RESONANCE ABSORPTION OF SOLAR p-MODES BY SUNSPOTS

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ABSTRACT

Braun, Duvall, and LaBonte have reported recently that the power in outgoing p-modes in the vicinity of sunspots is significantly less than the incoming power. We consider the possibility that the energy deficit is due to resonance absorption, which occurs when the sunspot boundary has a nonzero thickness. We use a simple planar analysis to examine the conditions required for resonance absorption, and to estimate the absorption coefficient. We find that resonance absorption can be significant under certain circumstances: the incoming waves must approach the sunspot magnetic field at angles less than about 40°, the sunspot density must be no more than half the exterior density, and the ratio of specific heats outside the sunspot must be substantially less than 5/3. The first of these conditions might be satisfied if the sunspot magnetic field is nearly vertical near the solar surface, since the p-modes propagate nearly vertically there. But, on the whole, resonance absorption probably cannot explain the substantial loss of p-mode power observed by Braun, Duvall, and LaBonte over a substantial range of frequency and azimuthal order. Nonetheless, some waves can be significantly absorbed by this process, and it might have observable consequences. One possibility is that the running penumbral waves are produced by resonant absorption of p-modes at the subsurface sunspot boundary.

Subject headings: Sun: oscillations — Sun: sunspots

I. INTRODUCTION

The possibility of probing the interior structure of the Sun by observing the surface manifestations of its internal oscillations is now well known (see, for example, reviews by Brown, Mihalas, and Rhodes 1985; Deubner and Gough 1984; Christensen-Dalsgaard 1983).

Recently, the effects of magnetic fields on the Sun's internal oscillations have been considered, the ultimate goal being to use the oscillations to determine the properties of the magnetic fields in the solar interior. Effects of magnetic fields on the global dispersion relation of the solar p-modes have been considered by Bogdan and Zweibel (1985) and Zweibel and Bogdan (1986) for the case where the magnetic field in the convection zone is assumed to be concentrated into thin "magnetic fibrils," and by Roberts and Campbell (1986) for the case where the magnetic field exists in a diffuse shell at the base of the convection zone. Roberts and Campbell (1986) suggest that magnetic effects may be responsible for observed frequency changes of the p-modes from 1980 to 1984 (Fossat et al. 1987; Woodard and Noyes 1985). There have also been a variety of studies of the effects of large-scale magnetic fields on global stellar pulsations (see, for example, Brunt et al. 1982; Chanmugam 1979; Dziembowski and Goode 1985; Nakagawa and Trehan 1970; and Roberts and Soward 1983).

There has recently been considerable interest in the detailed interaction between solar p-modes and sunspots or the magnetic fibrils, following the suggestion of Thomas, Cram, and Nye (1982) that the p-modes can be used to probe the magnetic structure of a sunspot below the visible surface.

Wilson (1980) calculated the response of a cylindrical magnetic flux tube to an incident planar sound wave in an unstratified (zero gravity) atmosphere. He notes that if the flux tube is thin, then there is an enhanced response to the incident sound wave when \( \omega^2/k_0^2 = v_A^2 \), where \( v_A^2 \equiv v_{A1}^2/v_{A2}^2 = (v_A^2 + v_S^2) \), \( \omega \) is the (angular) frequency of the incident sound wave, \( k_0 \) is the component of the wavevector along the magnetic field (which is assumed to be parallel to the tube axis), \( v_A \) and \( v_S \) are the Alfvén and sound speeds respectively, and the subscript 2 denotes the interior of the flux tube. The quantity \( v_T \) is called the tube speed. It represents the propagation speed of an axisymmetric sausage mode along a thin flux tube (Roberts 1985). This mode has the property that the sum of the thermal and magnetic pressure perturbations is, to lowest order, zero both inside and outside the tube. Since the incident sound wave imposes pressure perturbations to the surface of the tube, this mode is easily driven to large amplitude. This result was anticipated earlier by Roberts (1979).

Bogdan and Zweibel (1987) and Bogdan (1987a, b) have considered several differences between wave propagation in fibril magnetic fields and in diffuse magnetic fields which might be exploited in order to use the solar p-modes as probes of the variation of the magnetic structure of a sunspot with depth. They consider the reflection and transmission of sound waves, and the cascade of acoustic power to smaller scale lengths due to multiple scattering off the fibrils. They offer several specific predictions which can be observationally tested to determine the extent to which the sunspot fields are diffuse or are clumped into fibrils below the visible surface.

The 5 minute oscillations in sunspot umbræ were studied by Abdelatif, Lites, and Thomas (1986) and by Thomas, Cram, and Nye (1982, 1984). They find that the 5 minute oscillations have lower amplitude in the spots than in the surrounding photosphere, and that the umbra seems to respond to particular frequencies of the p-modes in the surroundings. They suggest that this result may be due to the umbra acting as a selective filter in admitting some frequencies more readily than others. Their analysis is based on simple considerations of wave transmission across a discontinuous umbral boundary, and resonance effects which can occur when the finite size of the umbra is taken into account, in close analogy with anti-reflection coatings on lenses (e.g., Hollweg 1984). The theoretical basis of their work can be found in Abdelatif and Thomas.
1 There is a technical problem associated with calling the driven mode an “Alfvén” mode. Strictly speaking, if an observer could sit on the resonant field line, he would measure nonzero total pressure perturbations. Since \( \delta p_{\mathrm{tot}} = 0 \) for the Alfvén mode, he would be tempted to declare that the motions on the resonant field line are not Alfvén waves. However, if he measured the velocity and magnetic field fluctuations perpendicular to the ambient magnetic field, he would find that these fluctuations, when suitably normalized, are very much larger than the total pressure perturbations. In some situations the perpendicular motions can approach infinity while \( \delta p_{\mathrm{tot}} \) remains finite; see eqs. (8) and (9). The very dominant transverse motions would correctly lead our observer to designate the motions as Alfvén waves. A similar discussion is valid when the total pressure perturbations resonantly drive the slow mode.

In the presence of a suitable dissipation mechanism, it is possible to find a steady state solution in which \( \delta p_{\mathrm{tot}} \) is purely harmonic in time. Classical viscosity or electrical resistivity can do the job, because large gradients develop across the resonant layer. If the viscosity and resistivity are small, then the net (i.e., integrated across the resonant layer) heating rate is independent of the viscosity and resistivity coefficients. Moreover, the net heating is precisely equal to the rate at which energy is pumped into the resonant layer in the dissipationless case. What happens is that gradients build up to the point where viscosity and resistivity can absorb all the energy which is being pumped into the resonant layer by \( \delta p_{\mathrm{tot}} \). The large gradients might also drive Kelvin-Helmholtz instabilities which could dissipate via a turbulent cascade (Hollweg and Yang 1988), but this remains to be demonstrated in detail.

If dissipation is present, one can also talk about a damped normal mode in which energy decays everywhere in the system at the same rate (see Hollweg 1987b and references therein), but this case requires that the viscosity or resistivity be in excess of some minimum value.

In this paper we will apply these concepts to the interaction of \( p \)-modes with sunspots. If the sunspot boundary were a true discontinuity, then one would have a standard calculation for the amounts of \( p \)-mode energy which are reflected by the sunspot or transmitted into its interior. When this problem is solved, it is found that \( \delta p_{\mathrm{tot}} \) is nonzero at the sunspot boundary, and the situation is ripe for resonance absorption if the boundary is in fact a thin transition layer. In that case the reflected and transmitted energy fluxes (measured outside the transition layer) will be less than the incident \( p \)-mode flux.

The energy deficit goes into Alfvén or slow modes in the resonant layer, as described above, or into heat if dissipation is present. There is also another possibility which could be particularly relevant for sunspots. If the total pressure perturbations are applied only to a spatially limited portion of the transition layer, then the Alfvén or slow modes which are produced there by resonance absorption will be able to propagate away from the region where they are generated. In the case of sunspots, the \( p \)-modes would generate Alfvén or slow waves in the upper portions of the convection zone, but these waves could then propagate along the magnetic field either deeper into the convection zone or up into the photosphere and chromosphere. It is intriguing to speculate that the running penumbral waves, which do appear to be confined to the outermost field lines, are generated in the convection zone by resonance absorption. This possibility will not be pursued further here, however, since it requires a two-dimensional analysis at least, in contrast to the one-dimensional calculations heretofore available.

This paper has in fact been motivated by a recent observational study by Braun, Duvall, and Labonte (1987, hereafter BDL). They studied the interaction of \( p \)-modes with sunspots by examining the \( p \)-mode behavior external to the spots, in contrast to Abdelatif, Lites, and Thomas (1986), who looked inside the umbral edge. Outside the spot the wave field involves motions terms of the form

\[
\psi = [AH_{1}^{12}(kr) + BH_{m}^{32}(kr)] \exp (i\theta + i\omega t),
\]
where $A$ and $B$ are functions only of $k$ and $\omega$, $H^{(1)}_m$ and $H^{(2)}_m$ are Hankel functions, $r$ and $\theta$ are the usual polar coordinates in a system centered roughly on the sunspot center, $k$ is a wave-number, and $\omega$ is the angular frequency. For $-5 < m < +5$ they find that the inward-propagating $p$-mode power exceeds the outward-propagating power. They calculate absorption coefficients, which tend to increase with $k$ for $0 < k < 1.5$ $\text{Mm}^{-1}$ (the absorption coefficients are obtained after summing over $m$ and integrating over the frequency interval 1.5 $\text{mHz} < f < 5$ $\text{mHz}$). As much as 50% of the $p$-mode power is lost. This appears to be a real loss of power which cannot be explained by the transmission resonances considered by Abdelatif and Thomas (1987). BDL suggest that the loss might be only apparent, and due to the scattering, by inhomogeneities, of the incoming waves into higher unobservable wave-numbers, as in the papers by Bogdan and Zweibel mentioned above. Further observations, spanning a greater range of $m$, $f$, and $k$, would be required to resolve this issue. Another possibility is that after the waves penetrate into the spot they are refracted up or down and then lost from the system. This could qualitatively explain the dependence on $k$. Waves with wavelengths comparable to or larger than a spot diameter would not undergo refraction in the usual sense, but short wavelengths could conceivably be refracted out of the system. This explanation would fail, however, in cases where the $p$-modes are evanescent inside the sunspot, and total reflection occurs. A third possibility is viscosity and other classical dissipation mechanisms. But given that the $p$-modes suffer remarkably little dissipation outside of the sunspots, it seems unlikely that the sunspots could add much to the damping. The fourth possibility is resonance absorption.

Our analysis of resonance absorption will be rather crude, but we believe that it suffices to delineate the basic parametric dependences. We will consider a planar rather than cylindrical system. This will be a good approximation only for large $m$ and large $r_r$, where $r_r$ is the sunspot radius, but it allows us to make immediate use of analytical results which are available elsewhere. We will also employ an approximation to obtain $\delta p_{\text{tot}}$. We first solve the standard reflection-transmission problem for the case where the sunspot boundary is a true discontinuity, and we calculate $\delta p_{\text{tot}}$ for this case. We then assume that this value of $\delta p_{\text{tot}}$ applies also to the case where the discontinuity is replaced by a thin transition layer with ignorable inertia. This value of $\delta p_{\text{tot}}$ is then used to calculate the rate at which energy is pumped into the resonance layer. The validity of this approximation has been discussed by Hollweg and Yang (1988). It basically requires $k_a < 1$, where $a$ is the thickness of the transition layer and $k$ is the projection of the wave vector $k$ onto the transition layer. But it also requires that the absorption coefficient be small, since the original calculation for $\delta p_{\text{tot}}$ ignores resonance absorption altogether. In view of the large absorption coefficients obtained by BDL, this is a serious restriction.

Our analysis will also drop gravity. The $p$-modes external to the sunspot are then pure sound waves. It turns out that the most important resonance will be the “Alfvén resonance.” The resonantly driven waves are then Alfvén waves which are non-compressive, and are unaffected by gravity anyway. But we will also encounter a “cusp resonance” which produces compressive slow waves that could be affected by gravity. Our analysis will be restricted to a single boundary. Any effects on $\delta p_{\text{tot}}$ of multiple boundary phenomena, such as the transmission resonances discussed by Abdelatif and Thomas, will be ignored.

However, we expect that multiple boundary effects will be unimportant if the waves are strongly evanescent inside the sunspot; this will in fact be the case in a significant fraction of the parameter space.

The plan of the paper is as follows. In § II we repeat the essential results obtained by Hollweg and Yang (1988). In § III we discuss the relevant parameter regimes for resonance absorption. In § IV we calculate $\delta p_{\text{tot}}$ and estimate the absorption coefficient. The results are discussed in § V.

II. BASIC EQUATIONS

The unperturbed transition layer is taken to lie parallel to the $x-z$ plane. The transition layer is essentially a tangential discontinuity, so $B_{\text{out}} = 0$ (henceforth the subscript zero denotes unperturbed background quantities, and the prefix $\delta$ denotes the fluctuations associated with the waves). The background magnetic field is thus $B_{in} = (B_{ox}, 0, B_{oz})$. There is no background flow, so $V_{in} = 0$ ($V$ is the plasma velocity). All background quantities are taken to vary only in the $y$-direction, so $B_{in} = B_{in}(y)$. The background must be in static equilibrium, so

$$p_{0} + B_{in}^2/8\pi = \text{constant},$$

where $p_{0} = p_{0}(y)$ is the thermal pressure. We shall also without loss of generality take the waves to propagate in the $x$-direction; thus $\partial/\partial z = 0$. Fourier analysis will be used from the start, so that the $x$- and $t$-dependences of all fluctuating quantities will be taken to be $\exp(ik_x x - i\omega t)$. Dissipation will be ignored. All units will be cgs.

The linearized mass conservation, momentum, and energy equations, as well as the linearized magnetic induction equation, are written with $\delta p_{\text{tot}}$ as a driving term:

$$B_{in} \cdot \delta V = k_a B_{0x} v_x^2 (\rho_0 \omega)^{-1} \delta p_{\text{tot}}/C,$$

$$B_{in} \cdot \delta B = \delta V (2i\omega)^{-1} dB_{in}/dy + 4\pi (C - v_x^2) \delta p_{\text{tot}}/C,$$

$$\nabla \cdot \delta V = (i\omega\rho_0)^{-1} \delta p_{\text{tot}}/C,$$

$$i\omega \rho_0 \delta \rho = 2\delta V \cdot \delta V,$$

where $\rho_0$ is mass density,

$$C \equiv \sqrt{\gamma - \gamma^2 k_a^2 v_x^2 / \omega^2},$$

and we have defined

$$v_x^2 = B_{in}^2/(4\pi \rho_0),$$

with $\gamma$ the usual ratio of specific heats. Proceeding along the same lines, we can calculate the velocity and magnetic field fluctuations perpendicular to $B_{in}$, but in the plane of the transition layer we have

$$B_{0x} \delta V_x - B_{0x} \delta V_x B_{0x} = (\omega/k_a)(B_{0y}/B_{0x}) \delta p_{\text{tot}}(\rho_0 A)$$

and

$$B_{0y} \delta B_x - B_{0x} \delta B_x B_{0y} = - \delta V_x \cdot \frac{B_{0x}^2}{B_{0x}} \frac{d}{dy} \left( \frac{B_{0x}}{B_{0x}} \right) \frac{\delta p_{\text{tot}}}{A},$$

where

$$A \equiv \omega^2/k_a^2 - v_x^2.$$
The quantity $C$ becomes zero when
\[ \frac{\omega^2}{k_x^2} = \frac{v_{Ax}^2}{v_s^2} \cdot \]
This is called the "cusp singularity," or "cusp resonance." If we regard $\omega/k_x$ as known, and determined by the dispersion relation of the external sound wave, then any field lines in the transition layer satisfying equation (11) will be in resonance with the external wave. It can be shown that the resonantly driven wave is a slow-mode wave on an infinitesimally thin magnetic slab. Note that the cusp resonance disappears in a cold plasma ($v_s^2 = 0$). Equations (8) and (9) exhibit resonance when $A = 0$, i.e., when
\[ (\omega/k_x)^2 = v_{Ax}^2. \]
This is called the "Alfvén resonance" or "Alfvénic singularity." As was the case with the cusp singularity, field lines satisfying equation (12) will be in resonance with the external sound wave, which can be regarded as the driver. Note that the cusp resonance refers to components along $B_0$, while the Alfvén resonance refers to components transverse to $B_0$. Moreover, the quantities $\delta p$, $\nabla \cdot \delta V$, and $\delta \rho$ exhibit singularities only at the cusp resonance.

Consider the Alfvén resonance, $A = 0$. It is convenient to work with the quantity
\[ \delta V_\perp \equiv (B_0 \delta V_x - B_{0z} \delta V_z)/B_0. \]
We rewrite equation (8) in the form
\[ \left( \frac{\partial^2}{\partial t^2} + k_x^2 v_{Ax}^2 \right) \delta V_\perp = - \frac{B_{0z}}{\rho_0 B_0} \frac{\partial^2 \delta p_{tot}}{\partial x \partial t}. \]
Equation (13) is the equation for a driven harmonic oscillator, with natural frequency $k_x v_{Ax}$. For convenience, take
\[ \delta p_{tot} = P \cos(k_x x - \omega t). \]
Thus $P = |\delta p_{tot}|$. Recall that this is the total pressure fluctuation imparted to the system by the external sound wave. We now consider an initial-value problem in which there are no velocity or magnetic field fluctuations on or near the resonant field line at $t = 0$. As discussed by Hollweg and Yang (1988), the $\gamma$-integral of the kinetic and magnetic energy in the resonant layer grows secularly at a rate
\[ H = nk_x P^2 (B_{0z}/B_0)^2/(4\rho_0 v_{Ax}) \]
after averaging over $x$ and $\omega t$. Here $v_{Ax} = |d\phi_{Ax}/dy|$, and all quantities in equation (15) are to be evaluated at the resonant field line; in obtaining (15) it has been assumed that $B_0$ and $\rho_0$ are constant across the (thin) resonant layer. Equation (15) gives the rate at which kinetic and magnetic energy is pumped into the (Alfvénic) energy-containing layer by the total pressure fluctuations associated with the external sound wave, per unit area in the $x$-$z$ plane. Energy conservation implies that this is the energy lost by the sound waves.

Note that $H = 0$ if $B_{0z} = 0$; there is no resonance absorption if the sound wavevector, the normal to the sunspot boundary, and $B_0$ are all coplanar.

A similar calculation can be performed for the cusp resonance, $C = 0$. However, we will find in the next section that the cusp resonance occurs only for rather extreme sunspot conditions, and we will henceforth ignore it.

III. PARAMETERS FOR RESONANCE ABSORPTION

Let region 1 be the uniform region outside the sunspot, in $y > a/2$. Let region 2 be the uniform interior of the sunspot, $y < -a/2$. The transition layer is the monotonically and smoothly varying region $-a/2 < y < a/2$. Note that $A_1 > 0$ and $C_1 > 0$, since $B_{01} = 0$. In order for there to be an Alfvén resonance inside the transition layer, we must have $A_2 < 0$. Similarly, a cusp resonance requires $C_2 < 0$. The goal of this section is to define the conditions required for $A_2 < 0$ or $C_2 < 0$ subject to the constraints imposed by pressure balance (eq. [1]) and the dispersion relation for the sound waves, viz.,
\[ \omega^2/k_x^2 + k_z^2 v_s^2 = v_{Ax}^2. \]

It is convenient to introduce a new coordinate system $(x', y' = y, z')$ illustrated in Figure 1. The $z'$-axis is taken parallel to $B_{02}$. It is also convenient to introduce the wavevectors
parallel and perpendicular to $B_{02}$, denoted $k_{\perp}$ and $k_x$, respectively. The angle of incidence of an incoming sound wave, in the x'-y' plane, is denoted $\phi$ in Figure 1. If both the boundary of the sunspot and its magnetic field were vertical, then $k_{\perp}$ would be the vertical wavenumber and $k_x$ would be the horizontal wavenumber; $k_{\perp} \cos \phi$ would then roughly correspond to $k$ in BDL, while $k_x \sin \phi$ would roughly correspond to their $m/r_e$. If either the sunspot boundary or $B_{02}$ were not vertical, then a further transformation of coordinates would become necessary, but we will not pursue this complication here. In terms of the coordinate system used in the previous section, we have

$$B_{02x} = k_x B_0 (k_{\perp}^2 + k_x^2 \sin^2 \phi)^{-1/2},$$  

$$B_{02\perp} = k_{\perp} B_0 (k_x^2 + k_{\perp}^2 \sin^2 \phi)^{-1/2},$$  

$$k_x = (k_{\perp}^2 + k_x^2 \sin^2 \phi)^{1/2}.    \tag{19}$$

Note that $k_x B_{02x} = k_{\perp} B_{02\perp} = B \cdot B_{02}$. (Some subscripts have been omitted, since $k_x = k_{x2}$ and $k_{\perp} = k_{\perp2}$.)

Consider now the requirement $A_2 < 0$, or $k_x^2 k_{\perp}^2 / \omega^2 > 1$. From equations (1), (16), (17), and (19) we rewrite this condition as the second inequality in

$$\frac{2}{\gamma_1} > \beta > \beta_{1}, \tag{20}$$

where

$$\beta \equiv \left( \frac{\rho_{01}}{\rho_{02}} - \frac{T_{02}}{T_0} \right)^{-1}$$  

and $T_0$ is related to the temperature by $T_0 \equiv \rho_0 / \rho_0$. We have defined

$$\beta \equiv 2 k_x^2 / (\gamma_1 k_x^2), \tag{22}$$

and the first inequality in formula (20) follows from the definition of $\beta$. Note that the condition for an Alfvén resonance is independent of $\phi$.

Consider next the requirement for a cusp resonance, $C_2 < 0$, or $(k_x^2 k_{\perp}^2 / \omega^2)^{1/2} > c_2 (c_2^2 + v^2)^{-1} > 1$. Proceeding as before, this condition is the second inequality in

$$\frac{2}{\gamma_1} > \beta > \beta_1 + \beta_c, \tag{23}$$

where we have defined

$$\beta_c \equiv 2 T_{01} / \gamma_2 T_{02}. \tag{25}$$

Note that the range of $\beta$ for a cusp resonance is more restricted than the range of $\beta$ for the Alfvén resonance. Note too that if $C_2 < 0$, then $A_2 < 0$; a cusp resonance implies an Alfvén resonance, but not vice versa. Finally, note that the condition for a cusp resonance is independent of $\phi$.

We now want to consider the properties of the waves in the sunspot. The dispersion relation for the fast and slow modes can be written compactly as

$$\omega^2 / k_x^2 = C_2. \tag{25}$$

If $C_2 < 0$ (cusp resonance somewhere in the transition layer), then $k_x^2 < 0$ and the waves are evanescent in the sunspot. The question of whether the waves are evanescent or propagating in the sunspot is relevant for the calculation of $\delta p_{\rm res}$ to be performed in the next section, since that calculation is basically a standard reflection/transmission problem. To this end, let $n_2^2 \equiv -k_x^2$. If $n_2^2$ is positive (negative), the waves will be evanescent (propagating). Using equations (1), (16), (19), (21), (22), and (25), we have

$$\frac{n_2^2}{k_x^2} = \sin^2 \phi + \frac{\beta - \beta_2 / \Gamma}{(2/\gamma_1 - \beta)}}. \tag{26}$$

where

$$\Gamma \equiv C_2 / \gamma_2 \tag{27a}$$

$$= 1 - (\beta - \beta_1) / \beta_c. \tag{27b}$$

Equation (26) shows that if $n_2^2 (\phi = 0) > 0$, then the waves will be evanescent in the sunspot for all $\phi$. This will be true if $\beta > \beta_2 / \Gamma$, since $2 / \gamma_1 > \beta$. If we confine our attention to situations where there is an Alfvén resonance (inequality [20]), then $\beta > \beta_2 / \Gamma$ if $\beta_1 < \beta_2 < \beta_c$, or if $\beta > \beta_1 + \beta_c$. If $2 / \gamma_1 < \beta_c$, then the waves will be evanescent in the sunspot, regardless of $\phi$, for all values of $\beta$ leading to an Alfvén resonance.

Suppose, on the other hand, that $\beta < \beta_2 / \Gamma$. Equation (26) then implies that the waves in the sunspot will be propagating for small $\phi$, but that they might again be evanescent for larger angles of incidence. If we again confine our attention to situations where there is an Alfvén resonance, we can distinguish two cases: (1) If $\beta_l < \beta_c$, then $n_2^2 (\phi = 0)$ is negative for $\beta < \beta_1 < \beta_c$. (2) If $\beta_c < \beta_l$, then $n_2^2 (\phi = 0)$ is negative for $\beta < \beta_1 < \beta_c$. In case (2), this case $n_2^2 (\phi = 0)$ is always negative at the smallest values of $\beta$ leading to an Alfvén resonance.

Finally, from equation (26) we see that the waves in the sunspot are propagating for all $\phi$ if $1 + (\beta - \beta_2 / \Gamma) / (2 / \gamma_1 - \beta) < 0$. This condition can be rewritten in the form

$$\beta_l + \beta_c (1 - \gamma_1 \beta_2 / 2) < \beta < \min (2 / \gamma_1, \beta_1 + \beta_c). \tag{28}$$

Again restricting our attention to cases where there is an Alfvén resonance ($\beta_l < \beta < 2 / \gamma_1$), we can easily show that there is an allowed range of $\beta$ satisfying equation (28) as long as $2 / \gamma_1 > \beta_l$.

These considerations are summarized in Figure 2. The density ($\rho_{01} / \rho_{02}$) and temperature ($T_{02} / T_{01}$) ratios are arbitrary except for the requirement that $\rho_{02} < \rho_{01}$ or $\rho_{01} / \rho_{02} > T_{02} / T_{01}$. The hatched line in each of the panels represents $2 / \gamma_1 = \beta_l$; it intersects the horizontal axis at $\rho_{01} / \rho_{02} = \gamma_1 / 2$. According to inequality (20), an Alfvén resonance ($A_2 < 0$) is possible only to the right of the hatched line; we will henceforth limit our attention to this region. The horizontal line in each panel represents $2 / \gamma_1 = \beta_1$; it intersects the vertical axis at $T_{02} / T_{01} = \gamma_1 / 2$. The other diagonal line represents $\beta = \beta_c$, and the hyperbola represents $2 / \gamma_1 = \beta_1 + \beta_c$. The waves in the sunspot have different behaviors in the five regions indicated in Figure 2 (upper left), as follows:

Region I.—The waves are always evanescent in the sunspot.

Region II.—The waves are evanescent for all $\phi$ for values of $\beta$ in the lower end of the allowed range. At larger values of $\beta$, the waves propagate in the sunspot for small values of $\phi$. At the largest allowed values of $\beta$, the waves propagate at all $\phi$.

Region III.—Same as region II, but the largest values of $\beta$ give a cusp resonance, and the waves are again evanescent at all $\phi$.

Region IV.—For the smaller values of $\beta$ in the allowed range, the waves are propagating in the sunspot if $\phi$ is small. At the largest allowed values of $\beta$, the waves propagate at all $\phi$.

Region V.—Same as region IV, except that the largest values of $\beta$ give a cusp resonance and the waves are again evanescent at all $\phi$.

Inspection of Figure 2 reveals that the existence of an Alfvén resonance implies that the waves will be evanescent in a cool...
Fig. 2.—Regimes for resonance absorption for four combinations of $\gamma_1$ and $\gamma_2$. Absorption by the Alfvén resonance is possible only to the right of the hatched line. Absorption by the cusp resonance is possible only to the right of the hyperbola. In region I the waves are always evanescent in the sunspot. In regions II-V the waves may be propagating or evanescent in the sunspot depending on the values of $\beta$ and $\phi$.

If a cool sunspot is to have a cusp resonance, Figure 2 (upper right) shows that rather large values of $\rho_{01}/\rho_{02}$ are required. We will not consider this case in what follows.

In the next section we will estimate $\delta\rho_{\text{tot}}$ in the transition layer by solving a standard reflection/transmission problem at a discontinuous sunspot boundary. If the waves are propagating in the sunspot, there is a slight complication which we consider here. From the dispersion relation (eq. [25]) we can calculate the $y$-component of the group velocity in the sunspot; holding $k_x$ fixed, we obtain

$$\frac{d\omega}{dk_y} = \frac{(k_y^2/\omega)(C_2/v_2)^2}{\beta_i + \beta_c - 2\beta}.$$  (29)

The group and phase velocities will have opposite signs if $2\beta > \beta_i + \beta_c$; this condition can also be written as $2(\beta - \beta_c) > \beta_i - \beta_c$ or $2(\beta - \beta_i) > \beta_c - \beta_i$. Consider first regions II and III in Figure 2, where $\beta_i < \beta_c$. From our previous discussion, the waves in the sunspot can be propagating only if $\beta_i < \beta < \beta_i + \beta_c$, and the waves in the sunspot can be propagating only if $\beta_i < \beta < \beta_i + \beta_c$. Thus a radiative boundary condition on the waves in region 2 requires $\omega/k_y > 0$.

**IV. RESONANCE ABSORPTION**

In region 1 there are an incident and a reflected sound wave, denoted by subscripts $i$ and $r$, respectively. The incident and reflected waves vary as $\exp(ik_x x - ik_y y - i\omega t)$ and $\exp(ik_x x + ik_y y - i\omega t)$, respectively, where $k_y$ has the same sign as $\omega$.

The sound waves obey

$$\rho_{0i}\frac{\partial V_i}{\partial t} = -V\delta\rho_{\text{tot}},$$  (30)

and it is readily shown that the sound waves impart the following pressure perturbation to the (discontinuous) sunspot boundary at $y = 0$:

$$\delta\rho_{1,y=0} = (\omega\rho_{01}/k_y)\delta V_y - \delta V_iy=0.$$  (31)

In region 2 there is a single wave which varies as $\exp(ik_x x + ik_y y - i\omega t)$. As shown in the previous section, if there is a propagating wave in the sunspot, then $k_y$ has the same sign as $\omega$. If the wave is evanescent in the sunspot, then $k_y = -in_2$ if we regard $n_2$ as real and positive. From equation (30) of
Hollweg and Yang (1988), the total pressure and velocity in region 2 are related by

$$\delta V_{y2} = \left(\omega k_y^2/\rho_{o2} k_x^2 A_2\right) \delta p_{tot,2}.$$  \hspace{1cm} (32)

We now require $\delta p_{rot}$ and $\delta V_{r}$ to be continuous across the boundary at $y = 0$. We then obtain

$$\delta V_{yi,y=0} = -\frac{1}{2} \left( k_y/\rho_{o1} - \frac{\omega k_y}{\rho_{o2} k_x^2 A_2}\right) \delta p_{tot,y=0}.$$  \hspace{1cm} (33)

Equation (33) gives the desired relationship between the incident sound wave and $\delta p_{tot}$, which is responsible for the resonance absorption. A somewhat more useful form is obtained after multiplying equation (33) by its complex conjugate:

$$V = -k_y/k_{r1}.$$  \hspace{1cm} (35)

and

$$p^2 = \frac{8v_{A1}^2(\beta/\beta_1 - 1)^2 F_y^2 \cos \phi}{\rho_{1}(U + V)^2 (1 - \gamma \beta^2/2)^{1/2}},$$  \hspace{1cm} (34)

where $F_y$ is the time-averaged energy flux density of the incident sound wave normal to the boundary, $P \equiv |\delta p_{tot,y=0}|$, and we have defined

$$U \equiv (\rho_{o2}/\rho_{o1})(1 - \gamma/\gamma_1) \cos \phi$$  \hspace{1cm} (36)

and

$$V \equiv -k_y/k_{r1}.$$  \hspace{1cm} (37)

In obtaining equation (34) we have used equations (16) and (22) and $k_x^2 A_2/c^2 = 1 - \beta/\beta_1$; the quantity $V$ can be determined from equation (26).

Equation (34) is rigorously correct for a discontinuous sunspot boundary. We now assume that it is still approximately correct for a thin transition layer, and we use it in equation (15) for $H$. If we further use equations (16) and (18) and the fact that $\omega^2 = k_x^2 v_{A0}$ at the resonant field line, we obtain

$$H/F_y = k_{r1}(\rho_{o1}/\rho_{o2})^{1/2} \rho_{o1}^{-1} R,$$  \hspace{1cm} (38)

where the subscript $r$ refers to the resonant field line and

$$R = 2\pi \left( \frac{\rho_{o2}}{\rho_{o1}} \right)^{3/2} \left( 1 - \gamma \beta^2/2 \right)^{1/2} (1 - \beta/\beta_1)^{3/2} \sin^2 \phi \cos \phi$$  \hspace{1cm} (39)

$$\left( U + V \right)^2.$$  \hspace{1cm} (36)

The quantity $\tau$ is a measure of the thickness of the transition layer; it is formally defined by $v_{Ax} = v_{A0}/\tau$. In deriving equations (37)–(38), it has also been assumed that the magnetic field does not change direction through the transition layer.

The quantity $H/F_y$ is the absorption coefficient. Note that the absorption coefficient scales linearly with $k_{r1}$ (recall that $\beta$ and $\phi$ are measures only of the direction of $k_1$). This may correspond to the linear dependence of the absorption coefficient on $\kappa$ found by BDL, but detailed comparison is difficult, since they summed over $m$ and integrated over frequency. Note also that the absorption coefficient vanishes for $\beta = 2/\gamma_1$ (i.e., $k_{r1} = 0$) and for $\beta = \beta_1$. We will see below that the absorption coefficient at intermediate values of $\beta$ increases with the allowed range of $\beta$, i.e., with $\gamma_1 - \beta$. Smaller values of $\gamma$, $\rho_{o2}/\rho_{o1}$, and $T_{o2}/T_{o1}$ lead to more absorption. Finally, note that the absorption coefficient is zero for $\phi = 0$ and $\phi = \pi/2$. If the sunspot boundary and the magnetic field were both vertical, then there would be no absorption for $m = 0$ (sin $\phi = 0$). However, this picture would become confused if the sunspot boundary were corrugated.

We now present several figures illustrating the behavior of $R$ as a function of $\phi$ for five values of $\beta$ given by $\beta = \beta_1 + m(2/\gamma_1 - \beta)/6$, where $m = 1, 2, \ldots, 5$.

Consider, first, cases where $\gamma_1 = \gamma_2 = \gamma$ and $T_{o2} < T_{o1}$; the waves are then evanescent in the sunspot if $A_2 < 0$ (i.e., region I of Fig. 2). Four cases are presented in Figures 3a–3d. In all cases $R$ is a smoothly varying function of $\beta$ and $\phi$, with a broad maximum near $\phi \approx \pi/4$ and $\beta \approx (2\gamma + \beta_1)/2$. The quantity $R$ is larger for smaller values of $\gamma$, $T_{o2}/T_{o1}$, and $\rho_{o2}/\rho_{o1}$. In addition, smaller values of $\gamma$, $T_{o2}/T_{o1}$, and $\rho_{o2}/\rho_{o1}$ increase the allowed range of $\beta$-mode propagation directions leading to resonance absorption, since the maximum value of $k_{r1}/k$ is $(1 - \gamma \beta^2/2)^{1/2}$; this quantity takes the values 0.57, 0.79, 0.66, and 0.72 in Figures 3a–3d, respectively. (It should be noted that the maximum values of $R$ occur when the $\beta$-modes impinge on the sunspot magnetic field at rather shallow angles. If we evaluate $k_{r1}/k = (1 - \gamma \beta^2/2)^{1/2}$ at $\beta = (2\gamma + \beta_1)/2$, we find $k_{r1}/k = 0.41, 0.56, 0.47$, and 0.51 in Figures 3a–3d, respectively. The maximum value of $R$ occurs when the angle between $k$ and $B_{o2}$ is about 30°.)

The behavior of $R$ is somewhat different in regions II and IV of Figure 2. In Figures 4a–4d we illustrate the behavior for $\rho_{o1}/\rho_{o2} = 2$, $\gamma_1 = 1.167$, and $\gamma_2 = 5/3$ (cf. Fig. 2, upper right), and four values of $T_{o2}/T_{o1}$. In Figure 4a, $T_{o2}/T_{o1} = 0.85$. The curves for $n = 1$–$3$ are similar to those in Figure 3 in that the waves are evanescent in the sunspot for all $\phi$. However, the $n = 4$ curve has propagating waves for $\phi \leq 12$ degrees, while the curve for $n = 5$ has propagating waves in the sunspot for all $\phi$. Note that $R$ is rather large in these cases. If $T_{o2}/T_{o1}$ is increased to 0.9, we obtain Figure 4b. The $n = 4$ curve has propagating waves for $\phi \leq 31^\circ$, while the $n = 5$ curve has propagating waves at all $\phi$. The maximum value of $R$ in curve 4 is $\approx 1.1$. It occurs at $\phi \approx 31^\circ$, where the waves in the sunspot make a transition from propagating to evanescent. The peak occurs close to the value of $\phi$ at which $k_{r2}$, and thus $V$, is zero (see eqs. [36] and [38]). This behavior occurs also in Figures 4c–4d, for $T_{o2}/T_{o1} = 1$ and 1.25, respectively. The peaks are again in close correspondence with the values of $\phi$ at which $k_{r2}$ is zero. (In Fig. 4d the curves for $n = 4$ and $n = 5$ have not been drawn because they give rather low values of $R$. This occurs because the waves are propagating for all values of $\phi$, and $k_{r2}$ is never close to zero). In Figures 4a–4d we again find that the largest values of $R$ occur when the angle between $k$ and $B_{o2}$ is roughly 30°.

The absorption coefficient is roughly $(k_{r1} \tau)R$, since we expect $(\rho_{o1}/\rho_{o2})^{1/2} \approx \rho_{o1}$, (eq. [37]). Consider as an example the sunspot observed by BDL on 1983 January 18. The umbral radius was about 8 Mm. If we take an average value of $m$ of about 2.5, the characteristic azimuthal wavenumber, $m_{11}$, is 0.3 Mm$^{-1}$. The radial wavenumbers leading to significant $\beta$-mode absorption were $\kappa \approx 0.5$–1.0 Mm$^{-1}$. The value of $k_{r1}$ is thus at least 1 Mm$^{-1}$. If we assume that $\tau$ is one-tenth of the sunspot diameter, we obtain $k_{r1} \tau > 1.6$. We will take $k_{r1} \tau = 2$ as a conservative rule of thumb, and the absorption coefficient will then be twice the values of $R$ in Figures 3 and 4. If $\gamma_1 = \gamma_2 = 5/3$, then Figures 3a–3c show that absorption coefficients of the observed magnitude can result only for the largest value of $\rho_{o1}/\rho_{o2}$ considered, i.e., $\rho_{o1}/\rho_{o2} = 3$. The situation is more promising if $\gamma_1 = \gamma_2 = 1.2$ (Fig. 3d). The values of $R$ are still somewhat low, but slightly larger values of $\rho_{o1}/\rho_{o2}$ could yield the observed absorption coefficients. The most interesting cases occur when $\gamma_1 < \gamma_2$, as in Figure 4. Figures 4a and 4b show that significant absorption can occur for modest values
of $\rho_{01}/\rho_{02}$ and for temperature ratios comparable to the surface values.

The above estimates of the absorption coefficient have to be viewed with caution, however. Our entire procedure has assumed that the total pressure perturbations in the transition layer can be estimated by replacing the transition layer with a true discontinuity. This requires the solutions for $\tau \neq 0$ to be not very different from the solutions for $\tau = 0$. But this requirement is clearly violated when the absorption coefficient is in the observed range of 40%–50%. A more exact solution must then be computed, and the results of this paper can only serve as a guide to the minimal conditions needed for significant absorption. It is also clear that $k_2\tau$ cannot greatly exceed unity, since in that case the wave propagation would be better approximated by the usual WKB geometrical optics approach. Finally, it should be kept in mind that Figures 2–4 refer only to waves undergoing resonant absorption. Waves having $\beta < \beta_1$ will not be absorbed. For the examples given in Figures 3–4, resonance absorption will occur only if the angle between $k_1$ and $B_0$ is less than $40^\circ$–$45^\circ$.

V. DISCUSSION

This paper has been motivated by the recent observations of Braun, Duvall, and LaBonte (1987) which showed that outgoing $p$-modes in the vicinity of sunspots have substantially less power than the incoming waves. The observations were restricted to a finite range of frequencies and azimuthal orders, and the possibility exists that the energy deficit occurs simply as a consequence of the scattering of energy outside the observed range.

In this paper we examined another possibility, viz., resonance absorption. This process seems to be rather common in nature, and there is every reason to believe that it should occur in the interaction of $p$-modes with sunspots. Our goal here has been to clarify the conditions required for its occurrence and to estimate the magnitude of the absorption.

Resonance absorption occurs when the boundary between two regions (in our case the sunspot and the exterior field-free region) is smooth rather than discontinuous; this is almost certainly the case in the Sun. It is further required that the quantity $1 - (k \cdot v_0/\omega)^2$ have at least one zero inside the sunspot boundary; we have here assumed that things vary smoothly and monotonically across the boundary, so that there is at most one zero. This condition puts a restriction on the propagation directions of the sound waves outside the sunspot; for reasonable parameters the angle between the incoming sound wavevector, $k_1$, and the sunspot magnetic field, $B_0$, must be less than $40^\circ$–$45^\circ$, so only a fraction of the sound waves can be resonantly absorbed at all. The resonance absorption vanishes at this maximum angle, reaches a broad
maximum at smaller angles of about $30^\circ$, and vanishes again as the angle between $k_1$ and $B_0$ approaches zero. The resonance absorption depends also on the angle between the normal to the sunspot boundary and the plane containing $k_1$ and $B_0$ ($\phi$ in Fig. 1); the absorption coefficient vanishes for $\phi = 0^\circ$ and $90^\circ$. In evaluating these restrictions, it should be kept in mind that the $p$-modes propagate nearly vertically near the solar surface, and thus the angle between $k_1$ and $B_0$ might in fact be small if the sunspot magnetic field is nearly vertical. If the sunspot parameters are in region I of Figure 2, then the resonance absorption coefficient has a broad maximum around $\phi = 45^\circ$. In regions II and IV of Figure 2 the absorption coefficient can exhibit distinct peaks near certain values of $\phi$. (Regions III and IV allow another kind of resonance, but the rather extreme parameters exclude it from further consideration.)

These geometrical considerations suggest the following. To make contact with the analysis of BDL, we assume the sunspot boundary to be locally planar, and we decompose $k_1$ into radial ($-\kappa$), vertical ($k_v$), and azimuthal ($m/r_s$) parts. If the magnetic field at the boundary has no azimuthal component, then there will be no resonance absorption for $m = 0$, since $\phi = 0$. The absorption will increase with intermediate values of $m$, but it will die off again at the largest values of $m$, since $m \to \infty$. For a given $m$, the resonance absorption will tend to maximize when

$$k_v = -\kappa B_{0r}/B_{0v},$$

since the angle between $k_1$ and $B_0$ is then minimized; the subscripts $r$, $\theta$, and $v$ will denote radial, azimuthal, and vertical components, respectively. Note that the resonance absorption will not be symmetric with respect to upward and downward propagation if $B_{0r} \neq 0$; this device could be used to investigate the flaring or convergence of magnetic field lines below the surface. Suppose, on the other hand, that the sunspot boundary is vertical, so that $B_{0r} = 0$. The angle $\phi$ will then be zero when

$$m/r_s = k_v B_{0v}/B_{0r}.$$

This expression is antisymmetric with respect to upward and downward propagation, and we would expect the absorption coefficient to show minima at equal positive and negative values of $m$, for a given value of $|k_v|$. The absorption coefficient would in this case show identical behavior upon changing the signs of both $m$ and $k_v$; this device could be used to deduce the presence of an azimuthal magnetic field below the solar surface. If the sunspot boundary is not vertical, then $\phi = 0$.
The magnitude of the resonance absorption coefficient is roughly \( k_1 \tau R \), where \( \tau \) is a measure of the thickness of the transition layer and \( R \) is a function of the direction of propagation of the incident sound wave, displayed in Figures 3-4. Since our analysis fails for large values of \( k_1 \tau \) (in which case ordinary geometrical optics would apply), it is clear that the observed absorption coefficients of 40%–50% can be obtained only if \( R \) exceeds a few tenths. Figures 3-4 show that this requires some rather special circumstances, such as a small specific heat ratio outside the sunspot (Figs. 3d and 4a–4d) or a sunspot with a low interior density (Fig. 3b). Even when these conditions are fulfilled, the resonance absorption is large only for a limited range of sound wave propagation directions. However, the principal restriction in this regard is that the angle between \( k_1 \) and \( B_0 \) be small, of the order of 30°. This requirement might in fact be fulfilled near the solar surface if \( B_0 \) is nearly vertical, since the p-modes propagate nearly vertically there. But, on the whole, resonance absorption probably cannot explain the large absorption coefficients reported by BDL for a wide range of frequency and azimuthal order. Refraction and/or scattering must still be regarded as viable candidates.

Nevertheless, the fact that some waves can experience substantial resonance absorption suggests that the resonance absorption might not be totally without consequence. One intriguing possibility is that the energy which is pumped into the resonant layer propagates upward along the magnetic field lines and becomes the running penumbral waves.

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