form of azimuthal rolls, then the interaction between the torsional oscillations and the convection would be mitigated if each roll were to drift intact as a whole. We wish to point out that the treatment of torsional surface waves as a modulation of the rotation due to giant cells imposes a very stringent constraint on the distribution of rotational velocity in the convection zone. Our hypothesis enables us to relax this constraint and perhaps even helps us understand why it is azimuthal rolls that develop, rather than cells of some other kind.

The question of whether the oscillations are indeed related to the 11-yr solar cycle needs further study. In particular, the phenomenon of parametric resonance is of interest. Since convectively neutral structures are highly sensitive to changes in the difference between the true and adiabatic temperature gradients, and since these changes inevitably occur in the presence of torsional oscillations, it is possible for global oscillations (of angular frequency $\omega$) to be excited if the resonance condition $2\omega = \omega_R$ is satisfied, where $\omega$ is the frequency of the global mode has twice the period of the initial torsional mode. This effect might possibly be responsible for the observed 22-yr cycle.

In the case of massive stars with a magnetic field, circumstances favoring the excitation of torsional-wave instability could develop during the stage of hydrogen depletion if the contracting convective core were to become surrounded by a zone with a superadiabatic temperature gradient.

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The internal rotation of the sun from helioseismological data

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Analysis of the Duvall–Harvey and Brown–Morrow measurements of the rotational frequency-splitting among 5-min solar acoustic-oscillation modes yields the angular velocity of the solar interior

($0.06 < r/R < 0.9$) near the equatorial plane as well as the latitudinal differential rotation ($0.4 < r/R < 0.9$), which is substantially weaker in the radiative than in the convection zone. The available helioseismological data are not yet adequate to establish the radial angular-velocity gradient in the latter zone. An initial data-reduction method is proposed which may give a more accurate determination of the sun's internal rotation.

1. INTRODUCTION

Rapid progress is now being made in acquiring and improving the observational data on the properties of the five-minute acoustic modes of solar oscillation, and as a result it has become feasible to solve the inverse problems of helioseismology to evaluate certain parameters describing the internal structure of the sun. For example, the distribution of the speed of sound in the solar interior has been established from the measured frequencies of oscillations identified as nonradial acoustic modes. The peaks in the acoustic-wave power spectrum have been found to exhibit fine structure—a frequency splitting attributable to the rotation of solar material. One can take advantage of the measured frequency differences between the fine-structure peaks to determine how the angular velocity depends on radius and latitude.

The first reasonably detailed information on this rotational frequency-splitting was reported in
1984 by Duvall and Harvey. In light of these data the radial dependence of the angular velocity in the equatorial plane was represented as a piecewise-constant function; in particular the sun’s core turned out to be rotating rapidly. The latest findings indicate how the angular velocity varies not only with radius but also with latitude.

I have sought to refine the equatorial angular-velocity profile in the solar interior based on the Duvall–Harvey results, and to determine the latitudinal differential rotation from Brown and Morrow’s data. The analysis outlined below relies on the WKB approximation for describing the acoustic modes theoretically and on the Abel inversion for solving the inverse problem. A fuller account of the solution method, the results of trial calculations, and an analysis of the observational data are being published elsewhere.

2. METHOD FOR SOLVING THE INVERSE PROBLEM

In a spherical coordinate system the wave pattern of the free oscillations has a spatial structure which can be described as a product of a function of the radius r and a spherical harmonic: $\tilde{a}_n(r) Y_l^m(\theta, \phi)$, where $Y_l^m(\theta, \phi) = (2\pi)^{-1/2} P_l^m(\cos \theta) \exp(i m \phi)$, with $P_l^m(\theta)$ denoting a normalized Legendre polynomial. The eigenfunction $\tilde{a}_n(r)$ oscillates in space and has n modes.

The angular index ("degree") l specifies the total horizontal component of the wave number at radius r; thus $k_h = L/r$, where $L = \sqrt{l(l+1)}$. The azimuthal index ("order") m is proportional to the projection of $\mathbf{k}_h$ on the unit vector $\hat{\mathbf{e}}_z$; thus $k_\phi/k_h = m/L$, taking 2 |m| + 1 values from $-l$ to +l. The three integer parameters $n$, l, m govern the eigenfrequencies $\nu_n$, $\nu_l$, $\nu_m$ of the modes.

With this definition of the modes, the azimuthal order m will depend on the choice of coordinate system. If the sun were perfectly spherically symmetric so that all directions were equivalent, the frequencies $\nu_n$, $\nu_l$, $\nu_m$ would be independent of m. Solar rotation, however, breaks the spherical symmetry and causes modes with different m to have slightly different frequencies. The frequency shift is caused mainly by a simple drift in the wave pattern of the oscillations.

Let us take the polar axis of the spherical coordinate system to coincide with the sun’s rotation axis. Then the m < 0 modes, propagating in the same sense that the sun is rotating, will accelerate, so their frequencies will increase. Correspondingly the frequencies of the m > 0 modes, propagating counter to the rotation, will diminish. Rotation will leave the frequencies of the m = 0 modes unchanged.

The frequency shifts $\delta \nu_{n,l,m} = \nu_{n,l,m} - \nu_{n,l,0}$ will be proportional to $m <\nu>$, where $<\nu>$ is the mean value of the sun’s angular velocity $\Omega(r, \theta)$, suitably weighted over the domain of propagation of each mode. (Gorkin and the author have given a qualitative description of the domains of propagation for all types of hydrodynamic modes.)

In the case of the acoustic modes the propagation domains are annular zones symmetric about the equator. In latitude these zones span a distance $\Delta \theta \approx \pi - 2 \arcsin(\mid m \mid /L)$, so that as the azimuthal order m increases the propagation zone will narrow toward the equator. In the radial direction the upper boundary of the propagation domain will be located near the surface (at $r = R$); the lower boundary, at the turning-point radius $r = r_t$, is determined by the criterion of total internal reflection of acoustic waves: that the horizontal phase velocity $\omega /k_h$ be equal to the local value of the speed of sound $c$.

Accordingly by measuring the shifts $\delta \nu_{n,l,m}$ for different modes one can obtain weighted mean values of the angular velocity in different batches of solar material. From these values the inverse problem of helioseismology is to recover the angular-velocity distribution inside the sun.

The radial extent of the region within which the angular velocity can be determined will depend

FIG. 1. a) The frequency-splitting parameter $\tilde{\nu}_l$, normalized to the angular velocity $\Omega_0$ at the solar surface, as a function of the turning-point radius $r_t$ of the acoustic modes, according to Duvall and Harvey’s data. Horizontal bars, averaging intervals for preliminary processing; vertical bars, measurement errors; curve, the resultant spline. b) Equatorial angular velocity $\Omega_0(r)$, normalized to $\Omega_0$, as a function of radius. Solid curve, the profile obtained by solving the inverse problem; dashed curves, the spread in $\Omega_0$ obtained in a simulation experiment for 20 different noise realizations.

FIG. 2. Profiles similar to those of Fig. 1 after excluding the two points for $\ell = 90$ and 100.
on the range of observed oscillation modes with respect to the parameter \( i \): modes with large \( i \)-values will be trapped in the sun's surface layers \( (r_t \approx R) \) for these modes), while modes with small \( i \) will penetrate deep into the interior \( (r_t \ll R) \). In particular, for the Duval-Harvey data\(^{*} \) analyzed below, \( i \approx 1-100 \), corresponding to a radial internal \( r_t/R \approx 0.06-0.9 \) to which the oscillations will penetrate. For the Brown-Morrow data\(^{2} \) this range is narrower: \( i \approx 15-99 \), so that angular velocities can be established only in the more limited interval \( r_t/R \approx 0.4-0.9 \).

If we write the coordinate dependence of the angular velocity in the form

\[
\Omega(\rho, \vartheta) = \Omega_0(\rho) + \Omega_1(\rho) \cos \vartheta + \Omega_2(\rho) \cos^2 \vartheta
\]

and consider the acoustic modes in the WKB approximation,\(^{4} \) we obtain a rotational frequency-splitting\(^{1} \)

\[
\delta \nu_{n,i,m} = -m \sum_{l=0}^{1} \delta h_{n,i,0,m} M_{k,1,i,m}
\]

where

\[
\delta h_{n,i,0,m} = \frac{2}{\bar{h}_{n,i}} Q_{n,i}(\rho) \Omega_{n,i}(\rho) \frac{\partial \bar{h}_{n,i}(\rho)}{\partial \rho},
\]

\[
Q_{n,i}(\rho) = \frac{1}{m^2} \left[ 1 - L_i^2 \right] / (2 \Delta v_{n,i})^2 / \left( 2 \Delta v_{n,i} \right)^{1/2},
\]

\[
M_{k,1,i,m} = \frac{m}{\rho} \left( P_i \right)^m \cos^5 \vartheta \sin \vartheta \partial \vartheta \partial \vartheta \approx 2 \left[ 1 - m^2 / L_i^2 \right]^{-1}
\]

for \( k = 0, 1, 2 \).

Analogous expressions have previously been derived both for the case\(^{3} \) \( k = 0 \) and in the general case\(^{1} \) \( \Omega = \bar{h}(\rho, \vartheta) \). The accuracy of the WKB approximation improves if one sets\(^{2} \) \( L = \frac{1}{2} \).

Equation (3) represents an integral equation for the functions \( \Delta h_{n,i}(\rho) \), which can be evaluated in terms of the parameters \( \bar{h}_{n,i}, \nu_{n,i} \) determined from Eq. (2) through preliminary processing of the observational data. With the change of variables \( x = L/(2 \nu_{n,i}) \), \( y = \nu_{n,i} \), Eq. (3) reduces to the standard Abel equation

\[
2 \int_{0}^{y} \frac{y \varphi(y)}{(y^2 - x^2)^{1/2}} dy = f(x),
\]

where

\[
\varphi(y) = 1 / 2 \Delta h_{n,i}, \quad y = d \ln r / d \ln \nu.
\]

By solving Eq. (6) we can use Eqs. (7) to find \( \Delta h_{n,i} \) as a function of \( \nu \), and accordingly, since \( \nu \) is a unique function of \( r \) for standard solar models, the \( \Delta h_{n,i} \) profile that we are seeking. The measurable parameter \( \bar{h}_{n,i} \) appearing on the right in Eq. (6) will be defined for discrete values of the variable \( x \). In practice it is convenient to treat \( x \) as a continuous quantity and to supplement \( \Delta h_{n,i} \) by interpolation throughout the range of continuous \( x \)-values. We have approximated the function \( \Delta h_{n,i}(x) \) by cubic splines,\(^{1} \)

To find \( \varphi(y) \) we have applied an algorithm\(^{10} \) which utilizes a modification of the Abel inversion that does not require differentiation of \( f(x) \):

\[
\varphi(y) = \frac{1}{\pi} \left[ \frac{f(y) - f(b)}{(b^2 - y^2)^{1/2}} - \frac{b}{y} \right] \int_{0}^{b} \left[ f(x) - f(y) \right] dx.
\]

An approximate analysis of the accuracy of the resulting solution has been carried out by means of a "quasirealistic experiment"\(^{12} \) entailing repeated calculations as random noise is introduced into the initial data.

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A. G. Kosovichev
3. RESULTS OF CALCULATIONS

The Duvall–Harvey data comprise average values for the rotational frequency–splitting of acoustic modes whose azimuthal order \( m = \pm \ell \) in the 2.1–3.7 mHz frequency range for \( \ell = 1–100 \). Averages were taken over all the observed modes having equal degrees \( \ell \). Altogether 37 values for the drift \( \delta \nu_g \) were obtained from 180 modes. Equation (5) with \( m = \pm \ell \) gives \( M_{\pm \ell, m, -m} = 1 \) and \( M_{\pm \ell, m, m} = 0 \) for \( k = 1, 2 \). Hence the frequency splitting for these modes depends chiefly on the equatorial angular velocity \( \Omega_0(r) \). This situation comes about because the propagation domain for modes with \( |m| = \ell \) occupies a narrow latitude belt near the equator.

Figure 1a plots the normalized values obtained for the splitting parameter \( \delta \nu \) versus the turning-point radius \( r_1 \) of the corresponding modes. Horizontal line segments indicate the intervals of averaging for the preliminary data processing; vertical bars, the measurement errors. The continuous curve represents the spline derived from these results.

The \( \Omega_0(r) \) profile obtained by solving the inverse problem is shown in Fig. 1b by a solid curve; the dashed curves demonstrate the spread in the angular–velocity values obtained in a simulation experiment entailing 20 different noise realizations. This solution displays a rise in angular velocity in the central core of the sun, an adjoining zone of slower rotation, and beyond a local peak as \( \Omega_0(r) \) rises gradually toward the surface. Such an \( \Omega_0(r) \) profile is qualitatively consistent with what Duvall et al.\(^5\) established in the class of piecewise-constant functions, but our solution exhibits small-scale variations in the convection zone that exceed the random measurement errors.

As is apparent from Fig. 1a, these variations are caused mainly by the two \( \Omega_0 \) values for \( \ell = 90 \) and 100 (the points at the far right). Duvall et al.\(^5\) have in fact drawn attention to the significant departure of these two values from the neighboring ones, perhaps because of systematic observational error. This interpretation is supported by our analysis of Brown and Morrow's observations\(^7\) (see below). If the two exceptional \( \Omega_0 \) values are excluded, the behavior of \( \Omega_0(r) \) in the sun's outer envelope becomes quite smooth (Fig. 2): the small-scale variations at \( r/R \geq 0.5 \) are now within the error margin.

There is also a minor local maximum in the angular velocity at \( r/R \approx 0.35 \) (Fig. 2b). This feature derives chiefly from the large \( \Omega_0 \) value observed for \( \ell = 11 \), although Libbrecht's observations\(^7\) do not confirm that value.

Brown and Morrow's data\(^7\) are presented in the form of coefficients \( A_k \) \((k = 1, 2, \ldots, 5)\) averaged over certain groups of modes, as obtained by expanding the measured quantities in a series of Legendre polynomials \( P_k \):

\[
\delta \nu_{m, -m} = \sum_{k=1}^{5} A_k P_k (m/L).
\]

The measurements here span the oscillation modes in the range \( \ell = 15–99 \), which are divided into 14 groups comprising all the modes in the 2.5–3.5 mHz frequency range for three successive \( \ell \)-values.

Comparing the expressions (8) and (2), we readily find that

\[
\Omega_0 = A_1 + A_2 + A_3, \quad \Omega_1 = -5A_4 - 14A_5, \quad \Omega_2 = 21A_5.
\]

Thus once the parameters \( \Omega_0 \) are known we can evaluate the \( A_k \) and determine the radial profiles \( \Omega_0(r) \) in the rotation law (1). However, when computations are made from Eqs. (9) the measurement errors in the \( A_k \) are summed, and the parameters \( \Omega_0 \) obtained are subject to large uncertainty, especially the latitudinal differential-rotation parameters \( \Omega_1(r), \Omega_2(r) \).

Accordingly, along with Eq. (1) let us represent the angular velocity in the form

\[
\Omega(r, \theta)/2\pi = A_1(r) + [A_1(r) P_2 (\cos \theta)] + A_2(r) P_2 (\cos \theta)/\sin \theta,
\]

in which the coefficients \( A_2(r) \) are directly related to the measured quantities \( A_k \) by expressions like (3). Like \( \Omega_0(r) \), \( \Omega_1(r) \), \( \Omega_2(r) \), the functions \( A_1(r), A_2(r) \) characterize the latitudinal differential-rotation profile in the solar interior. At the surface, spectroscopic measurements\(^1\) give the values \( A_1 S = 435 \) nHz, \( A_2 s = 21 \) nHz, \( A_3 s = -4 \) nHz.

Figure 3 shows the results obtained when Eqs. (9) are utilized to determine the equatorial rotation from the Brown–Morrow measurements. Compared with the Duvall–Harvey results, the former has a considerably smoother \( \Omega_0(r) \) profile with no abrupt changes in the convection zone. Nevertheless the two sets of results agree qualitatively: the angular velocity diminishes gradually into the solar interior until the level \( r/R = 0.5–0.6 \) is reached, after which it increases again. Perhaps the smoother \( \Omega_0(r) \) profile when the Brown-Morrow data are used reflects the larger intervals for averaging the measurements with respect to the parameter \( \ell = L/2 \pi \nu_n, \ell \).

Neither the Brown-Morrow nor the Duvall–Harvey data can yield the angular-velocity gradient in the convection zone—an important parameter for solar dynamo theories.

We have performed analogous calculations for \( A_1, A_2, A_3 \) separately. In Fig. 4 we present the initial data and the results for \( A_3 \). The measured values \( A_3 \) thus far are too inaccurate to indicate with any confidence how the latitudinal differential rotation varies with depth. Still, one can distinctly see the decrease in \( A_3 \) in passing from the convection zone to the radiative zone at \( r/R = 0.7 \), signifying that the angular velocity varies more weakly with latitude in the deep layers of the sun.

In closing we would point out that the uncertainty involved in recovering the internal rotation law derives in part from the large span of the intervals over which the observational data are averaged. The uncertainty can be reduced if in the preliminary reduction the oscillation modes are grouped not separately by frequencies \( \nu_n, \ell \) and indices \( \ell \) but according to the parameter \( L/\nu_n, \ell \). An asymptotic approach enables the angular velocity of internal rotation to be evaluated not only in the parametric form (1) or (10) but also in the most general form \( \Omega(r, \theta) \), for which the rotational frequency-splitting has to be measured as a function of two variables: \( L/\nu_n, \ell \) and \( L/m \). In that event the inverse problem would reduce to the solution of a two-dimensional Abel equation.\(^9\)


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Excitation-potential dependence of solar Fe I line widths

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The measured halfwidths and equivalent widths of unblended neutral-iron lines in the solar spectrum are shown to correlate with the lower-level excitation potential.

1. INTRODUCTION

Boyarchuk and Savanov\(^1\)\(^2\) point out that when the solar iron abundance is determined from the ion lines, it is 50% higher, on the average, than the abundance indicated by the lines of neutral iron atoms. What the source of the discrepancy may be remains an open question.

To seek an answer I have compared all 22 lines of the Fe II ion that occur in a list of unblended solar lines\(^3\) based on the Liege photometric atlas\(^4\) against the Fe I lines in the same list that have approximately the same wavelengths \(\lambda\) and central depths \(d\). In turns out that the full halfwidths \(h\) of the Fe II lines are 10-20% greater than the corresponding Fe I halfwidths. One gets the impression that the Fe II lines have been further broadened by some mechanism which is overlooked in the traditional procedure for determining element abundances by fitting theoretical equivalent widths \(w\) to the measurements.

Blackwell et al.\(^5\)\(^6\) have drawn attention to the dependence of their computed iron abundances upon the excitation potential \(\chi\) of the lines: as \(\chi\) increases, the abundance \(A_{Fe}\) derived from the corresponding lines becomes higher, except in fact for the anomalous behavior of lines having \(\chi = 2.2\) eV. Furthermore, the role of the damping constant \(\gamma\) in the broadening of lines tends to increase with \(\chi\), presumably because the theoretical damping constant \(\gamma_d\) due to van der Waals interaction forces does so.

But neither the papers of Blackwell et al.\(^5\)\(^-\)\(^7\) nor any others yet published examine what might cause the \(\chi\)-dependence of the derived abundance \(A_{Fe}\). In view of this situation as well as the natural circumstance that in the Grotrian diagram the ion energy levels lie above those of the atoms, we have sought to ascertain whether the observed halfwidths \(h\) of the Fe I lines depend on their excitation potentials. In scanning the literature we have yet to come across a definitive answer. (It is worth emphasizing that we refer to the observed halfwidths, not those computed theoretically.)

That the matter should still be undecided is not surprising, for one requires a rather large number of clean, unblended lines with different measured parameters \((h, d, \chi)\) and different \(\chi\)-values. The list of unblended lines compiled by Rutten and van der Zalm\(^8\) now provides such a sample.

![Graph](https://via.placeholder.com/150)

**FIG. 1.** Correlation between the observed relative halfwidth \(h/\lambda\) of solar Fe I lines having a central depth \(d\) \(\geq 0.3\) and their excitation potential \(\chi\).