EVOLUTIONARY MODELS OF THE ROTATING SUN

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ABSTRACT

We have developed a new rotating stellar evolution code and applied it to the Sun. Although we follow the basic philosophy of Endal and Sofia in the initial conditions and transport of angular momentum, the coding is totally independent, and the details of many of the redistribution mechanisms have been reformulated. We begin with a hydrostatic fully convective pre-main-sequence model. We then evolve this model to the age of the Sun. As the model evolves, we account for angular momentum loss via a magnetic wind and angular momentum redistribution by rotationally induced instabilities. We adjust our parameters to satisfy the usual global properties of the standard solar model and require that the solar models constructed with this program match the observed solar rotation rate. The resulting models have an oblateness in agreement with observed upper limits. The rotation curves of our models of the present-day Sun show two main features: (1) the outer layers \( r > 0.6 R_\odot \) exhibit minimal radial differential rotation and (2) a rapidly rotating central core \( r < 0.2 R_\odot \) is preserved. These basic features persist through a wide range of model parameters. The amount of differential rotation depends primarily on the properties ascribed to the angular momentum redistribution process. The spin-down of young cluster stars places a lower limit on the time scale for angular momentum redistribution at the base of the surface convection zone. Transport of angular momentum leads to rotationally induced mixing which reproduces the observed Li depletion in the Sun and solar analogs in open clusters. It also produces a depletion of Be in our solar models and leads to CNO abundance anomalies in subgiant and giant branch models. The amount of rotationally induced mixing is different for models with different initial angular momenta; the observed range in Li abundance in cluster stars of similar masses is a consequence of a spread in the initial angular momentum. Angular momentum loss limits the amount of angular momentum with which a star can arrive on the main sequence, and makes the surface rotation rate of old main-sequence stars almost independent of their initial angular momentum. The rotation curves in our models of the present-day Sun are in qualitative agreement with the estimates from current oscillation data for \( r > 0.6 R_\odot \) and have systematically more differential rotation within \( 0.4 R_\odot \). However, the rotation periods of a small sample of low-mass subgiants, estimated from their chromospheric activity, are difficult to explain without differential rotation in their main-sequence progenitors. The rotation period estimates for these stars are consistent with our evolved models; direct measurements of subgiant rotation periods will place strong constraints on rotating stellar models.

Subject headings: stars: evolution — Sun: abundances — Sun: interior — Sun: rotation

I. INTRODUCTION

One of the most remarkable things about the theory of stellar evolution is how well relatively simple models succeed in explaining the properties of stars. As more data have become available, however, the limits of the models have become more apparent. The abundances of light elements such as \(^7\)Li are not what standard stellar evolution predicts for a variety of stars. The most embarrassing example is the Sun, where an observed depletion of 200 relative to that found in meteorites finds no ready explanation in the standard models (D'Antona and Mazzitelli 1984). In addition, evolved stars frequently exhibit anomalous surface abundances which are indicative of deep mixing of processed material. These stars seem to inhabit regions of the H-R diagram where stellar evolution theory predicts no such mixing; they exist in blatant disregard of its strictures. The most likely culprit is not a fundamental error in the physics but the neglect of processes which induce mixing. Rotation is a property which virtually all stars possess, yet it is not treated in standard stellar models; it is a good candidate to be the mixing agent because differential rotation will generate mas motions which induce mixing. The study of rotating models will also shed light on star formation, and could alter age estimates for star clusters.

The Sun is a good test of rotating models for a number of reasons. We know its vital statistics (age, mass, composition, radius, and so on) far more precisely than we do for any other star. Solar seismology can in principle tell us about the internal rotation of the Sun— a probe not currently available for other solar-type stars. Another limit can be placed on the internal solar rotation by the precession of the perihelion of Mercury, which sets an upper limit on the quadrupole moment of the Sun if general relativity is to be believed. The Sun therefore serves to constrain the general properties of rotating models. On the other hand, rotating models of other stars, which sample different ages, masses, and chemical compositions, will also constrain the possible rotational properties of the Sun. This two-way interaction between the Sun and stars is an essential feature of the study of rotating stars.

In this paper we investigate the properties of rotating solar models computed with a newly developed rotating stellar evolution code. In § II we describe the initial conditions, input physics, and numerical techniques used in our rotating models.
Section III explores the consequences of parameter variations. In § IV we select the set of parameters which best reproduce currently available constraints on the Sun and describe the evolution of our best solar model. We discuss tests of the solar rotation curve in § V. We present our conclusions and detail additional work in progress in § VI.

II. METHOD

There are several problems which are not present in non-rotating stellar evolution that must be solved in order to compute the evolution of rotating stellar models. The most important of these are (a) computing the effects of rotation on the structure of the star; (b) determining a reasonable set of initial conditions; (c) modelling angular momentum loss due to a magnetic wind; and, (d) calculating the transport of angular momentum due to rotationally induced instabilities and estimating their effect on the abundance profiles of the various important chemical species. Of particular interest are the trace elements such as Li, Be, and $^3$He, which can act as probes of rotational mixing.

a) Effects of Rotation on the Structure

A rotating star generates mass motions which are inherently three-dimensional. Existing stellar evolution codes, however, are only one-dimensional due to the prohibitive computational expense of a full three-dimensional treatment. A reasonable one-dimensional solution is still the method originally proposed by Kippenhahn and Thomas (1970) as modified by Endal and Sofia (1976, hereafter ESI). This method allows standard evolution codes to be modified to include the effects of rotation with a minimum of difficulty. In particular, we have started with the Yale evolution code as described by Prather (1976) with the modifications stated by Seidel, Demarque, and Weinberg (1987).

In a standard nonrotating model the independent variable is the mass contained within a spherical surface, $M_i$. In the ESI method the independent variable is the mass contained within an equipotential surface, $M_p$, which need not be spherical. Given the angular momentum distribution we solve for the shapes of the equipotential surfaces. This then yields distortion terms which can be added to the standard stellar structure equations. The rotation-corrected equations of stellar structure may then be solved in the usual way. It is worth noting that the changes in the structure due to rotation are quite small. As a result, this method is highly accurate even for rapidly rotating stars.

The structure equations and correction terms used are the same as those in ESI; however, there are several differences between the current treatment and the ESI method that are worth noting here. ESI interpolated in tabular data to find the local gravitational acceleration and its inverse on an equipotential surface are needed to find the rotational correction terms for the equations of stellar structure. To determine these quantities, the potential is divided in three parts: a spherically symmetric portion, a cylindrically symmetric component, and a distorted component due to the departure from sphericity. The distorted potential was directly integrated from the center to a fitting point near the surface. Several static envelopes with luminosity and effective temperature close to those of the model are then integrated from the surface to the fitting point; interpolation between them is then used to find the surface boundary conditions. This causes problems for rotating models because the envelope integration depends on the angular momentum distribution of the model and the rotation rate in the outer layers. ESI used a four-dimensional envelope grid $(L, T, \omega, \rho_{\text{center}})$ rather than the standard two-dimensional grid $(L, T)$. They also assumed that the interior model they were integrating to was a polytrope rotating as a solid body; thus the parameter $\rho_{\text{center}}$ in the envelope grid. We chose a fitting point far enough out in the model that the contribution to the potential due to the envelope was negligible and solid body rotation in the envelope (at the rate of the final model point) was a reasonable assumption. Typical envelope masses were in the range of $10^{-9}$ solar masses. As the interior potential changed during the convergence process, new envelopes were generated. A new set of envelopes was generated for each model. This is more accurate and not significantly more time consuming than the ESI approach.

b) Initial Conditions

The above method allows us to compute the effects of a given angular momentum distribution upon the structure of a model. The angular momentum distribution at a given time, however, is a function of its previous history. We thus need to find reasonable initial conditions and evolve the angular momentum distribution from this initial state to the desired age.

The rotational properties of a star are strongly influenced by what occurs in the pre-main-sequence phase of evolution. Because of this we choose a fully convective hydrostatic model on the upper Hayashi track for our starting model. Hydrodynamic calculations of protostellar evolution by Larson (1969) indicate that a small core may never become convective. We have not considered such models since this core contains less than 1% by mass of the star. In addition there are mechanisms to be discussed below that will minimize differential rotation in such a core. The rotation law enforced in fully convective regions is uncertain on both theoretical and observational grounds. However, the indication from solar oscillation data is that the surface convection zone of the present day Sun rotates approximately as a solid body (Brown and Morrow 1987a).

We enforce this physically reasonable solid body rotation law in all convective regions in the models. In principle, other rotation laws are also possible, with the extreme limit being constant specific angular momentum. Our choice was dictated by the observations. The important factor here is that turbulence quickly enforces the limiting rotation law in convection zones. If this assumption holds, the star retains no memory of the distribution prior to the fully convective state and the starting model is reasonable. We therefore have only one parameter for our initial conditions: the total initial angular momentum $J_0$.

It has been known for some time that the surface rotational velocities of stars are closely related to their masses (Kraft...
The observed surface rotation velocities for stars later than spectral type F are much lower on average than the rotation velocities for stars earlier than spectral type F. The existence of angular momentum loss in these low-mass stars (as discussed below) makes their surface rotation velocities a poor guide to their initial angular momenta. The rapidly rotating massive stars, on the other hand, are thought to suffer minimal angular momentum loss. Kawaler (1987) has shown that the data for the massive stars is consistent with an average initial angular momentum which is a constant fraction of the equatorial breakup velocity on the main sequence. Extending this relationship to low-mass stars provides an estimate of the initial angular momentum of the Sun,

\[ J_0 = 1.63 \times 10^{50} \text{ g cm}^2 \text{ s}^{-1} \]

(Kawaler 1987). Real variations in the surface angular velocity among stars of the same mass are observed at the high-mass end of the Kraft curve (i.e., Wolff, Edwards, and Preston 1982). We also ran sequences with a smaller \( J_0 \) (5 \( \times \) \( 10^{49} \)) and a larger (5 \( \times \) \( 10^{50} \)) to see the consequences of the variation of this somewhat uncertain parameter.

c) Angular Momentum Loss

The observed rotation velocities in low-mass stars (spectral type mid-F and later) are much lower than the typical rotation velocities of higher mass stars (Kraft 1970). The location in mass of this discontinuity coincides with the development of a substantial surface convection zone, suggesting that angular momentum loss is a ubiquitous feature in the early evolution of low-mass stars. When advancing the models in time we must modify the total angular momentum to allow for this angular momentum loss from the surface. Endal and Sofia (1981; hereafter ES3) used an empirical fit to the model of Belcher and MacGregor (1976) for angular momentum loss by a magnetic stellar wind. We use a more general parameterization (Kawaler 1988):

\[ \frac{dJ}{dt} = -K_w \omega^2 + 4aN/3 \left( \frac{R}{R_\odot} \right)^{2-N} \left( \frac{M}{M_\odot} \right)^{-N/3} \left( \frac{\dot{M}}{10^{-14}} \right)^{1-2N/3} \]

where \( K_w \) is a constant that combines scale factors for the wind velocity and magnetic field strength; in practice, we adjust \( K_w \) (within sensible bounds obtained from scaling arguments) to give the solar surface rotation velocity at the solar age. In the derivation of the above equation, it has been assumed that \( B_\sigma \), the surface magnetic field strength, is proportional to \( \omega^a \). We choose \( a = 1 \), in accord with observations of stellar magnetic fields (Linsky and Saar 1987). The wind index is denoted by \( N \) and is a measure of the magnetic field geometry; \( N \) can vary from 0 to 2, with an \( N \) of 2 corresponding to radial fields and smaller values representing more complex field geometries. The dependence on the mass loss rate is very weak; we have found that modest changes in \( \dot{M} \) produced no appreciable change in the final configuration. For our solar models we therefore took the mass loss rate \( \dot{M} \) to be \( 10^{-14} M_\odot \text{ yr}^{-1} \) (effectively removing the mass loss term from eq. [1]). There are thus two free parameters in the wind model: the constant \( K_w \) and the wind index \( N \).

d) Angular Momentum Transport

So far we have not discussed angular momentum transport in our models. Models without angular momentum transport are not realistic on two accounts. First, rotating solar models evolved without angular momentum redistribution have an oblateness much greater than the observed upper limit (Hill and Stebbins 1975). Second, as the wind torques down the surface convection zone to a fraction of its initial angular momentum, a large shear at the base of the convection zone is generated. Laboratory experiments on rotating fluids show that such a configuration is unstable (Zahn 1987 and references therein). These instabilities will drive mass motions which redistribute angular momentum and mix the stellar material. Thus the treatment of redistribution of angular momentum, and any resulting mixing in radiative zones, is an important ingredient for any rotating models.

As mentioned above, we enforce rigid rotation in convective regions. Local conservation of angular momentum in radiative zones generates differential rotation due to differential contraction in the pre-main sequence; angular momentum loss will also generate differential rotation at the base of the surface convection zone. We apply local conservation of angular momentum in radiative regions and the angular momentum loss formula in § IIc to initially determine the angular velocity gradients and then analyze their rotational stability.

The mechanisms which redistribute angular momentum can be divided into two categories according to the time scales involved. Those which occur on a time scale which is much shorter than the evolutionary (nuclear or thermal) time scale are referred to as “dynamical” instabilities; those whose time scales are comparable or longer are called “secular” instabilities. When dynamical instabilities are triggered, the model is instantaneously readjusted to a state of marginal stability by radial exchange of angular momentum. For secular instabilities, the time scales were explicitly computed, and angular momentum redistribution was treated as a diffusion process.

The instabilities considered are the same as those in Endal and Sofia (1978; hereafter ES2). To minimize duplication, we refer to that paper for the detailed discussion of the criteria for stability and the diffusion coefficients used in the calculations. The only equations given here are those which differ from ES2. Like ES2, we do not address distribution of angular momentum by processes for which no estimates of diffusion coefficients are available, such as by the ABCD instability (Spruit, Knobloch, and Roxburgh 1983), magnetic fields, and the triply diffusive instability. Similarly, we do not include turbulent diffusion (Schatzman 1969) because it has extreme and controversial consequences for nonrotating models that are not supported by current observations (Schatzman and Maeder 1981; Ulrich and Rhodes 1983).

i) Dynamical Instabilities

Three dynamical instabilities are included in our calculations: convection, the Solberg-Holiland instability, and the dynamical shear instability. As mentioned previously, we enforce rigid rotation in convective regions. The Solberg-Holiland criterion refers to stability against axisymmetric adiabatic perturbations (Waszynski 1946). It is of secondary importance in our models. Dynamical shear, we find, is usually the most effective dynamical instability. The stabilizing force against this shear is a density gradient; thus in the absence of a density gradient, no differential rotation can be tolerated (Zahn 1974). Since equipotential surfaces are surfaces of constant density in our models (the departures are second order) the action of dynamical shear ensures that the rotation velocity is constant on equipotential surfaces. The condition for stability against dynamical shear (in the equatorial plane) in the radial
direction is

\[ R_c = \frac{\rho}{P} (V_{ad} - V) \left( \frac{g}{d \ln \rho} \right)^2 > \frac{1}{4} \]  (2)

(Zahn 1974). The controlling factor in the above condition is \( V_{ad} - V \). At the boundary between radiative and convective regions, this term is zero by definition, and this mechanism tolerates no differential rotation. Within radiative zones, however, this term increases substantially, allowing large gradients in angular velocity to be stable. Therefore, besides removing latitude dependence, the main effect of dynamical shear is to severely limit radial differential rotation in the radiative region adjoining convection zones.

### ii) Secular Instabilities

Because of the longer characteristic time scales for the secular instabilities compared to the dynamical instabilities, the angular momentum and composition redistribution were computed using the coupled nonlinear diffusion equations:

\[ \rho \frac{L}{M} \frac{d \omega}{dt} = f_o \frac{d}{dr} \left( \rho \frac{V}{M} \frac{d \omega}{dr} \right) \]  (3)

for the transport of angular momentum and

\[ \rho \frac{d X_i}{dt} = f_i \frac{d}{dr} \left( \rho \frac{V}{M} \frac{d X_i}{dr} \right) \]  (4)

for composition transport of species \( X_i \). The moment of inertia per unit mass is \( I/M \), and \( D \) is the diffusion coefficient, which is the product of a length scale and characteristic velocity. The length scale is the velocity scale height or the radius whichever is smaller. The velocity used for the diffusion coefficient \( D \) is the sum of the circulation velocities generated by each of the processes discussed below. None of these velocities are known precisely; the velocity estimate for secular shear is particularly uncertain since it is based on dimensional arguments.

Because of these inherent uncertainties in the diffusion coefficients we introduce the adjustable parameter \( f_o \) to investigate the effects of these uncertainties. ESZ used the same diffusion coefficients for composition transport as angular momentum transport. This need not be true in general; we incorporate \( f_i \) as a separate parameter to allow for angular momentum transport by processes which mix material less efficiently than they transport angular momentum. For example, values of \( f_i < 1 \) could be used to mimic the effects of angular momentum transport by magnetic fields.

When a secularly unstable region develops, we determine characteristic velocities and length scales for the entire unstable region. After transforming to an equally spaced grid in radius using an osculatory spline interpolation routine, we solve the system of diffusion equations for the new angular momentum distribution and run of composition using a fully implicit scheme, iterating on the angular velocities, compositions, and diffusion coefficients. Because the diffusion coefficients depend on the composition gradients, the angular momentum and composition transport must be solved for simultaneously.

The system of diffusion equations must be supplemented by appropriate boundary conditions. The boundary condition for the diffusion equations at a stable radiative boundary is that the diffusion coefficient vanishes. The boundary condition at the interface between an unstable region and a convective zone is more complicated. The local circulation velocities at this boundary define a diffusion velocity. Determining the velocity scale height is more difficult. The diffusion velocities are typically of order \( 10^{-5} \text{ cm s}^{-1} \). Since velocities in convective zones are many orders of magnitude larger, the velocity scale height is discontinuous at the radiative-convective boundary. Because of this, a transition region must develop in real stars. In the absence of a detailed understanding of this region, we model the transition from large convective velocities to small diffusive ones by means of a small boundary layer. There are two properties of this layer which are of key interest. The velocity scale height is proportional to the assumed extent of the boundary layer; it also depends on the slope of the velocity curve in the boundary region. By extensive tests, we have found that different characterizations of this region have little or no effect on the behavior of the models (Pinsonneault 1988). We use a boundary region of 0.05 pressure scale heights.

The three secular instabilities treated in the current work are, in ascending order of importance, Eddington circulation, the Goldreich-Schubert-Fricke (GSF) instability, and the secular shear instability. Eddington circulation is a consequence of the fact that purely radiative thermal equilibrium cannot be maintained in a rotating star (von Zeipel 1924). This generates large-scale mass motions which reduce angular velocity and composition gradients. The characteristic velocity of these currents has been estimated by Kippenhahn and Möllerhof (1974) as

\[ v_{E0} = \frac{V_{ad}}{\delta (V_{ad} - V)} \frac{\omega^2 r L_y}{(2\pi r^2 r^2 - 2r^2 M_y - 3)} \]  (5)

where

\[ \delta = \frac{\partial \ln \rho}{\partial \ln T_{eq}} \]  (6)

and the subscript \( \psi \) denotes quantities on equipotentials. Also, in the above equation, \( \epsilon \) is the specific energy generation rate and the radius is take as \( r_\psi \), the mean radius of the equipotential surface (see ES1). Gradients in mean molecular weight tend to choke these circulation currents (Mestel 1953). The net velocity due to Eddington circulation with this effect taken into account is

\[ v_e = |v_{E0}| - \left| \frac{v_\mu}{|v_\mu|} \right| v_{E0} \]  (7)

Kippenhahn (1974) estimates \( v_\mu \) as

\[ v_\mu = f_\mu \frac{\varphi H_p}{\delta (V_{ad} - V)} \frac{|V_\mu|}{\mu} \]  (8)

where \( H_p \) is the pressure scale height, and

\[ \varphi = \frac{\partial \ln (\rho)}{\partial \ln (T_{eq})} \]  (9)

This value of \( v_\mu \) is based on an estimate of the thermal relaxation time of a blob \( (\tau_{KH}) \) with a mean molecular weight different from its surroundings. Because this is a function of the blob geometry we use the local Kelvin-Helmholtz time scale and introduce a parameter \( f_\mu \) which allows us to vary the sensitivity of Eddington circulation to \( \mu \) gradients.

Eddington circulation velocities are proportional to \( \omega^2 \). As a result, this process is important in rapidly rotating regions. It becomes negligible compared to other mechanisms in slowly rotating regions such as the outer layers of the Sun. It is less sensitive to \( \mu \) gradients than the secular shear; as a result,
Eddington circulation (and the GSF instability) dominate angular momentum transport in the deep interior. Because the Eddington circulation velocity is independent of the angular velocity gradient, it provides a mechanism for transport of angular momentum across regions of the star which do not contain large angular velocity gradients.

The GSF (Goldreich and Schubert 1967; Fricke 1968) condition requires rotation on cylinders for stability. Because the dynamical shear instability removes any latitude dependence of the angular velocity (§ IId[ii]), only solid body rotation will be stable against both. The velocity associated with GSF (ES2) has the same dependence as Eddington circulation on gradients of the mean molecular weight. Unlike Eddington circulation, the characteristic GSF velocity is strongly dependent on angular velocity gradients. These gradients develop at the base of the surface convection zone and at the boundary of a rapidly rotating core. Because the secular shear instability is dominant at the base of the convection zone, the effects of GSF are largest in the deep interior regions.

The secular shear instability (Zahn 1974) arises because radiative diffusion tends to remove the stabilizing effects of a density gradient. This effect is maximized in fluids with low viscosity. As a result of the low Prandtl number (ratio of the viscosity to the thermal diffusivity) typical in stars the secular shear sets very stringent limits on differential rotation in stars. However, because the secular shear is highly sensitive to gradients in the mean molecular weight, and because like the dynamical shear it tolerates more differential rotation far from a convective region, it is ineffective in the deep interior (which is far from a convective region and where these gradients appear). Thus the secular shear is important in the outer layers, in particular between the base of the convection zone and the nuclear burning core. The amount of differential rotation tolerated by the secular shear is proportional to a critical Reynolds number. Because this number is uncertain, we treat it as another adjustable parameter ($R_{crit}$).

III. CONSEQUENCES OF PARAMETER VARIATIONS

a) Observational Constraints and Tests of the Models

i) Surface Rotation as a Function of Time

We now wish to find the combination of parameters which gives a rotating model in closest agreement with observations of both the Sun and other stars. To do so we need to determine how each of the free parameters affects the observable properties of the models. In stars other than the Sun, the currently measurable quantities which are related to the stellar rotation are the surface rotational velocities, abundances, and chromospheric activity levels. In the last few years rotational velocities and trace element abundances for a number of solar-type stars of different masses, compositions, and ages have been determined (i.e., Stauffer and Hartmann 1987 and references therein; Hobbs and Pilachowski 1986a, b; Balachandran and Lambert 1989). The surface velocities of stars as a function of age and mass strongly constrain the models for the spin-down of the surface layers. In particular, the rotation of young cluster stars provides an important test of the models. We will therefore examine the effects of parameter variations on the surface rotational velocities as a function of time.

ii) Light Element Abundances

Rotationally induced mixing will alter the run of composition as a function of depth of the various elements. In the Sun, $^{7}$Li is destroyed at temperatures only slightly higher than those at the bottom of the surface convection zone; the surface abundance of $^{7}$Li is therefore a highly sensitive function of the properties of mass transport at the base of the convection zone. Beryllium survives to higher temperatures than lithium; any reduction in the surface abundance of Be provides evidence of rotationally induced mixing to a greater depth. The peak in the $^{3}$He abundance profile in nonrotating models occurs at $M_{r}/M_{\odot} \sim 0.6$ in solar models, and some CNO processing occurs in the deeper interior. Each of these elements provides a probe of rotationally induced mixing in a different region of the Sun, as shown in Figure 1.

While the rotating model of the current Sun does not show CNO cycle anomalies produced at the surface for any combination of parameters, the abundance profiles of these elements are significantly altered. This will produce surface composition anomalies in the subgiant and giant branch phases of evolution. Such anomalies are in fact observed in globular star clusters such as M92 (Smith 1987). We will examine this effect in future work; for the purposes of this paper we merely identify parameters that have an effect on the profile of the CNO cycle elements.

Although an enhancement of the surface $^{3}$He abundance is a feature of all of our solar models, the observational measurement of $^{3}$He (or $^{4}$He!) is extremely difficult. The best tracers of rotational mixing in the Sun are therefore the light elements Li and Be. The Sun is deficient in lithium by a factor of 200 relative to the cosmic abundance (i.e., Müller, Petremann, and de la Reza 1975; Boesgaard 1976; Duncan 1981) and in beryllium by a factor of 2–5 (Molaro and Beckmann 1984; Ross and Aller 1974; Reeves and Meyer 1978). The effect of parameter variations on the surface abundances of these elements is our second major test.

iii) Solar Rotation Curve

In addition to the above observations in both the Sun and other stars, it is possible in principle to determine the current internal solar rotation curve by means of helioseismology (Brown and Morrow 1987b). We therefore explore the differences in rotation as a function of depth in models of the present-day Sun. Unfortunately, the models differ the least in the outer layers where the rotational inversion techniques are most reliable. (Gravity modes, if detected, would provide a probe of the rotation of the deep interior.)

iv) Reference Solar Model

For comparison purposes, we computed a reference model with our initial guess as to the most likely set of values for all free parameters. This fiducial set of values was chosen to reproduce the observed solar lithium abundance, surface rotation rate, luminosity, and radius at the solar age. (The “best” solar model described in § IV, on the other hand, also satisfies additional constraints derived from observations of other stars). We then ran sequences with parameters different from this reference model. We summarize the surface and central properties of the reference model in Table 1; surface abundances in this table are given in terms of their initial value. The effects of parameter variations on these model properties are summarized in Table 2. All cases have $K_{u} = 4.9$, $f_{u} = 0.05$, $f_{s} = 1$, and $R_{cut} = 15,000$ unless otherwise noted. All models were evolved to an age of 4.7 Gyr with $Y = 0.24$, $Z = 0.02$, and $Z = 1.371$. In Table 2, we list the values of the parameters in columns (2)–(4); columns (5)–(9) show the differences in various quantities from the reference model.
b) Structural Effects of Rotation

The direct effects of rotation on the mechanical and thermal structure of the model of the present-day Sun are not large for any choice of parameters. This is because the initial solid body rotation curve concentrates most of the angular momentum in the outer layers of the star. As a result, the structure of the deep interior is not greatly altered even if there is no angular momentum transport there. Angular momentum loss at the surface also ensures that a minimal amount of angular momentum survives in the outer layers of the model. At an earlier age the total angular momentum in the models is larger and the structural effects of rotation on the models are larger. We show the effects of rotation on the evolutionary track in the H-R diagram in Figure 2. For main-sequence stars the position in the H-R diagram is a weak function of rotation and the various rotating models differ little from one another. For pre-main-sequence stars, however, the position in the H-R diagram does depend on the degree of rotation in the models. This will complicate attempts to assign masses to very young stars based on theoretical evolutionary sequences. A summary of the changes in the surface and central properties of the Sun due to rotation is listed in Table 3.

c) Consequences of Varying the Initial Angular Momentum

The surface rotational velocity of our models of the present-day Sun is insensitive to changes in the initial angular momentum $J_0$. This is shown in Figure 3 where we display the surface rotation rate as a function of time for three models differing only in $J_0$. We can therefore calibrate our models so that they have the observed solar rotation rate at the solar age independent of the initial solar angular momentum. The reason for this insensitivity is the strong dependence of the wind on the surface rotation rate. Models which begin with large initial angular momentum spin-down faster than those with smaller $J_0$; by $(2-3) \times 10^8$ yr they will have similar surface velocities (see also Kawaler 1988). An important consequence of this is that angular momentum loss in the pre-main-sequence phase of evolution places an upper limit on the total angular momentum a star can retain by the beginning of the main sequence. Models with the average $J_0$ predicted by our extrapolation of the Kraft curve have a rotation curve which is similar to those with a $J_0$ 3 times larger very early in their evolution. We therefore conclude that the critical value of $J_0$ is at or below this value. Models which begin with less than this limit will retain

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**TABLE 1**

**Properties of the Reference Solar Model at an Age of 4.7 Gigayears**

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>$\log \omega_{\text{surf}}$</td>
<td>-5.5220</td>
</tr>
<tr>
<td>$\log \omega_{\text{core}}$</td>
<td>-3.8049</td>
</tr>
<tr>
<td>Li</td>
<td>0.00613</td>
</tr>
<tr>
<td>Li$_0$</td>
<td></td>
</tr>
<tr>
<td>Be</td>
<td>3.76455</td>
</tr>
<tr>
<td>Be$_0$</td>
<td>0.01101</td>
</tr>
<tr>
<td>$^3$He</td>
<td>1.0349</td>
</tr>
<tr>
<td>$^3$He$_0$</td>
<td></td>
</tr>
<tr>
<td>$T_{\text{eff}}$</td>
<td>3.76455</td>
</tr>
<tr>
<td>$L$</td>
<td>0.01101</td>
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</table>
Fig. 2.—Evolution in the H-R diagram of rotating and nonrotating solar models. All sequences have $Y = 0.24$, $Z = 0.02$, and $\alpha = 1.371$. The solid line is a nonrotating model; the nearly indistinguishable long-dashed line is a slow rotator (case A in Table 2) whose evolution is indistinguishable from the nonrotating model; and the short-dashed line is a rapid rotator whose initial angular momentum is 10 times the reference solar model (case F in Table 2).

Fig. 3.—Surface rotation velocity as a function of time for three models differing only in initial angular momentum. The solid line is the reference solar model (case A in Table 2) with $J_0 = 5 \times 10^{48}$ g cm$^2$ s$^{-1}$. The long-dashed line is a model with $J_0 = 1.63 \times 10^{49}$ g cm$^2$ s$^{-1}$ (case B). The short-dashed line is a model with $J_0 = 5 \times 10^{49}$ g cm$^2$ s$^{-1}$ (case C). An order of magnitude in $J_0$ produces only a small variation in the surface rotation velocity on the main sequence ($\log t > 8$).
less rotation in the interior; models which begin with more will have the same rotation curve as models with this critical $J_0$. Even the low $J_0$ case rotates almost as rapidly in the central regions as the high $J_0$ case by the age of the Sun, which is a measure of how rapidly stars lose sensitivity to their initial conditions (Fig. 4).

Since the final rotation curve is insensitive to $J_0$, stars which start out rotating faster lose more angular momentum from the interior; thus they transport more angular momentum from the core outward. Angular momentum transport also results in transport of composition; thus models that start out with large $J_0$ will mix more material between the interior and surface. This effect is particularly dramatic for the fragile light elements lithium and beryllium. Figure 5 shows the surface $^7$Li and $^9$Be abundances as a function of time for the three $J_0$ cases. Nuclear burning of lithium in the surface convection zone produces the initial depletion between $10^6$ and $10^7$ yr. This does not occur for beryllium. Note that the high $J_0$ case burns more lithium than the lower $J_0$ cases even though angular momentum redistribution is not important at this age. This results from the effect of rotation on the structure; the rapid rotator behaves like a lower mass star in which the surface convection zone retreats more slowly. The surface lithium abundance for all sequences drops during main-sequence evolution as mixing of lithium-poor material to the surface dilutes the surface abundance and the lithium-rich material is transported down and destroyed. At this point the higher $J_0$ models undergo more angular momentum transport and the difference between their surface lithium abundances and that of the slow rotator increases. The difference between the high and intermediate $J_0$ models does not increase after $10^8$ yr; this is because the rapid rotator has almost the same rotation curve as the intermediate model by this time.

In nonrotating models it is difficult to produce a range in surface lithium abundance among stars of the same mass, composition, and age. On both observational and theoretical grounds we do not expect large star to star variations in their initial Li abundance (Boesgaard 1976; Duncan 1981). Due to the rapid pre-main-sequence burning of lithium, very young clusters with a small spread in age can have a large spread in lithium at constant mass. As a result, a correlation between more rapid rotation and higher lithium is possible in young clusters with a small spread in age can have a large spread in lithium and beryllium.

**TABLE 2**  
RESULTS OF PARAMETER VARIATIONS

<table>
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<tr>
<th>Run</th>
<th>$J_0$</th>
<th>$N$</th>
<th>$f_a$</th>
<th>$\Delta \log \omega_{surf}$</th>
<th>$\Delta \log \omega_{cen}$</th>
<th>$Li_{ref}$</th>
<th>$Be_{ref}$</th>
<th>$^3He_{ref}$</th>
</tr>
</thead>
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<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
<td>(8)</td>
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<td>+0.0</td>
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<td>1.0</td>
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<td>+0.1413</td>
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<td>0.567</td>
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<td>+0.0189</td>
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<td>+0.0</td>
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<td>+0.1413</td>
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<td>1.076</td>
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<td>+0.0</td>
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</table>

* $f_a$ modified with respect to the secular shear only.

**TABLE 3**  
STRUCTURAL EFFECTS DUE TO ROTATION

<table>
<thead>
<tr>
<th>$J_0$</th>
<th>$\log T_{surf}$</th>
<th>$\log L/L_0$</th>
<th>$\log T_{cen}$</th>
<th>$X_{cen}$</th>
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<tr>
<td>0.0</td>
<td>3.76451</td>
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<td>7.17375</td>
<td>0.38715</td>
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<td>3.76455</td>
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<td>5.0</td>
<td>3.76460</td>
<td>0.01041</td>
<td>7.17260</td>
<td>0.39490</td>
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</table>
Fig. 4.—Angular velocity as a function of radius at two ages for three models differing only in their initial angular momentum $J_0$. The solid line is the reference solar model (case A in Table 2) with $J_0 = 5 \times 10^{49} \text{ g cm}^2 \text{s}^{-1}$ at an age of $3 \times 10^7 \text{ yr}$. The long-dashed line is a model with $J_0 = 1.63 \times 10^{50} \text{ g cm}^2 \text{s}^{-1}$ (case B) at the same age and the short-dashed line is a model with $J_0 = 5 \times 10^{49} \text{ g cm}^2 \text{s}^{-1}$ (case C) at the same age. The dash-dotted lines indicate the range of rotation curves in the present-day Sun. An order of magnitude variation in $J_0$ produces a large change in the rotation curve at an early age but only a small change in the rotation curve of the present-day Sun.

Fig. 5.—Surface $^7\text{Li}$ and $^9\text{Be}$ abundances as a function of time for three models differing only in their initial angular momentum $J_0$. Abundances are logarithmic on the scale where hydrogen = 12.0. The upper set of lines (starting at 3.3) are $^7\text{Li}$ abundances; the lower set of lines (starting at 2.0) are $^9\text{Be}$ abundances. The solid line is the reference solar model (case A in Table 2) with $J_0 = 5 \times 10^{49} \text{ g cm}^2 \text{s}^{-1}$. The long-dashed line is a model with $J_0 = 1.63 \times 10^{50} \text{ g cm}^2 \text{s}^{-1}$ (case B). The short-dashed line is a model with $J_0 = 5 \times 10^{49} \text{ g cm}^2 \text{s}^{-1}$ (case C). An order of magnitude variation in $J_0$ produces a large range in $^7\text{Li}$ abundance and a smaller range in $^9\text{Be}$ abundance; this range increases with increasing age.
A distribution in \( J_0 \) will produce a spread in the surface lithium abundance among stars of the same mass, composition, and age. On the main sequence, models with a higher \( J_0 \) lose more angular momentum. This leads to more angular momentum transport than occurs in slower rotators and thus more lithium depletion, resulting in a spread in Li which increases with increasing age. By an age of a few gigayear, however, the rotation curve in the outer layers, is similar for all values of \( J_0 \) and the rate of Li depletion will be the same in them. The spread in Li will therefore reach some limiting value which depends on the mass, composition, and initial distribution of \( J_0 \).

The dependence of the surface lithium abundance on \( J_0 \) creates serious problems for the use of lithium as an age indicator for individual objects, as we can see in Figure 5. Stars of the same mass, composition, and lithium abundance can have greatly different ages. In addition, the lithium abundance is a strong function of both mass and composition. If the age, mass, and composition of a star is known (i.e., for cluster stars or the Sun), the lithium abundance is an excellent indicator of the initial angular momentum. The average lithium abundance of an ensemble of stars of a given mass and composition is a reasonable indicator of the age of the ensemble that is independent of the initial angular momentum distribution.

These conclusions receive support from observations of lithium in open cluster stars. Late-type stars in Alpha Perseus with an age of \((2-3) \times 10^7\) yr exhibit a range in Li between a cosmic abundance of 3.3 (on the logarithmic scale where \( H = 12.0 \)) to a lower envelope consistent with an age spread (Balachadran and Lambert 1988). The G stars in the Hyades (age = 350 Myr) show a small range in Li, although the sample of G stars observed is small (Cayrel et al. 1984; Boesgaard and Tripicco 1986). NGC 752, which is an older cluster (1.7 Gyr) has a substantially larger range in Li abundances among stars of the same effective temperature (Hobbs and Pilachowski 1986a). A dispersion in Be is also present in our models; however, the number of observations of stellar Be is limited. Observations of surface Be abundances in young cluster stars would provide another important constraint on the models because Be probes angular momentum transport in deeper layers of the star.

The abundance profiles of the CNO cycle elements are affected by different values of \( J_0 \) as well. This is shown in Figure 6 for \(^{13}\)C. As we look at species which are mixed later in the Sun's lifetime the dependence on \( J_0 \) weakens, however; Li is more sensitive than Be which in turn is more sensitive than \(^{13}\)C. The differences in the composition profiles of the CNO cycle elements will manifest themselves on the giant branch in the form of varying amounts of CNO processed material being mixed to the surface as the surface convection zone deepens and dredges up this material.

**d) Consequences of Varying the Angular Momentum Loss Rate**

The particular choices of parameters for the angular momentum loss rate determines when spin-down occurs and how rapidly and completely it proceeds. The wind index \( N \) determines when spin-down becomes significant; the constant \( K_w \) determines how rapidly it proceeds (see Kawaler 1988). With all other parameters fixed models with a large \( N \) will reach a lower maximum surface velocity and will do so earlier.
in their evolution as compared to small $N$ models. Models with large $N$ also retain less of their initial angular momentum (Fig. 7).

The wind model exerts a dominant influence on the surface rotation rate at the solar age. In particular, the surface rotation velocity at a given time is proportional to $K_w^{-3/4} a N$ for mature main-sequence stars; therefore we calibrate our solar models by adjusting $K_w$ such that the final model matches the current solar rotation rate. The surface lithium depletion is also proportional to $K_w^{-3/4} a N$. This occurs because the rotation rate in the region immediately below the surface convection zone is directly proportional to the surface rotation rate. The amount of angular momentum transport (and thus lithium depletion) is therefore proportional to the surface rotation rate. The effect of changing $K_w$ is less for the transport of other species because the amount of angular momentum transport in the deep interior is less affected by changes in the surface rotation rate.

The wind index $N$ is determined by the geometry of the external magnetic fields which could in principle vary in time and from star to star; we therefore retain it as a free parameter. As we will show in § IV, fully calibrated solar models have rotation curves which are almost independent of the wind index. The effects of the wind index on the properties of the best solar model are restricted to indirect feedback effects which we discuss in § IIIf.

(e) Effects of Changing Angular Momentum Transport

Changes in the remaining parameters represent different models for the physics of angular momentum transport. The factor $f_a$ changes the time scale of angular momentum transport; similarly, $f_e$ changes the time scale of composition diffusion. The boundary region affects the degree to which the surface convection zone is coupled to the interior. The sensitivity to gradients in mean molecular weight determines how far into the star the GSF and Eddington instabilities can penetrate. Finally, the critical Reynolds number affects the stability condition and time scale estimate for the secular shear. Unlike the initial angular momentum and wind parameters, our freedom to vary these parameters is based on our lack of knowledge of the physics of angular momentum redistribution. Although the critical Reynolds number is uncertain there is no reason for it to be different in different stars. We therefore require these parameters to be the same in all stars.

(i) Time Scale of Angular Momentum Redistribution $f_a$.

The two most important parameters that govern angular momentum transport are $f_a$ and $f_e$. Changes in $f_a$ have a marked effect on the surface rotation rate of stars as a function of time (Fig. 8). The surface rotation rate is determined by structural changes due to evolution, the loss of angular momentum in the solar wind, and the amount of angular momentum mixed into the convection zone from below. The wind parameters determine the rate of angular momentum loss and $f_e$ controls the amount of angular momentum transport from below. Initially the global contraction dominates all other effects and the surface layers spin up. Eventually angular momentum loss due to the wind dominates and the surface angular velocity begins to drop. For $f_a$ of unity, angular momentum transport from the interior becomes effective at about the same time. The whole star, effectively, spins down;
Evolutionary Models of Rotating Sun

Fig. 8.—Surface rotation velocity as a function of time for three models differing only in the time scale for angular momentum transport $f_\mu$. The diffusion coefficients for angular momentum redistribution are multiplied by $f_\mu$. The solid line is the reference solar model with $f_\mu = 1.0$ (case A in Table 2). The long-dashed line is a model with $f_\mu = 0.1$ (case M). The short-dashed line is a model with $f_\mu = 10.0$ (case N). The value of $f_\mu$ determines the time dependence of the surface rotation velocity, especially for stars less than $10^9$ yr old.

Angular momentum transport is fueled by angular momentum loss at the surface and differential rotation generated by differential contraction. As gradients in mean molecular weight are generated in the deep interior this region soon becomes rotationally detached from the rest of the star. Because the central regions do not begin with much angular momentum, however, the timing of this detachment does not affect the surface rotation velocity as a function of time.

As the diffusion coefficients for angular momentum transport are decreased (smaller $f_\mu$) the spin-down of the surface convection zone is initially more rapid. This is because a smaller $f_\mu$ provides weaker coupling between the surface and the interior; the wind has a smaller reservoir of angular momentum to draw upon. At the same time the interior regions spin up faster than they would have with more efficient transport. On the main sequence the surface layers have spun down to a much lower rate than the higher $f_\mu$ case and thus are losing angular momentum at a greatly reduced rate. There is therefore a transition period where the reduced angular momentum loss from the surface is roughly balanced by the increased angular momentum transport from below. During this period the surface angular velocity changes only slowly and the layers below the surface convection zone are spinning down. When the interior has spun down sufficiently the surface layers can torque down again. The final rotation period at the solar age is only slightly higher than that for the higher $f_\mu$ case.

As the diffusion coefficients for angular momentum transport are increased (higher $f_\mu$) the coupling between the surface and interior is made more efficient, and the star has a larger pool of angular momentum to draw upon at all times. As a result, it reaches a higher maximum surface rotation rate later in its evolution. This increase is achieved at the expense of the interior, which retains much less of its initial angular momentum. A high $f_\mu$ model thus begins its main-sequence life with a more rapidly rotating surface and less internal angular momentum than a low $f_\mu$ model. The surface rotation rate therefore declines more rapidly than the low $f_\mu$ case and reaches almost the same value at the solar age.

The rotation curves of models of the present day Sun are also strongly dependent on $f_\mu$, as shown in Figure 9. In the outer layers the time scale for angular momentum transport is much less than the age of the Sun for any reasonable value of $f_\mu$. The rotation curve is therefore almost flat for $r > 0.5 R_\odot$. Interior to $0.5 R_\odot$, however, gradients in mean molecular weight inhibit angular momentum redistribution. As $f_\mu$ is decreased, central gradients in mean molecular weight are allowed more time to develop. As a result, the central regions of the Sun rotate much more rapidly for smaller values of $f_\mu$. In fact, there is almost no transport of angular momentum in the deep interior for $f_\mu = 0.1$. As $f_\mu$ is increased, more angular momentum is lost early when $\mu$ gradients are minimal and the models retain less angular momentum.

By changing $f_\mu$ we are simultaneously changing the time scale for all of the instabilities considered in this paper. In fact the physical uncertainty in the time scale estimates for Eddington circulation and the GSF instability is smaller than that for
the secular shear mechanism. We therefore computed models in which only the secular shear time scale estimate was changed to see which of the effects of changes in $f_s$ were due to the shear and which were caused by the other mechanisms. The spin-down of the surface layers and the $^7$Li depletion were found to be sensitive mainly to the time scale for the secular shear (Fig. 10). This is because the shear is the dominant mechanism in the outer layers. The rotation in the inner 30% (by radius) of the model of the present Sun, however, is the same as for the case in which the secular shear velocity was unaltered. This is because transport in the deep interior is mostly due to Eddington circulation and the GSF instability. The spin-down and lithium depletion, then, constrains only the time scale for the secular shear. The rotation of the deep interior of the current Sun constrains the time scales for operation of the other mechanisms, as does the mixing of CNO processed materials.

The depletion of the surface Li is affected by $f_o$ as well. In Figure 11 we show the surface Li abundance as a function of time for three values of $f_o$. Because the low $f_o$ case transports less angular momentum, it loses less lithium; the opposite is true for the high $f_o$ case. The difference between the rotating model CNO cycle composition profiles and the nonrotating ones decreases with decreasing $f_o$ for the same reason.

ii) Time Scale for Composition Transport $f_c$

Changing $f_c$ changes the efficiency of composition transport with respect to angular momentum transport. As $f_c$ is reduced the effect of angular momentum transport on a given composition gradient is reduced. Possible values of $f_c$ range from 0 to 1, with $f_c = 0$ implying no material mixing due to angular momentum redistribution and $f_c = 1$ giving the same time scale for reduction of angular velocity and composition gradients. In general the amount of composition transport is proportional to $f_c$. The depletion of lithium is an even stronger function of $f_c$; (Li depletion approximately proportional to $f_c^2$) because the temperature at the base of the surface convection zone is hotter early in the lifetime of the Sun. As a result we adjust $f_c$ so that the models have a surface lithium abundance 200 times smaller than the initial value. As $f_s$ is made smaller, the required value of $f_c$ rises. Therefore, for a given $J_0$, there is a minimum allowed value of $f_s$. The requirement that our solar models reproduce the observed solar lithium depletion thus places an upper limit on the time scale for angular momentum transport in our models.

A value of less than one for $f_c$ implies that the time scale for depletion of lithium is longer than the time scale for angular momentum transport at the base of the surface convection zone. It is certainly possible that either the angular momentum is redistributed in such a way as to minimize composition transport (such as very small scale turbulence) or that we are neglecting a mechanism, such as magnetic fields, which efficiently transports angular momentum but does not mix material. Even if some other mechanism is operating, however, the data from other stars discussed in § IV requires that it operate on a time scale which is comparable to that of the mechanisms we treat in this paper.

iii) Sensitivity to $\mu$ Gradients $f_o$

Eddington circulation and the GSF instability are inhibited by mean molecular weight gradients. Changes in this level of inhibition (parameter $f_o$) exert a strong influence on the rota-

![Fig. 9.—Angular velocity as a function of radius for three models of the present-day Sun differing only in the time scale for angular momentum transport $f_s$. The solid line is the reference solar model with $f_s = 1.0$ (case A in Table 2). The long-dashed line is a model with $f_s = 0.1$ (case M). The short-dashed line is a model with $f_s = 10.0$ (case N). The value of $f_s$ determines the amount of differential rotation with depth present in our models of the present-day Sun.](image)
Fig. 10.—Angular velocity as a function of radius in the present-day Sun for three models differing only in the time scale for angular momentum transport. The solid line is the reference solar model with no change in the time scale for any mechanism (case A in Table 2). The short-dashed line is a model with the velocity estimate for the secular shear (only) multiplied by 0.25 (case G in Table 2). The long-dashed line is a model with the velocity estimates for all mechanisms multiplied by 0.25 (case D in Table 2). Note that if the case with the secular shear velocity reduced is calibrated to give the same surface rotation rate as the reference solar model, the difference between the two disappears entirely. This is not true for the case with all velocities reduced. The difference between the long-dashed line and the other two cases interior to 0.3 $R_{\odot}$ results from the change in the time scale for the GSF instability.

Fig. 11.—Surface lithium abundance as a function of time for three models differing only in the time scale for angular momentum transport $f_\alpha$. The solid line is the reference solar model with $f_\alpha = 1.0$ (case A in Table 2). The long-dashed line is a model with $f_\alpha = 0.1$ (case M). The short-dashed line is a model with $f_\alpha = 10.0$ (case N). The value of $f_\alpha$ has a strong influence on lithium depletion.
Fig. 12.—Angular velocity as a function of radius in the present-day Sun for three models differing only in the sensitivity of angular momentum transport to gradients in mean molecular weight $\mu$. The degree to which transport is inhibited by these gradients is proportional to $f_{\mu}$. The solid line is the reference solar model with $f_{\mu} = 1.0$ (case A, Table 2). The long-dashed line is a model with $f_{\mu} = 0.1$ (case P). The short-dashed line is a model with $f_{\mu} = 10.0$ (case Q). The degree of rotation in the inner 30% (by radius) of the present-day Sun is strongly affected by the value of $f_{\mu}$; rotation in the outer layers is not.

The changes due to differences in $R_{\text{crit}}$ are difficult to separate from changes in $f_{\omega}$. The exception is the slope of the rotation curve in the outer layers; a smaller critical Reynolds number corresponds to a slower increase in angular velocity with depth. Oscillation data will in principle allow us to determine this property of the Sun and thus fix the value of $R_{\text{crit}}$.

iv) Critical Reynolds Number $R_{\text{crit}}$

The critical Reynolds number, $R_{\text{crit}}$, affects both the velocity estimate for the secular shear (ES2) and its stability condition (the amount of differential rotation being tolerated is proportional to $R_{\text{crit}}$ and the velocity goes as $R_{\text{crit}}^{-1/2}$). The secular shear is the dominant mechanism in the outer layers of the Sun on the main sequence. The time scale for the secular shear is affected by $R_{\text{crit}}$; hence the rate of lithium depletion is affected by decreasing $R_{\text{crit}}$ in the same way as by an increase in $f_{\omega}$. A smaller $R_{\text{crit}}$ leads to more transport earlier and a greater depletion of lithium. The effect on the surface rotation velocity as a function of time is shown in Figure 14. The effect on the rotation curve of the present-day Sun is given in Figure 15. Note that changes in $R_{\text{crit}}$ affect the outer layers of the model more than the deep interior, unlike changes in $f_{\omega}$. The slope of the rotation curve in the outer 50% by radius of the solar model is roughly proportional to $R_{\text{crit}}$ because marginal stability against the shear is effectively enforced there.

So far we have only discussed the effects of changing one parameter at a time from the set of values in our reference solar model. In general, nonlinear feedback effects will result from changing several parameters simultaneously. For example, our models predict a spread in the surface lithium abundance at a given age due to a spread in the initial angular momentum; the size of this spread will depend on the properties of angular momentum loss. To construct the best solar model we must know how simultaneous changes in these parameters can affect our tests of the models.

The most important rotation parameters are: the initial angular momentum ($J_0$); the wind index ($N$), which determines the dependence of angular momentum loss on the surface rotation velocity; and $f_{\omega}$, which allows us to adjust the time scale for angular momentum transport. The properties of the surface layers are sensitive to $J_0$ only during pre–main-sequence evolution; the interior is always sensitive to $J_0$ to some extent. Similarly, the value of $N$ is most important during pre–main-sequence evolution; angular momentum transport on the main sequence is controlled by $f_{\omega}$. Because these parameters affect the models at different stages of evolution and in different regions of the interior we can separate their effects on the three
Fig. 13.—Mass fraction of $^{13}$C as a function of mass in the present-day Sun for three models differing only in the sensitivity of angular momentum transport to gradients in mean molecular weight $f_{\alpha}$. The degree to which transport is inhibited by these gradients is proportional to $f_{\alpha}$. The solid line is the reference solar model with $f_{\alpha} = 1.0$ (case A, Table 2). The long-dashed line is a model with $f_{\alpha} = 0.1$ (case P). The dashed-dotted line is a model without composition mixing due to angular momentum transport, which is indistinguishable from a model with $f_{\alpha} = 10.0$ (case Q).

Fig. 14.—Surface rotation velocity as a function of time for two models differing only in the critical Reynolds number at which the secular shear sets in ($R_{\text{crit}}$). The solid line is the reference solar model with $R_{\text{crit}} = 15,000$ (case A in Table 2). The dashed line is a model with $R_{\text{crit}} = 1000$ (case R). A decrease in $R_{\text{crit}}$ affects the rotation as a function of time in a similar way to an increase in $f_{\alpha}$.
Fig. 15.—Angular velocity as a function of radius in the present-day Sun for two models differing only in the critical Reynolds number at which the secular shear sets in \((R_{\text{crit}})\). The solid line is the reference solar model with \(R_{\text{crit}} = 15,000\) (case A in Table 2). The long-dashed line is a model with \(R_{\text{crit}} = 1000\) (case R). The short-dashed line is a model with \(R_{\text{crit}} = 1000\) calibrated to give the solar rotation period at the age of the Sun. A decrease in \(R_{\text{crit}}\) flattens the rotation curve from 0.5 \(R_\odot\) to 0.7 \(R_\odot\) and thus reduces the rotation interior to 0.5 \(R_\odot\).

major tests of our rotating models: the range of stellar properties due to a spread in initial angular momentum, the spin-down of the surface, and the rotation curve of the current Sun.

Our models show differences in composition mixing among otherwise identical models, resulting from differences in their initial angular momentum. The magnitude of the differences depends on both the rates of angular momentum loss and transport. The dispersion in the amount of Li depletion from a range in \(J_0\) has two phases. The surface convection zone retreats more slowly in a rapid rotator than a slow rotator during early evolution; this effect of rotation on the structure means that the rapid rotator will become more depleted in lithium than the slow rotator early in its life. Later, more angular momentum is transported into the surface convection zone from the interior in the rapid rotator than in the slow rotator. The associated mixing leads to a more severe depletion in the rapid rotator.

A larger wind index means that stars spin down faster and earlier. Therefore the structural effects of rotation in all models, including the rapid rotators, will be minimized early, and they will all arrive on the main sequence with about the same surface lithium abundances. On the main sequence, different amounts of angular momentum transport will still lead to different rates of lithium depletion; the total range in Li abundance, however, will be smaller. The opposite is true for a smaller \(N\). This is illustrated in Figure 16. The effect operates in the same sense when changing \(f_w\). This is because decreasing the efficiency of angular momentum transport leaves more angular momentum at the beginning of the main sequence in the rapid rotators. The magnitude of these differences is much less for changing \(f_w\) than for changing \(N\), however. The size of the spread in lithium provides what is probably the best test for the true value of \(N\). We discuss this point further in §IV.

The change in the CNO abundance profiles is also affected by the values of \(N\) and \(f_w\). More angular momentum loss early (high \(N\)) means that more angular momentum is transported when CNO processing is minimal; as a result, less is transported when CNO processed material is created later. The reverse is true for a smaller \(N\). On the other hand, as \(f_w\) is lowered the total amount of angular momentum lost from the nuclear burning region decreases and the time scale for composition transport increases. A smaller \(f_w\) therefore gives less transport of the CNO cycle elements.

While the surface rotation velocity depends only weakly on \(J_0\) after \(10^8\) yr, the sensitivity of the spin-down to \(f_w\) is different for different values of \(J_0\). When \(f_w\) is decreased sufficiently a plateau in the spin-down occurs as discussed in § III(e); in this plateau the surface layers are only spinning down slowly while the layers below the surface convection zone are torquing down more rapidly. Slow rotators begin with a weaker wind, and thus their surface layers are being spun down more slowly. Because the plateau is caused by the wind time scale being shorter than the time scale for angular momentum transport, it is less pronounced in slow rotators than in rapid ones. The feedback between the choice of \(N\) and \(f_w\) is minor because they are important in different phases of the star's life. The most
rapid rotators of a given spectral type in clusters provides a sample of stars with about the same \( J_0 \); their spin-down will be used to determine the best value of \( f_\sigma \) in § IV.

IV. THE BEST SOLAR MODEL

a) Choice of Parameters

We adjust the mixing length and helium abundance to match the observed solar radius and luminosity in the usual way. There are seven adjustable parameters in our solar model which are related to rotation, two of which can be fixed. The constant in the angular momentum loss formula \( (K_w) \) is adjusted so that the models have the solar angular velocity \( (3 \times 10^{-6} \text{ s}^{-1}) \) at the solar age. We demand that \( K_w \) remain within one order of magnitude of the value given in § II. The efficiency of composition transport relative to angular momentum transport, \( f_c \), is reduced until the lithium depletion at the surface relative to its initial abundance is a factor of 200; we take the initial abundance to be 3.3 on the logarithmic scale where the hydrogen abundance is 12.0. This leaves us with five free parameters: \( J_0, N, f_\sigma, f_\varphi, \) and \( R_{\text{crit}} \).

i) Choice of Optimal \( J_0 \)

If the massive stars are a reasonable guide to the properties of the low-mass stars we can expect a range in initial angular momentum \( (J_0) \) of about one order of magnitude among stars of the same mass and composition. Where does the Sun lie in this range? To answer this, we first constrain the other model parameters as discussed below. With the parameters of angular momentum loss and transport fixed, a range in initial angular momentum will lead to a range in the amount of composition transport in our models as discussed in § IIIc. For the solar model the most sensitive test of composition transport is lithium. We enforce a fixed amount of lithium depletion in our solar model; the amount of angular momentum transport depends on \( J_0 \). A star with a higher \( J_0 \) will lose more lithium; one with a lower \( J_0 \) will lose less. If the solar Li abundance is higher than average relative to other stars of solar mass, composition, and age, then the Sun began with less than the average angular momentum. Conversely, a low solar Li abundance relative to these stars is evidence for a higher than average \( J_0 \). The Sun appears to be depleted in Li at least as much as field G stars (Duncan 1981), which argues against a low solar \( J_0 \). A comparison with younger solar analogs is also instructive. In Figure 17 we plot the expected range in Li as a function of time for solar models given an order of magnitude spread in \( J_0 \) about the average value. In this diagram the solid line is the solar lithium depletion as a function of time and the dashed lines represent the expected range of Li depletion in solar analog as a function of time given the solar \( J_0 \). We also plot representative Li abundances for solar-type stars in the Hyades (Cayrel et al. 1984), NGC 752 (Hobbs and Pilachowski 1986a), and M67 (Hobbs and Pilachowski 1986b). A low solar \( J_0 \) case consistently has a range of Li abundances at the ages of these clusters that is lower than those observed. If the Sun has a high \( J_0 \), it is difficult to reproduce the observed depletion in the most Li-poor solar analog in NGC 752; for this reason, we
choose an average value for the solar $J_0$. However, all the other stars in this sample are consistent with either an average or a higher than average solar $J_0$; therefore we cannot confidenty exclude a higher than average value for the solar $J_0$.

The value of $J_0$ for the Sun is important for our models of other stars because we use the depletion of lithium in the Sun to set $f_c$, the time scale for composition transport due to rotationally induced mixing. A proper determination of $J_0$ therefore is essential if we wish to interpret lithium depletion in other stars. Also, the relatively low value of $f_c(0.046)$ required by our models may be a consequence of angular momentum transport by magnetic fields. Models of the internal solar magnetic fields generated by differential rotation will therefore be a sensitive function of the solar $J_0$ as well.

A larger sample of Li abundances for G stars in intermediate age clusters would provide us with a much tighter constraint on the solar $J_0$. Observations of Be in solar analogs would provide an independent test which would restrict the allowed range of $J_0$ even further. Because the main-sequence rotation is insensitive to $J_0$, these trace elements provide the best means of determining where our Sun fits in relative to other stars. 

**iii) Choice of Optimal N**

In the calibrated present-day solar model the rotation curve is virtually independent of the wind index. The value of $N$ used does affect other properties of the model, however. With a reasonable value of the constant in the wind law, $K_w$, we cannot spin down the surface to the observed solar rotation rate with a wind index less than 1.3. This leaves us with a possible range for $N$ between 1.3 and 2. Our tests for the best value of $N$ are (1) the slope of the observed main-sequence spin-down relation; (2) the size of the spread in Li as a function of age; and (3) the spin-down of young cluster stars as a function of mass. All three of these tests point toward a value of the wind index of $\sim 1.5$.

Skumanich (1972) found that the surface angular velocity of main-sequence low-mass stars is roughly proportional to $t^{-1/2}$. In our models the slope of this relationship depends on the wind index. Because a high $N$ case reaches a lower maximum surface velocity, it does not need to decrease as much as a lower $N$ case. As can be shown analytically (Kawaler 1988; Mestel 1984) the wind index which corresponds to a Skumanich law spin-down is $N = 1.5$.

An efficient wind ($N = 2$) will decrease the range in lithium produced by a given distribution of $J_0$ at all times. Models with $N = 2$ have a spread in Li which is systematically smaller than the observed value. A value less than or equal to 1.5 produces a spread consistent with what is observed.

The best test for the parameters describing angular momentum loss lies in the fact that stars of different mass spin down at different rates. A sequence of decreasing mass at constant age is

![Figure 17](image-url)

**Fig. 17.—Spread in surface lithium abundance as a function of time due only to a range in the initial angular momentum $J_0$. The solid line is the best solar model with $J_0 = 1.63 \times 10^{50} \text{ g cm}^2 \text{s}^{-1}$. The crosses are the observed solar analogs from clusters of known age (see text). The upper dashed line is a model with $J_0 = 5 \times 10^{49} \text{ g cm}^2 \text{s}^{-1}$ and the other parameters the same as the best solar model. The lower dashed line is a model with $J_0 = 5 \times 10^{50} \text{ g cm}^2 \text{s}^{-1}$. The range of possible Li abundance is between the dashed lines.**
a sequence of increasing depth of the surface convection zone. Lower mass stars will have a larger pool of angular momentum available to be lost to the wind at an early age. As a result, stars with thin convection zones, such as G stars, will spin down first. This will lead to a break in a plot of rotation velocity as a function of mass which is observed in young clusters (Stauffer, Hartmann, and Latham 1987; Stauffer and Hartmann 1987). On the pre-main-sequence the mass at which this break occurs will be primarily a function of N. In future work we will look at the spin-down of models of late-type stars at different ages for comparison with young clusters to more precisely determine the rate of angular momentum loss in young stars.

iii) Choice of Optimal $f_a$

A value of $f_a$ different from unity implies a time scale for angular momentum transport different from the estimates in § II. Because we consider three mechanisms for transport, there are really three different $f_a$'s. Here we discuss the constraints we can place on the time scales of the three mechanisms we consider: Eddington circulation, the GSF instability, and the secular shear. We take the time scale for the secular shear to be the surface spin-down of the models, while changes in the time-scale of the GSF instability affect the rotation of the inner 50% by radius of the models.

Eddington circulation is not the dominant mechanism in any phase of evolution in our models. As a result, we cannot constrain the time scale over which it operates because solar models are insensitive to its behavior. In massive stars (without a magnetic wind) Eddington circulation will be more important; their rapid rotation leads to a much shorter time scale for its operation. In addition, the absence of angular momentum loss at the surface will minimize differential rotation in these stars. This will lessen the importance of instabilities which require differential rotation to operate, such as the shear and GSF mechanisms. For the best solar model we do not modify the time scale estimate for Eddington circulation.

The GSF instability is important in rapidly rotating regions with strong differential rotation. Because it dominates angular momentum transport in the inner 50% by radius of the present Sun, the transport of CNO cycle elements and the rotation curve in the deep interior are the best tests for the time scale of the GSF instability. As the time scale for the GSF mechanism is increased, less CNO processed material is mixed up and the interior retains more angular momentum. Evolved stars will provide a constraint on the rotation curve in the deep interior, as would better oscillation data; for reasons discussed in § IVc, we do not modify the time scale estimate for the GSF instability for our best solar model.

The secular shear mechanism dominates angular momentum transport in the regions of the model with small $\mu$ gradients. The time scale over which the shear operates therefore has a strong influence on the spin-down of the models. The best constraint on the time scale of the shear lies in the behavior of rapid rotators in the Pleiades and Hyades star clusters. G stars in the Hyades rotate at almost the same rate as those in the Pleiades even though the Hyades is several hundred million years older (Stauffer, Hartmann, and Latham 1987; Stauffer and Hartmann 1987). A larger fraction of the K stars, on the other hand, rotate rapidly in the Pleiades than in the Hyades. In other words, stars in the Pleiades behave as if their surface convection zones are spinning down alone; stars in the Hyades act as if the entire star is spinning down (Kawaler 1988). This places stringent limits on the time scale for transport at the base of the convection zone; it may not be highly efficient in the Pleiades stars but must be effective in the Hyades stars. Models of the spin-down of both K and G stars will provide strong limits on how rapidly angular momentum transport may occur; this conclusion holds even if we have neglected mechanisms which may be important (such as magnetic fields). For the purposes of this paper we look only at the spin-down of G stars.

In Figure 18 we show the spin-down of rapid rotators for 2 values of $f_a$ (modified with respect to the secular shear only). Superposed on this diagram are the observed ranges of rotation velocities in $\alpha$ Persei, the Pleiades, and the Hyades for solar analogs (solid vertical lines) and all G stars (dashed vertical lines). Models with $f_a$ greater than 0.25 have a spin-down between the ages of the Pleiades and Hyades which is too rapid for solar analogs. Although this conclusion depends on the age assigned to the Hyades cluster, the spin-down of other G stars will be even more difficult to explain than that of solar analogs if the time scale for angular momentum transport in the base of the surface convection zone is too rapid. We therefore conclude that our estimate for the time scale of the secular shear mechanism is too short by a factor of at least 4. Because the time scale estimate for the shear is based on dimensional arguments, this is not unreasonable.

iv) Choice of Optimal $f_a$ and $R_{crit}$

We take the sensitivity to gradients in mean molecular weight to be the standard value ($f_a = 1$). The reason we do this is that $f_a$ does not affect the currently observable properties of the solar model. However, the sensitivity to $\mu$ gradients will have important consequences for the later stages of evolution. The timing and amount of dredge-up of CNO processed material in subgiant and giant branch stars will depend very strongly on how easily rotational mixing can penetrate gradients in mean molecular weight. In the future we plan to compute such models, in order to place stronger constraints on this parameter.

We take the critical Reynolds number to be 15,000. As discussed in § IIIe, we cannot separate the effects of a smaller $R_{crit}$ from a larger $f_a$. We therefore cannot exclude a smaller $R_{crit}$ in conjunction with a smaller $f_a$. The best test for $R_{crit}$ is the slope of the rotation curve in the outer 50% by radius of the Sun. Because the secular shear is strong, the rotation curve in this region is a curve of marginal stability against it. The GONG project should eventually provide this slope.

b) The Evolution of the Best Solar Model

The parameters for the best solar model are summarized in Table 4. The model progresses through a series of stages as it evolves; the rotation curve as a function of time and mass fraction is shown in Figure 19. Initially the model spins up as it contracts; the time scale for contraction is much less than the time scale for angular momentum loss or transport. By 10 million yr (curve b) the surface has spun-up enough for the angular momentum loss due to the wind to dominate the contraction. At this point the surface begins to torque down while the interior continues to spin-up. At 30 million yr (curve c) the model undergoes a series of changes. First of all, the convection zone attains its present-day value and stops receding. Contraction also effectively ceases and nuclear burning becomes the main energy source. Because the time scale for the wind has been less than the time scale for transport from the interior, a
shear at the base of the surface convection zone has developed. As the surface spins down, the time scale for the wind increases while the time scale for transport of angular momentum from below decreases as a result of the increasing shear. Between 30 and 50 million yr, the time scales for transport and loss become roughly equal. The model then readjusts to a situation in which the interior is being torqued down while the surface layers spin down only slowly. By an age of $\sim$120 million yr (curve d) the entire star is spinning down. The surface then resumes its spin-down. By an age of 440 million yr (curve e) the bulk of the angular momentum loss has already occurred and the outer layers rotating slowly.

Nuclear burning of hydrogen in the core prevents the deep interior from spinning down as a solid body. As the solar model evolves on the main sequence, it develops gradients in mean molecular weight which slow down and eventually halt angular momentum transport in the deep interior. The time scale for transport in the outer layers increases as they spin down. By 1.2 Gyr (curve f) angular momentum transport from the inner regions has been greatly reduced. The outer layers by this time exhibit an almost flat rotation curve; however, nonzero differential rotation at the base of the surface convection zone does persist to the present day. Once the core is no longer rapidly losing angular momentum, it is free to spin up as it converts hydrogen to helium and contracts. The outer edge of the core contains enough of a $\mu$ barrier to detach from the surface for transport purposes; this produces an inflection point in the rotation curve at $\sim$0.3 $R_\odot$. By an age of 4.7 Gyr (curve g) the inner 10% by radius of the model has spun up substantially and the weakly coupled core has grown.

TABLE 4
PARAMETERS FOR BEST SOLAR MODEL

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Reason for Choice of Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_o$</td>
<td>$1.63 \times 10^{29}$ g cm$^2$ s$^{-1}$</td>
<td>Solar lithium abundance relative to solar analogs</td>
</tr>
<tr>
<td>$K_u$</td>
<td>7.9</td>
<td>Adjusted to fit solar rotation rate at age of the Sun</td>
</tr>
<tr>
<td>$N$</td>
<td>1.5</td>
<td>Skumanich law</td>
</tr>
<tr>
<td>$f_t$</td>
<td>0.046</td>
<td>Adjusted to match solar lithium depletion</td>
</tr>
<tr>
<td>$f_m$</td>
<td>0.25 (secular shear only)</td>
<td>Spin-down of young cluster stars</td>
</tr>
</tbody>
</table>
remaining outer layers have spun down almost as a solid body. The rotation curve between the surface and the core is roughly a curve of marginal stability against the secular shear. Although the details of this picture depend on the model parameters, the models all undergo these general phases.

As the model evolves it also undergoes chemical mixing. The depletion of the surface Li as a function of time is shown in Figure 17; also indicated is the range of stellar Li for solar analogs, which covers one order of magnitude by the age of the Sun. The range in the surface Be abundance expected in other stars is shown in Figure 5. The range in Be is ~0.3 dex for stars older than 300 Myr. As an example of the future history of the Sun, we follow the evolution of our best solar model from 4.7 Gyr to the base of the giant branch. We illustrate giant branch rotation as a function of mass fraction and time in Figures 20 and 21.

Until core exhaustion at the center, the evolution on the main sequence after 4.7 Gyr proceeds in much the same fashion as before. The exhaustion of hydrogen at the center, however, precipitates a series of drastic changes in structure. A hydrogen burning shell outside a helium core develops. As in the non-rotating case, the model evolves to the giant branch and the depth of the surface convection zone increases dramatically. At the same time the core is rapidly contracting and the outer layers are expanding. This leads to interesting rotational effects.

As the model expands, rotation at the surface initially slows down. As high angular momentum material is dredged up by the advancing surface convection zone, this spin-down stops and the surface rotation velocity remains roughly constant until the convection zone reaches its maximum depth near the base of the giant branch. The convection zone also dredges up ³He and CNO processed material and dilutes the surface Li and Be. Transport is enhanced at the base of the convection zone; however, the time scale for evolution on the subgiant branch is shorter than the time scale for transport. As a result a shear develops at the base of the convection zone; a µ barrier also appears which slows transport at the base. Models with different \( J_0 \) have different amounts of CNO processed material mixed to the surface; we illustrate this by showing the \( ^{12}\text{C} / ^{13}\text{C} \) ratio as a function of \( T_{\text{eff}} \) on the subgiant branch for models differing only in \( J_0 \) (Fig. 22).

At the base of the giant branch, there is a rapidly rotating helium core with strong differential rotation across the hydrogen-burning shell which surrounds it. The degree to which angular momentum transport can penetrate the steep µ barrier of the hydrogen-burning shell as the star evolves up the giant branch will determine the rotation rates of horizontal branch and white dwarf stars. Details of the further adventures of a rotating solar model will be presented in a subsequent paper.
Fig. 20.—Angular velocity as a function of mass fraction and time in the core of the best solar model after main-sequence turnoff. Curve (a) is the run of angular velocity at main-sequence turnoff ($10.36 \times 10^9$ yr); (b) is at an age of $11.13 \times 10^9$ yr; (c) is at an age of $11.28 \times 10^9$ yr; (d) is at an age of $11.36 \times 10^9$ yr; (e) is at the base of the giant branch ($11.43 \times 10^9$ yr); and (f) is a giant branch model ($11.58 \times 10^9$ yr).

Fig. 21.—Angular velocity as a function of mass fraction and time in the envelope of the best solar model after main-sequence turnoff. Note the enlargement of the vertical scale as compared with Fig. 20. Curve (a) is the run of angular velocity at main-sequence turnoff ($10.36 \times 10^9$ yr); (b) is at an age of $11.13 \times 10^9$ yr; (c) is at an age of $11.28 \times 10^9$ yr; (d) is at an age of $11.36 \times 10^9$ yr; (e) is at the base of the giant branch ($11.43 \times 10^9$ yr); and (f) is a giant branch model ($11.58 \times 10^9$ yr).
EVOLUTIONARY MODELS OF ROTATING SUN

Fig. 22—$^{12}\text{C}/^{13}\text{C}$ abundance ratio as a function of effective temperature on the subgiant branch for two solar models differing only in their initial angular momentum $J_0$. The solid line is the best solar model with $J_0 = 1.63 \times 10^{50}$ g cm$^2$ s$^{-1}$. The model with $J_0 = 5 \times 10^{50}$ g cm$^2$ s$^{-1}$ (case I in Table 2) is indistinguishable from case B with respect to its $^{12}\text{C}/^{13}\text{C}$ abundance ratio. The long-dashed line is a model with $J_0 = 5 \times 10^{49}$ g cm$^2$ s$^{-1}$ (case G) and the same physics as the best solar model. An order of magnitude variation in $J_0$ produces changes in the $^{12}\text{C}/^{13}\text{C}$ abundance ratio which appear on the subgiant branch; these changes will become larger during the giant branch phase of evolution.

V. TESTS OF THE SOLAR ROTATION CURVE

Angular momentum transport in our models is remarkably efficient in reducing differential rotation in the solar interior; the possible range of rotation curves for models with transport is compared to a model with the same surface rotation velocity but without transport in Figure 23. The solar rotation curve for $r > 0.6 R_\odot$ is almost flat in our models. Interior to 0.6 $R_\odot$, however, the degree of differential rotation depends on our choice of parameters. The initial angular momentum and properties of angular momentum loss do not greatly affect the rotation curve. Therefore the parameters which determine the shape of the solar rotation curve are those which affect angular momentum transport: namely, the time scales of the instabilities and their sensitivity to gradients in mean molecular weight.

The sensitivity of the GSF instability and Eddington circulation to $\mu$ gradients ($f_\mu$) determines the rotation rate for $r < 0.3 R_\odot$ relative to the rate for $0.3 R_\odot < r < 0.6 R_\odot$ as per Fig. 14, § IIIe). The time scale for the secular shear is fixed by the spin-down of young cluster stars; Eddington circulation is not the dominant mechanism in any part of the Sun. This leaves the time scale for the GSF mechanism as a free parameter; the amount of differential rotation for $r < 0.6 R_\odot$ is inversely related to the time scale over which the GSF mechanism operates. We denote the factor by which we multiply the GSF velocity estimates by $f_{\text{GSF}}$. Values of $f_{\text{GSF}}$ greater than 10 affect the surface spin-down in a way which worsens agreement with observation. If $f_{\text{GSF}}$ is decreased below 0.1 the GSF mechanism is made less efficient for angular momentum transport than Eddington circulation. This leaves a wide range of possible rotation curves which satisfy all other constraints of the solar model (see Fig. 11, § IIIe for the allowed range).

a) Oblateness and Quadrupole Moment of the Sun

Our solar models have an oblateness consistent with the observed upper limit and thus a quadrupole moment consistent with general relativity. The primary contribution to the oblateness is in the outer layers of the Sun; all of our models rotate slowly in the outer 50% by radius of the Sun. For this reason the oblateness does allow us to distinguish between the rotating solar models. We also find a small change in the solar neutrino rate (of order a 10% reduction in the number of SNUs). This change is not sufficient to explain the solar neutrino problem. We will analyze the global properties of our rotating solar models in more detail in a future paper.

b) Preliminary Solar Rotation Curve from $p$-mode Oscillations

Preliminary inversion of the available $p$-mode data suggests a very flat rotation curve from the surface down to $\sim 0.2 R_\odot$ (i.e., Duvall, Harvey, and Pomerantz 1986), at which point the inversion procedure based on $p$-mode oscillation loses all sensitivity. It is worthwhile considering the implications of such a flat rotation curve on the choice of parameters for our solar evolutionary sequence, if indeed it is borne out by future observations of the Sun in the GONG program. Our model with $f_{\text{GSF}} = 10$ has a rotation rate twice the surface value at 0.5 $R_\odot$, rising to a factor of 3 at 0.4 $R_\odot$ and a factor of 6 by 0.3 $R_\odot$. Interior to 0.3 $R_\odot$ the shape is determined by the sensitivity of angular momentum transport to gradients in mean...
molecular weight. The slope of the rotation curve $0.5 R_\odot < r < 0.7 R_\odot$ is sensitive to the value of the critical Reynolds number ($R_{\text{crit}}$) at which the secular shear mechanism sets in. We have chosen a conservative value of 15,000 for our models; depending on the geometry of the experiment values as small as 1000 have been found in the laboratory. By choosing a smaller value for $R_{\text{crit}}$ and simultaneously requiring a longer time scale for the secular shear we can greatly reduce the amount of differential rotation in this region without violating our constraints on the surface behavior. This change would still leave our model with a rotation rate 4 times the surface value at $r = 0.3 R_\odot$.

If the Sun's rotation curve is indeed flat to 0.2 $R_\odot$ then there must be additional mechanisms not considered in our current model which reduces differential rotation even further. Such mechanisms are restricted by the requirement that they not affect the spin-down of young cluster stars or lithium transport. These constraints imply that if we are missing some mechanism for redistributing angular momentum it must operate over a time-scale comparable to the ones considered in this paper. The precise interpretation of the oscillation data is also in some dispute, and the inversion techniques become less reliable deeper in the Sun (see Hill 1987 for a discussion of the uncertainties in the rotation of the surface convection zone as an example).

c) Evolved Stars as Probes of the Solar Rotation Curve

As stars evolve past the main sequence their structure changes in a way that reveals their internal rotation. By studying evolved stars we can learn about the rotation curves of low-mass stars and by implication the solar rotation curve as well.

i) Subgiants

When a star exhausts the hydrogen at its center it leaves the main sequence and becomes a red giant. As it travels from the main sequence to the giant branch its radius increases dramatically; at the same time the surface convection zone deepens considerably. Therefore if stars rotate as solid bodies at all times the rotation period of low-mass subgiants should increase rapidly as their radius increases; for field subgiants this would appear as a large increase in period with decreasing temperature. If stars possess differential rotation below the main-sequence surface convection zone then as they expand they pick up higher angular momentum material from below. This will lessen the increase in period across the subgiant branch.

For the purposes of this paper we restrict ourselves to low-mass stars. Stars more massive than 1.5 $M_\odot$ begin the subgiant phase as rapid rotators because they don't experience angular momentum loss on the main sequence. As they develop a deep surface convection zone they experience substantial angular momentum loss. The extremely rapid increase in the rotation periods of massive subgiants is further evidence that angular momentum redistribution does not occur on a very short time scale in stars (Endal 1987).

Stars less massive than 1.5 $M_\odot$, on the other hand, begin the subgiant phase as slow rotators and further angular momentum loss across the subgiant branch is negligible in them. As a
result, the rotation periods of low-mass subgiants are dependent mainly on the rotation as a function of depth at main-sequence turnoff and only to a limited degree on the parameters for angular momentum loss. To see what different degrees of differential rotation will do to the periods of subgiants we considered the behavior of two cases. The first is a model which begins at the age of the Sun rotating as a solid body and in which we enforce solid body rotation at all times. If solid body rotation is not enforced at all times, a solar model with a solid-body rotation curve will produce rotation periods which are longer at all times on the subgiant branch. The second is our solar model with $f_{\text{GSR}} = 1$. The predicted subgiant branch rotation periods as a function of $\log T_{\text{eff}}$ of these two cases are shown in Figure 24. In this figure we also show the rotation periods of low-mass subgiants inferred from their chromospheric activity (Noyes et al. 1984). Solid body rotation predicts periods which are too long by more than a factor of 3. Our best solar model with $f_{\text{GSR}} = 1$ has periods very close to those which are measured for these stars. We therefore do not change $f_{\text{GSR}}$ for our best solar model; any change will worsen agreement with the subgiant data.

The rotation of these subgiants does not support the hypothesis that stars have solid body rotation enforced on a short time scale. The spin-down of young cluster stars is also very difficult to explain without differential rotation. There is no evidence for old low-mass stars rotating as rapidly as these subgiants must have been if they were rotating as solid bodies. Additionally, the stars in the sample have well determined parallaxes, which implies that they are not more massive evolved stars (which might be expected to rotate more rapidly). Although some of the subgiants in the sample are binaries (as are most stars) the reddest in the sample ($\delta$ Eri) is a single star.

The sample of subgiants studied by Noyes et al. (1984) is small, however, and it is conceivable that the relationship between chromospheric activity and rotation is different for these subgiants than for the main-sequence stars on which their calibration relied. Gilliland (1985) demonstrated that evolutionary effects could lead to a change in the period estimates of the subgiants in the Noyes et al. sample ranging from zero near main-sequence turnoff to a factor of 2 increase at the base of the giant branch. Including this effect decreases the disagreement between solid body rotation and the periods of the subgiants, but still leaves a significant discrepancy. Gilliland, however, adopted the Noyes et al. assumption that chromospheric activity is proportional to the rotation period divided by the convective turnover time. A number of authors (Simon and Fekel 1987; Basri, Laurent, and Walter 1985; Rutten 1987) have questioned this assumption. If the chromospheric activity is proportional to the rotation period times a function of color (only), then the period estimates of Noyes et al. for these subgiants are correct. There is no agreement in the literature on this question. For these reasons, direct measurements of the rotation periods of low-mass subgiants are needed. The differences between the rotation periods in solid body models and
those of models with differential rotation are large enough to allow us to see if there is evidence for solid body rotation in other low-mass stars.

We want to emphasize that the initial angular momentum and properties of angular momentum loss considered in this paper cannot produce large star-to-star variations in the rotation curve for a given mass, composition, and age among old main-sequence stars. It is possible, however, that other intrinsic stellar properties, such as the strength of internal magnetic fields, could differ from star to star and that the Sun is simply a peculiar star. We note, however, that all external measures of solar magnetic activity indicate that the Sun is normal among dwarfs of its age and composition.

ii) Horizontal Branch Stars

Rotation velocities have been observed in a number of horizontal branch stars (Peterson 1985). Because stars suffer mass loss as they ascend the giant branch, this places a lower limit on the amount of rotation present 0.2-0.3 M\(_\odot\) below the surface of the main-sequence progenitors of these stars. The velocities observed (of order 30 km s\(^{-1}\)) would seem to require either very rapidly rotating main-sequence progenitors or strong differential rotation which survives until at least the horizontal branch phase.

iii) White Dwarf Stars

The fact that some white dwarfs rotate at a measurable rate is difficult to reconcile with solid body rotation at earlier times in the evolution of low-mass stars. Pilachowski and Milkey (1987) measured rotation velocities of \(\sim 20\) km s\(^{-1}\) in six of the 20 white dwarf stars that they observed. A solar model rotating as a solid body would eventually become a white dwarf with a surface rotation velocity \(\sim 5\) km s\(^{-1}\) if angular momentum was conserved locally. If it was required to rotate as a solid body through the asymptotic giant branch phase of evolution, the rotation velocity it would have as a white dwarf would then be much less than 1 km s\(^{-1}\). Our best solar model would have a surface rotation velocity as a white dwarf \(\sim 75\) km s\(^{-1}\) if angular momentum was conserved locally. Although this is more rapid than the stars Pilachowski and Milkey observed, the star has many opportunities to spin down between the main sequence and the white dwarf phase and no easy ways to spin up.

VI. SUMMARY

a) Uniqueness of Rotating Stellar Models

Rotation has been invoked to explain many problems in stellar evolution. However, in the absence of self-consistent models it has been difficult to establish its precise role. This has not been through neglect. Consideration of the effects of rotation introduces several new parameters not present in standard stellar evolution. Unlike standard stellar evolution, the initial conditions are important for rotating models; these initial conditions are connected to the complex star formation process. The process of angular momentum loss is intertwined with stellar magnetic fields (a subject more poorly understood than rotation). And angular momentum redistribution depends on the uncertain properties of astrophysical fluid mechanics. In recent years, however, observers have measured rotation velocities and trace element abundances for a number of stars, two key properties that allow us to compare theoretical models of rotating stars with observations.

Observations of relatively low rotation velocities in T Tauri stars (Stauffer and Hartmann 1987) imply that protostars shed most of their angular momentum very early in their lifetime. Furthermore, angular momentum loss places a strong upper limit on the total angular momentum a low-mass star can possess when it reaches the main sequence. This is confirmed by the absence of rapidly rotating, moderately old, low-mass main-sequence stars. Rotating stellar models can therefore be computed starting in the pre-main-sequence without needing to know the details of star formation.

Similarly, we can empirically constrain angular momentum loss. Rotation velocities of solar analogs of different ages are fitted well by a relatively simple parameterization based on extrapolating the observed solar rate of angular momentum loss back in time (see also Kawaler 1988). In the future we plan to examine the spin-down of young cluster stars of different ages and masses to place stronger constraints on angular momentum loss.

The rotation rate of the surface convection zone of a low-mass star results from the influx of angular momentum from the interior and the outflux of angular momentum lost to the stellar wind. For very young stars, angular momentum inflow is negligible, and we can calibrate the rate of angular momentum loss in these stars. For a given wind law, the rotation of older stars allows us to determine the time scale for angular momentum transport at the base of the surface convection zone. We find that the observed rotation rates of G stars in the Hyades and Pleiades require a time scale for angular momentum transport at the base of the surface convection zone which is longer than the time scale for the wind but shorter than the evolutionary (nuclear) time scale. Models of stars of different masses will enable us to place stronger limits on this time scale.

The rotation of evolved stars also allows us to directly probe rotation as a function of depth in main-sequence stars. As stars evolve from the main sequence to the giant branch, their surface convection zone deepen and the rotation of the matter that resided below the surface convection zone on the main sequence is revealed as the depth of the surface convection zone increases. Due to mass loss, the rotation rates of horizontal branch and white dwarf stars directly probe the angular momentum in regions of the star which were hidden below the surface on the main sequence. The properties of evolved stars therefore serve to constrain the time scale for angular momentum transport in the interior of stars. Our models are consistent with the rotation periods of subgiants inferred from their chromospheric activity. Because the precise relationship between rotation period and activity is a matter of debate, a more quantitative answer will require direct observations of rotation periods of subgiants. Measurements of this type, as well as measurements of rotation periods of more evolved stars, will allow us to empirically determine the time scale of angular momentum transport throughout the interior of stars.

b) Properties of Rotating Stellar Models

i) Effects on Stellar Structure

Rotation changes the structure of stars directly (by altering the gravitational potential) and indirectly (by mixing of fresh hydrogen into the core by rotationally induced instabilities, for example). We find that the indirect effects are more important than the direct effects; the direct effects of rotation in our model of the present-day Sun are small. In massive stars the effect will be even more dramatic because of much lower rate of angular momentum loss in these stars. This change in the
main-sequence lifetime may lead to a change in the turn-off masses ascribed to cluster stars. It is not clear that the cluster ages will differ from those of isochrones based on nonrotating models, however (see Law 1981). By affecting the lifetime by different amounts in different stages of evolution, rotating models may also have a luminosity function different from nonrotating models. We plan to investigate these effects in the future.

ii) Rotationally Induced Mixing

Rotationally induced mixing is a natural explanation for the observed depletion of Li and Be in low-mass stars. We find that models calibrated to reproduce the solar lithium depletion also produce the observed amount of lithium depletion in solar analogs. Our solar model also has a beryllium depletion of slightly more than a factor of 3, in agreement with the observed solar depletion of a factor of 2–5 (Molaro and Beckmann 1984; Ross and Aller 1974; Reeves and Meyer 1978). These light elements serve as excellent probes of the amount of rotationally induced mixing a star has undergone.

Our models also explain how stars of the same mass, composition, and age can have different surface abundances. The initial angular momentum is a fundamental physical property which is observed to vary from star to star in the massive stars. By the age of the Sun, our solar models have a rotation curve (as a function of depth) which is almost independent of the initial angular momentum. The amount of rotationally induced mixing is proportional to the amount of angular momentum transport. As a result, the amount of rotationally induced mixing is proportional to the initial angular momentum.

This poses problems for the use of lithium as an age indicator. Although on average a star with a lower lithium abundance will be older than one with a higher lithium abundance, the age at which a given star reaches a given lithium abundance depends sensitively on that star's initial angular momentum. For stars of known age, mass, and composition, however, lithium and beryllium are excellent indicators of the initial angular momentum.

Rotationally induced instabilities also mix CNO processed material closer to the surface than is possible in a nonrotating model. Our models do not have surface CNO anomalies on the main sequence; however, changes in the surface $^{12}\text{C}/^{13}\text{C}$ and C/N abundance ratios begin to appear in the subgiant branch and become steadily larger on the giant branch. This is in accord with observations. Because the amount of mixing is proportional to the initial angular momentum, stars at the same position on the giant branch can have different $^{12}\text{C}/^{13}\text{C}$ and C/N ratios. We will compute giant-branch models to quantify this effect; these models will also allow us to determine the sensitivity of rotational instabilities to gradients in mean molecular weight.

c) Properties of the Rotating Sun

Our rotating solar models have an oblateness consistent with the observed upper limit. This is a consequence of a general feature of all of our solar models: namely, that they all rotate slowly in the outer layers where the contribution to the oblateness is greatest. We can calibrate the angular momentum loss formula by requiring that our models have the solar rotation rate at the solar age. We cab also adjust the efficiency of composition transport relative to angular momentum transport such that the Sun has the observed lithium depletion. To determine the other parameters of the rotating Sun, it is necessary to turn to observations of other stars.

As stated earlier, lithium is an indicator of a star's initial angular momentum. Relative to solar analogs the solar lithium abundance is no more than average, arguing for an average solar initial angular momentum (or higher). Angular momentum transport at the base of the surface convection zone is controlled by the time scale for the secular shear instability. The spin-down of young cluster stars implies a time scale for the secular shear 4 times longer than the time scale estimate from ES2.

The amount of differential rotation with depth in the interior of our solar models depends on the time scale of angular momentum transport in the interior and the sensitivity of the angular momentum redistribution mechanisms to gradients in mean molecular weight. Models of CNO anomalies in giant branch stars will provide the best test for the sensitivity to gradients in mean molecular weight, which will affect the amount of differential rotation for $r < 0.3 \, R_\odot$. The amount of differential rotation between 0.3 $R_\odot$ and 0.5 $R_\odot$ is controlled by the time scale for the GSF instability. Our solar models have rotation curves which are nearly flat to 0.6 $R_\odot$ and rise to between 4 and 15 times the surface rate by 0.2 $R_\odot$. Inversion of the available $p$-mode oscillation data suggests a nearly flat rotation curve to 0.2 $R_\odot$, which would argue for a short time scale for the GSF mechanism.

However, the rotation periods of subgiant branch stars as estimated from their chromospheric activity are difficult to explain without differential rotation in their main-sequence progenitors. The precise relationship between rotation period and activity is debatable, with an uncertainty in the derived rotation periods of up to a factor of 2. It is therefore difficult to constrain precisely the internal rotation of low-mass stars on the main sequence from the rotation of these subgiants. For this reason, direct measurements of low-mass subgiant rotation periods are needed. Such measurements will provide a crucial test of rotating stellar models, and will therefore shed light on the internal solar rotation. However, maintenance of solid body rotation on the main sequence would result in subgiant rotation periods that are much longer than any published rotation period estimates for these low-mass subgiants. The rotation of evolved stars in general (as well as the spin-down of young cluster stars) provides no evidence for solid body rotation in low-mass stars. The extremely rapid spin-down of high-mass subgiants (Gray and Nagar 1985) is also easy to explain if the time scale for angular momentum transport is longer than the time scale for angular momentum loss. Some might argue that there is some internal property of the Sun (such as strong internal magnetic fields) which makes it peculiar relative to other solar analogs. We feel that there is no observable surface property of the Sun which would suggest that it is peculiar.

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