we remark on the window effect (§ IV) by which a mode that is trapped (reflected) by the chromosphere at one field strength may propagate within the atmosphere at a higher field strength. In this respect, an active region may act as a sink for $p$- and $f$-modes. And observationally $p$- and $f$-modes are found to be absorbed in and around sunspots (Braun, Duvall, and LaBonte 1987, 1988). The relevance of the window effect to this phenomenon is not yet clear, but the fact that absorption is observed to occur in the vicinity of spots (where the magnetic field is predominantly horizontal) encourages further study.

In conclusion, we have indicated how magnetic fields in the solar atmosphere may influence oscillation frequencies in a systematic way. Only a greater number of data will show how significant such effects might be. We await developments with interest.

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**APPENDIX**

**THE DISPERSION RELATION IN THE LIMIT OF SMALL WAVENUMBER**

The dispersion relation is

\[
2\Omega^2 \frac{U'}{U} + (m + 1) \frac{\Omega^2}{kz_0} - (1 + \Omega^2) = \frac{(m + 1)(1 - \Omega^2)(m\Omega^2 - kz_0)}{\Gamma\chi},
\]

where $X$ is defined in equation (27). We are interested in the form of the solution of equation (A1) in the limit of small $kz_0$. © American Astronomical Society • Provided by the NASA Astrophysics Data System
First note that (Abramowitz and Stegun 1965)
\[ U'(-a, m + 2, 2kz_0) = aU(1 - a, m + 3, 2kz_0) \] (A2)
and that
\[ U(A, B, x) = \frac{\pi}{\sin \pi B} \left[ \frac{M(A, B, x)}{\Gamma(1 + A - B) \Gamma(B)} - x^{1-B} \frac{M(1 + A - B, 2 - B, x)}{\Gamma(A) \Gamma(2 - B)} \right]. \] (A3)

Thus
\[
2 \frac{U'}{U} = - \frac{(1 + m)}{kz_0} \left[ M(-1 - a - m, -1 - m, 2kz_0) - (2kz_0)^{m+1} \frac{M(1 - a, m + 3, 2kz_0) \Gamma(1 - a) \Gamma(-1 - m)}{\Gamma(-1 - a - m) \Gamma(m + 3)} \right] 
\times \left[ M(-1 - a - m, -m, 2kz_0) - (2kz_0)^{m+1} \frac{M(-a, m + 2, 2kz_0) \Gamma(-a) \Gamma(-m)}{\Gamma(-1 - a - m) \Gamma(m + 2)} \right]^{-1}. \] (A4)

At this point it is convenient to treat the \( p \)- and \( f \)-modes separately.

\textit{a) \( p \)-Modes}

Write the dispersion relation (A1) in the form
\[
2 \frac{U'}{U} = \frac{(m + 1)}{kz_0} + \frac{\Omega^2 + 1}{\Omega^2} + \frac{(m + 1)(1 - \Omega^4)(m\Omega^2 - kz_0)}{\Omega^2 X}, \] (A5)
and expand the right-hand side in a Taylor series in \( kz_0 \) to yield
\[
2 \frac{U'}{U} = 1 - \frac{(a_8 \Omega^8 + a_4 \Omega^4 + a_0)}{\Omega^2(\Omega^4 - 1)} + O(kz_0), \] (A6)
where
\[ a_8 = \frac{\gamma}{4\beta(\beta + 1)} \left( 1 + \frac{2\beta}{\gamma} \right)^2, \quad a_4 = -1 - \frac{\gamma}{\beta} + 2(\Gamma - 1)a_8, \quad a_0 = 1 + \frac{(\beta - 2\beta \Gamma - \Gamma^2)}{(\beta + 1)\Gamma^2}. \] (A7)

Observe that this expansion of the right-hand side of equation (A5) breaks down if \( \Omega^2 \) is close to zero or unity, or if \( \beta \) is too small. The case \( \Omega^2 = 1 \) corresponds to the \( f \)-mode and is considered below.

We see from equation (A6) that \( 2U'/U \) is finite as \( kz_0 \to 0 \). For this to be compatible with equation (A4), it is necessary that \( \Gamma(-a) \) tend to infinity in such a way that \( (kz_0)^m \Gamma(-a) \) remains finite. In particular, then, we require that
\[ a \to n - 1, \quad n = 1, 2, 3, \ldots. \] (A8)

From the definition of \( a \) under adiabatic conditions (see eq. [10]), we see that
\[ \Omega^2 \to \Omega_n^2 = 1 + \frac{2n}{m}, \] (A9)
as obtained earlier (eq. [12]).

To obtain the first-order correction to this result, set
\[ \Omega^2 = \Omega_n^2 + \delta_0(2kz_0)^s, \] (A10)
with \( s \) and \( \delta_0 \) to be determined. Then
\[ a = n - 1 + \frac{m\delta_0}{2} (2kz_0)^s. \] (A11)

Noting that
\[ \Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin \pi x}, \] (A12)
we see that
\[ \Gamma(-a) \sim \frac{(-1)^s(2kz_0)^{-s}}{\Gamma(n(m\delta_0/2))} \text{ as } kz_0 \to 0. \] (A13)

Substituting this result in equation (A4) combined with equation (A6) leads to the choice \( s = m + 2 \), provided that \( m \) is not an integer. Finally, noting that
\[ \frac{(-1)^s \Gamma(-m - n)}{\Gamma(-m)} = \frac{\Gamma(1 + m)}{\Gamma(1 + m + n)}, \] (A14)
we may then determine $\delta_0$ as

$$\delta_0 = -\frac{\Gamma(1 + m + n)}{m(n + 1)\Gamma(1 + m)\Gamma(2 + m)\Gamma(n)} \left[ \frac{a_6 \Omega^4 + a_4 \Omega^2 + a_0}{\Omega^4 - 1} - \frac{m \Omega^2}{m + 2} \right],$$  

(A15)

and result (29) of the main text follows.

b) $f$-Mode

The above analysis breaks down for the $f$-mode, which has $\Omega^2$ close to unity. Returning to equation (A4), we expand the $M$-functions in their power series in $(kz_0)$. Thus,

$$\frac{2U'}{U} = -\frac{m + 1}{kz_0} \left[ 1 - \frac{2(1 + a + m)kz_0}{m(m + 1)} - \frac{4(1 + a + m)(1 + a)(kz_0)^2}{m^2(m^2 - 1)} + b_0(kz_0)^{m+1} \right],$$  

(A16)

where

$$b_0 = \frac{2^{m+1}(\gamma - a)(\gamma - m)}{\gamma(m + 2)\Gamma(-1 - a - m)}. $$  

(A17)

(This expansion is not valid for the $p$-modes, since $a$ is an integer and then $b_0$ is large.) In writing equation (A16), we have neglected terms of order $(kz_0)^3$ and $(kz_0)^{m+2}\Gamma(-1 - a)$. This requires that $1 < m < 2$; otherwise, additional terms should be added to equation (A16).

Relation (A1) may now be written in the form

$$\frac{2}{m} (1 + a + m) = \frac{m + 1}{kz_0} \frac{2\Omega^2}{\Omega^4 - 1} + \frac{4(1 + a + m)(1 + a)\Omega^2}{m^2(m - 1)} (kz_0) - b_0(m + 1)\Omega^2(kz_0)^m \right] X = \frac{(m + 1)}{\gamma} (1 - \Omega^4)(m \Omega^2 - kz_0).$$  

(A18)

From the definition of $a$ under adiabatic conditions (see eq. [10]), we have

$$1 + a + m = \frac{m}{2} (\Omega^2 + 1).$$  

(A19)

Thus equation (A18) becomes

$$\left[ (\Omega^4 - 1) + \frac{2(1 + a)\Omega^4 + \Omega^2}{\gamma(m - 1)} (kz_0) - b_0(m + 1)\Omega^2(kz_0)^m \right] X = \frac{(m + 1)}{\gamma} (1 - \Omega^4)(m \Omega^2 - kz_0).$$  

(A20)

Expanding $X$ in a Taylor series in $kz_0$ gives

$$X = -\frac{m(\Omega^4 - 1)kz_0}{\gamma} \frac{(kz_0)^2}{\Omega^2} (d_8 \Omega^8 + d_4 \Omega^4 + d_0) + \cdots,$$  

(A21)

where

$$d_8 = \frac{\beta}{(\beta + 1)\Gamma^3}, \quad d_4 = \frac{2\beta(\Gamma - 1)}{(\beta + 1)\Gamma^3} - \frac{(\beta + 1)}{\beta\Gamma}, \quad d_0 = \frac{(\beta + 1)}{\beta \Gamma} (\Gamma - 1)^2 \frac{\Gamma^3}{\Gamma^3}.$$  

(A22)

Thus $X \to 0$ as $kz_0 \to 0$, and so from equation (A18) $\Omega^4 \to 1$ as $kz_0 \to 0$. Set

$$\Omega^2 = 1 + f_0(kz_0)^s,$$  

(A23)

with $s$ and $f_0$ to be determined. Then, from the definition of $a$, we have

$$1 + a = \frac{s}{m} f_0(kz_0)^s.$$  

(A24)

Thus, the terms in square brackets in equation (A20) are of order $(kz_0)^s, (kz_0)^s+1, \text{ and } (kz_0)^m$, respectively. These terms multiplied by $X$ are to be balanced by the right-hand side of equation (A20), which is of order $(kz_0)^m$. With the expansion (A21) for $X$ we see that $X$ is of order $(kz_0)^2$. Hence balance is achieved by taking $s = m + 2$, and then

$$b_0(m + 1)(d_8 + d_4 + d_0) = -\frac{(m + 1)}{\gamma} 2 f_0 m.$$  

(A25)

Thus $\Omega^2 = 1 + f_0(kz_0)^{m+2}$, with

$$f_0 = \frac{(1 + 2\beta)^2}{\beta(\beta + 1)m \Gamma(m + 2)}.$$  

(A26)

Equation (36) of the main text then follows.
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