RADIATIVE TRANSFER IN INHOMOGENEOUS ATMOSPHERES:
A STATISTICAL APPROACH

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ABSTRACT

We present a procedure for calculating the statistical properties of the radiation which emerges from a multicomponent gas when the absorption and emission coefficients vary statistically along the direction of propagation. We derive a relation describing the evolution of the intensity distribution through the gas and, from that, obtain a transfer equation for the expected value of the intensity, which is analogous to the standard transfer equation for a continuous medium and to which it reduces in the limit of a homogeneous medium. General solutions for this transfer equation, and the analogous transfer equation for the variance, are found for a special class of situations. As a representative example we consider the transfer of radiation through a spherical atmosphere consisting of radial structures, with an exponential height distribution, which are immersed according to a given probability distribution in an ambium, itself inhomogeneous, whose properties also vary with height.

Subject headings: radiative transfer — stars: atmospheres

I. INTRODUCTION

Unless there is compelling evidence to the contrary, the most natural basis to adopt for analyzing the radiation from an astronomical source is to suppose that the medium is homogeneous. This assumption has the virtue of simplicity, it introduces a minimum number of model parameters, and it allows direct application of well-established theory. As the scope and accuracy of data bases expand, however, we find typically that a simple homogeneous model cannot account consistently for the full range of the data, and this has led to pressure for a more general analytical approach which will allow for the presence of structural patterns—inhomogeneities—in the emitting medium.

While this matter is of concern in many branches of astrophysics, it is particularly relevant to the analysis of stellar radiation. Thus, our direct experience with images of the Sun can only lead us to expect that similar complex structural patterns will occur quite generally in stellar atmospheres and will have a more or less profound influence on the radiation emitted. Nor can we ever hope to resolve the stellar surface to a scale where inhomogeneities are not important in the analysis; indeed, we cannot necessarily expect to do this even for the Sun. In this paper we shall present a formalism for handling the transfer of radiation through a gas which contains randomly distributed inhomogeneities. The approach is designed to be general, in the sense that it can accommodate an arbitrary number of inhomogeneities each occurring, at any given point, with an arbitrary probability. However, because our specific motivation in developing this theory derived from attempts to obtain a consistent interpretation of solar observations (in the far-infrared at a total eclipse), the examples in the text may reflect that bias. No implication is intended that the application is limited to solar analyses, however, still less to those dealing with eclipse data.

Many workers who are either developing or using solar atmospheric models have been prepared to finesse the influence of inhomogeneities with quite satisfactory results. Thus, detailed and careful studies by Vernazza, Avrett, and Loeser (1973, 1981; hereafter VAL), extending over a decade and more, have given us a homogeneous reference solar atmosphere which predicts with good accuracy the continuum flux at the center of the solar disk over a wavelength range covering the space ultraviolet through to microwaves. Variations from this basic model, also homogeneous, have been successfully developed to describe mean conditions in resolved features such as network boundaries and interiors.

The empirical success of the VAL models has led to their widespread acceptance for studies of the solar atmosphere and for the determination of such conceptually useful quantities as the "height of formation" of the radiation at different wavelengths, i.e., the height at which the corresponding optical depth reaches unity in a smooth model. Their broad success, too, has provided a strong practical force for their continued use—at least for radiation formed in the photosphere, the chromosphere, and the transition region. Even so, the homogeneous models have not been uniformly successful. Thus, it has been clear for a long time that the solar microwave radiation cannot be consistently explained in this way, and, indeed, attempts to incorporate inhomogeneities in the modeling have been made for this express reason (see Simon and Zirin 1969; Lantos and Kundu 1972; Kalaghan 1974). A recent and compelling example along the same lines has emerged from airborne observations, made at the 1981 total solar eclipse by Becklin, Lindsey, and coworkers, at several continuum wavelengths in the range from 30 to 200 μm. These workers found the chromosphere to be extended considerably above the level predicted by VAL (Lindsey et al. 1986), which indicated strong dynamic or magnetic perturbations extending down to the...
temperature minimum (see Hermans and Lindsey 1986) and naturally raised the prospect of rough structure at those levels. This, in turn, led Lindsey (1987) to develop an approach for studying the sort of inhomogeneous structure which was hypothesised at these levels—basically spicular in nature, like the overlying chromosphere. Braun and Lindsey (1987) applied these ideas to arrive at a spicular model of the low chromosphere.

In this present study we shall generalize and refine that work. Braun and Lindsey assumed the atmosphere to be composed of a single type of structure, randomly distributed in space with no intervening ambient medium. An important generalization is to incorporate within the theory an arbitrary number of different types of structure within an ambium that also contributes significantly both to the emission and absorption of the radiation. This is essential for modeling the solar chromosphere in situations where emission from the corona or transition region material is important (see Ahmad and Kundu 1981). It applies to many other solar applications—for instance, to magnetic flux tubes in the underlying photosphere.

We should make it clear that we are not considering here the transfer of radiation through any atmosphere whose characteristics are specified precisely as a function of position along a ray path. Rather, our interest is in seeking a characterization of the radiation emerging from a gas composed of structural elements which are randomly arranged in the line of sight according to a given statistical law, and whose radiative properties are, similarly, specified only statistically. A particular example, which we shall often have implicitly in mind, is the case where more-or-less-equivalent structures of one kind (such as chromospheric spicules) are immersed in an intervening ambient material (such as the corona)—both of which may interact significantly with the radiation as it evolves along its optical path through the inhomogeneous medium.

Through this paper we shall confine attention to a restricted case where both the emission ($\epsilon$) and absorption ($\kappa$) coefficients at any point in the atmosphere are determined uniquely by the material at that point, and not by its distribution throughout the atmosphere. Thus, the values of $\epsilon$ and $\kappa$ which characterize a given type of material will be supposed to share the same statistical distribution with position along the path of the radiation. A situation to which this case applies, and one of great importance for modeling stellar atmospheres, is that of local thermodynamic equilibrium (LTE) where $\epsilon$ and $\kappa$ are related through the Plank function. We may contrast this with a non-LTE, or scattering, source function for which the emissivity at a given point is determined, at least in part, by the radiation intensity at that point which is, in turn, a statistical variable not having a unique correlation with the local value of the absorption coefficient. The general problem where emission and/or absorption at a point in the medium are themselves dependent on the intensity at that point appears significantly more difficult to address and is not considered further at this stage. The theory which we present here allows us to incorporate inhomogeneities in a straightforward way provided the radiation is formed in LTE. The far-infrared is consistent with this restriction, and observations in that wavelength region, therefore, should be very valuable for inferring the characteristics of a multicomponent solar atmospheric model without the added complexity introduced by departures from LTE.

The basis for the formulation of the transfer problem for inhomogeneous atmospheres is presented in § II. Our approach is to subdivide the atmosphere into equal finite cells and to develop the intensity as a statistical (or random) variable whose properties are determined by the statistical characteristics of the source function, and the cellular optical depth, for each of the atmospheric components.

Section III contains the development of a radiative transfer equation for an inhomogeneous atmosphere. For simplicity we first discuss the case where the source functions, optical depths, and cellular occupancy probabilities for the inhomogeneities are discrete, and constant in value at all points along the line of propagation. This case is then generalized to allow variations of these parameters along the radiation path.

As long as the physical parameters of the problem can take only a finite set of values, the random variable describing the emergent intensity must be a discrete set. When the source function or optical depth of an element of a specific type is described by a continuous probability distribution (so that it can take any value within a given range) then the emergent intensity will clearly be a continuous random variable. A generalization to continuous probability distributions is discussed in § IV.

A numerical illustration of the application of the theory is given in § V. Here we consider transfer through an inhomogeneous spherical atmosphere. In particular, we compute the mean and variance of the intensity of the emergent radiation in the region immediately below the limb. This geometry is modeled on that found at a total eclipse of the Sun to which this theory is to be applied in detail in a subsequent paper.

II. FORMULATION OF THE PROBLEM

Consider a medium made up of a number of "structural elements" which are more or less equivalent to one another and dispersed in an ambient medium. Both the ambium and the structures will, in general, emit and absorb radiation at the wavelengths of interest to us, and we wish to describe the characteristics of the flux of radiation which emerges in a given direction from such a gas. We need not enquire too deeply into the definition of a "structure"—suffice it to say that we mean a finite region over which the physical conditions are similar, but quite different from the corresponding conditions outside this region. Nor need we limit ourselves to only two media (structure and ambium). We adopt this for expository purposes, although the example given in § V adopts a model with three structural elements—the generalization of the formalism to an arbitrary number of components is straightforward.

Suppose, following Figure 1, that the radiation whose evolution we wish to follow propagates at an angle to the $z$ direction and let us subdivide the $z$ axis into cells of a size equal to the characteristic dimension of a structure.² Note that the rectangular geometry of Figure 1 can readily be generalized—as is done in § V where we use a spherical model. We shall require that, if a structural element is found in a cell, it shall fall completely into that cell, i.e., without overlap from or into its neighbors. In other words we suppose that the probability, at a given height, that a cell will be occupied by material of type $i$ is independent of the occupancy state of any other cells and, in particular, those which are adjacent to it. Note, however, that a ray which proceeds at an angle to the horizontal (as illustrated in Fig. 1) may pass from one medium to another in passage through a given cell if the probability of occurrence of that cell...
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Fig. 1.—Propagation of radiation through the atmosphere. The radiation progresses in the direction of increasing \( m \) and emerges from the gas after the \( n \)th cell.

The medium decreases with height (so that the ray may pass out of the top of the medium before leaving the cell).

As in Figure 1, again, we suppose that radiation of a specified intensity \( I_0 \) (which may be zero) is incident on one face of the atmosphere (at cell 1). We shall seek the (statistical) characteristics of the radiation intensity, \( I_{em} \), which emerges from the final cell. For example, \( I_0 \) may be the intensity incident from a more or less homogeneous underlying gas ("the photosphere") and \( I_{em} \) that which emerges after interaction with the "chromospheric" and "coronal" gases made up of spicules and interspicular material. In the case of observations above the solar limb, e.g., at a total eclipse, the \( z \) direction is parallel to the solar tangent and \( I_0 \) is in general zero.

In the usual examples of radiative transfer, when we specify a priori the values of the radiative properties (optical depths and source functions) at all points along the line of sight, it is straightforward to integrate the transfer equation to obtain the emergent intensity whether or not the gas is homogeneous. However, in our problem these quantities are specified only statistically, i.e., they are "random variables" (rather than physical variables having specified values at each point on the line of sight), and a calculus of probabilities must be used to work with them. To solve our problem, therefore, we must seek the statistical law which describes the random variable by which the emergent intensity is represented, using the given specification of the random variables representing the emission and absorption coefficients. From that we shall be able to compute such observable quantities as the expected value and variance of the emission from the inhomogeneous gas.

A random variable is characterized by a table (or more generally a continuous relationship) giving the set of all possible values which the associated physical variable can take, together with a set of probabilities associated with each of these values. To illustrate, consider the trivial case in which there are only two cells along the line of sight, and the optical depths \( \tau \) and source functions \( S \) can each take only one of two possible values (one for the structural element, one for the ambium) throughout the cell. In this case the random variable \( I_{em} \) takes one of four values corresponding to the value which the emergent intensity assumes for each of the four possible occupancies of the two cells, viz. structures in both cells, a structure in the first and ambium in the second, ambium in both cells, and ambium in the first with a structure in the second. The associated probabilities would be \( p_1^2, p_1p_2, p_2^2, p_2p_1 \), where \( p_1 (= 1 - p_2) \) is the probability of finding a structure in a cell. Similarly, if we had three cells and, again, \( S_i \) and \( \tau_i \) could each take only one of two discrete values, then \( I_{em} \) would have \( 8 (= 2^3) \) components each having an associated probability.

The optical depth and the source function in each cell are random variables which we shall denote by \( \tau \) and \( S \), respectively. Our approach will be to derive a recurrence relationship of the random variable \( F^{(m)} \) representing the statistical description of the intensity of the radiation emerging from the \( m \)th cell—in terms of \( F^{(m-1)} \) and the sources and sinks of the radiation field as represented by \( I_0 \) and the random variables \( \tau \) and \( S \).

Thus, consider the random variable \( F^{(1)} \) representing the radiation coming out of the first cell. At the point where the ray enters, this cell will contain either a structural element, or ambient material and we assign probabilities \( p_1, p_2 (= 1 - p_1) \), respectively, to these two possibilities. In either case the specific intensity of the radiation emerging from the cell may be written:

\[
F^{(1)} = I_0 \exp (-\tau) + \int_0^\tau S(t)e^{-t-\tau}dt. \tag{1}
\]

We shall suppose that the optical path length through a cell takes only one of two values, \( \tau_1, \tau_2 \) and that the corresponding source functions are \( S_1(t), S_2(t) \). The random variables \( F^{(1)}, \tau, \) and \( S \) are then described by Table 1 which gives the values and probabilities of each of them for each of the two possible cell occupancies shown in the leftmost column.

If we assume, for example, that the source function in a cell is not a function of position within that cell, either for the structural or interstructural component, then from equation (1), we see that the expected value of the random variable \( F^{(1)} \) would then take the simple form:

\[
\langle F^{(1)} \rangle = \sum_{i=1}^2 p_i I^{(1)}(i) = I_0(p_1e^{-\tau_1} + p_2e^{-\tau_2}) + [S_1p_1(1 - e^{-\tau_1}) + S_2p_2(1 - e^{-\tau_2})]. \tag{2}
\]

### Table 1

**Specification of the Random Variables \( F^{(1)}, \tau, S \)**

<table>
<thead>
<tr>
<th>Cell Contents</th>
<th>Probability of Occupancy</th>
<th>Optical Depth</th>
<th>Source Function</th>
<th>Intensity Emerging from Cell 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structure .....</td>
<td>( p_1 )</td>
<td>( \tau_1 )</td>
<td>( S_1(t) )</td>
<td>( I_0e^{-\tau_1} + \frac{1}{2}S_1(t)e^{-\tau_1}dt )</td>
</tr>
<tr>
<td>Ambium .....</td>
<td>( p_2 )</td>
<td>( \tau_2 )</td>
<td>( S_2(t) )</td>
<td>( I_0e^{-\tau_2} + \frac{1}{2}S_2(t)e^{-\tau_2}dt )</td>
</tr>
</tbody>
</table>

\[3^\text{We represent random variables in boldface type to differentiate them from the corresponding physical variables.}\]
By a simple extension of the above we find that the random variable \( F^m \), representing the intensity of the radiation emerging from the \( m \)th cell, can be represented in terms of that from cell \((m - 1)\) by the equation:

\[
F^m = F^{m-1} \exp (-\tau^m) + S^m[1 - \exp (-\tau^m)],
\]

with \( F^0 = I_0 \), and where we assume, again, that the source functions do not vary across a cell. The evolution, from one cell to the next, of the expected value of \( F^m \)—can be derived directly from equation (3). Obviously this expected value will depend on the distribution function for \( \tau, S \)—the form of this dependence is developed in the following section.

III. THE EXPECTATION AND VARIANCE OF THE INTENSITY

We shall first consider the formulation of an algorithm to express the progression from cell to cell of the expected value of the intensity in a statistically specified inhomogeneous gas. For clarity we start with a simple case in which the random variables \( S \) and \( \tau \) can each take only one of two values \((S_t, \tau_t)\) in a given cell. We further suppose, for this first case, that the parameters \( S_t, \tau_t \) are the same at every point along the line of sight where an element of type 1 is encountered, and that the same applies to \( S_2, \tau_2 \). We then generalize to the situation where the latter restriction is dropped, but still \( S^m_t, \tau^m_m \) can each take only two values. The further generalization to the case where the probability distributions for the radiative parameters can take a continuous range of values is deferred to § IV.

Higher moments of the distribution of the intensity can be discussed along similar lines. We shall consider here only the second moment (or equivalently the variance) as having most practical importance after the mean. We shall find that the distribution function for \( \tau, S \)—the form of this dependence is developed in the following section.

\[\begin{align*}
\langle F^m \rangle &= \sum_{k=1}^{2^m} p^{(m)}(k) F^m(k), \\
\end{align*}\]

we find, after a little consideration and making use of equation (3), that the expected value of the radiation emerging from the \( m \)th cell may be written:

\[
\langle F^m \rangle = \sum_{k=1}^{2^m} p^{(m)}(k) F^m(k).
\]

An interesting alternate form is obtained if we define, by \( \Delta \langle F^m \rangle \), the change in the expectation value of \( F \) in passing through the \( m \)th cell, i.e.,

\[
\langle F^m \rangle = \langle F^{m-1} \rangle + \Delta \langle F^{m-1} \rangle.
\]

With this definition equation (8) can be expressed

\[
\Delta \langle F^{m-1} \rangle = -(1 - \alpha) \langle F^{m-1} \rangle - \mathcal{E},
\]

and, of course, equations (11) and (12) become identical in the limit of a homogeneous medium.

In fact, the form of equation (8)—or equation (11)—is fundamental to inhomogeneous transfer independently of the geometrical assumptions regarding the distribution of material (or structures) along the line of sight. Just as we find for continuous transfer in the context of equation (12), the difference from one physical situation to another resides in the expressions for the quantities \( \alpha \) and \( \mathcal{E} \). The point will become clearer as we develop some examples in subsequent sections. We note also that, according to equation (11), we may introduce the concept of an "equivalent smooth atmosphere" for determining the expectation intensity \( \langle F \rangle \) in this inhomogeneous atmosphere by identifying the quantities \( \mathcal{E} \) and \((1 - \alpha)\), respectively, with the source function and the element of opacity.

\[\begin{align*}
4 \quad \text{The generalization of this equation to an atmosphere with } r \text{ components (rather than 2 as assumed here) is obtained simply by replacing } 2 \text{ and } 2^{m-1} \text{ in the sums by } r \text{ and } r^{m-1}.\n\end{align*}\]
A successive application of equation (8)—with \( \langle F_{m}^{0} \rangle = I_0 \)—allows us to study the evolution of the mean value of the radiation intensity as it propagates through the gas, and so to compute the mean emergent radiation. It is straightforward, however, to resolve this recurrence relationship into the closed form:

\[
\langle F_{m}^{0} \rangle = \alpha_{m}I_0 + \mathcal{F} (1 - \alpha_{m}) ,
\]

(13)
a result that is easy to confirm by induction.

Note, in particular, that if \( m \) becomes arbitrarily large (as for a semi-infinite plane-parallel atmosphere), we find

\[
\langle F_{m}^{0} \rangle = \mathcal{F} (1 - \alpha_{m}) .
\]

(14)

where the product over \( \alpha_{j} \) is set to unity when \( k = m \).

b) Generalization for Variations along the Line of Sight

A more general result is obtained when the quantities \( p_{i}, \tau_{i}, S_{i} \) are allowed to vary from cell to cell along the line of sight but with \( S_{i} \) again constant within a cell. Again, in this case \( F_{m}^{0} \) will have \( 2^{m} \) components since the \( \tau_{i}, S_{i} \) pairs are assumed to have only two possible values for each cell. The generalization of equation (6) is easily found to be.\(^5\)

\[
\langle F_{m}^{0} \rangle = \sum_{i=1}^{2^{m}} p_{i}(m) \times \left[ \sum_{j=1}^{2^{m-1}} p_{j}^{(m-1)} I_{j}^{(m-1)}(1) + S_{i}(m)(1 - e^{-\tau_{i}(m)}) \right] ,
\]

(15)

which yields, as the generalization of equation (7),

\[
\langle F_{m}^{0} \rangle = \left[ p_{1}(m)e^{-\tau_{1}(m)} + p_{2}(m)e^{-\tau_{2}(m)} \right] \langle F_{m-1}^{0} \rangle
\]

+ \left[ p_{1}(m)S_{1}(m)(1 - e^{-\tau_{1}(m)} + p_{2}(m)S_{2}(m)(1 - e^{-\tau_{2}(m)}) \right] .
\]

(16)

We may then express the generalization of equation (8) in the form

\[
\langle F_{m}^{0} \rangle = \alpha_{m}\langle F_{m-1}^{0} \rangle + (1 - \alpha_{m})\mathcal{F}_{m} ,
\]

(17)
or (see eq. [13]),

\[
\Delta \langle F_{m-1}^{0} \rangle = -(1 - \alpha_{m})\langle F_{m-1}^{0} \rangle - \mathcal{F}_{m} ,
\]

(18)

where

\[
\alpha_{m} = p_{1}(m)e^{-\tau_{1}(m)} + p_{2}(m)e^{-\tau_{2}(m)} = \langle e^{-\tau_{1}(m)} \rangle ,
\]

\[
\mathcal{F}_{m} = \frac{p_{1}(m)S_{1}(m)(1 - e^{-\tau_{1}(m)} + p_{2}(m)S_{2}(m)(1 - e^{-\tau_{2}(m)})}{p_{1}(m)(1 - e^{-\tau_{1}(m)} + p_{2}(m)(1 - e^{-\tau_{2}(m)})}
\]

(19)

The analogous result to equation (13) is readily obtained as

\[
\langle F_{m}^{0} \rangle = I_0 \prod_{i=1}^{m} \alpha_{i} + \sum_{k=1}^{m} (1 - \alpha_{k})\mathcal{F}(k) \prod_{j=k+1}^{m} \alpha_{j} ,
\]

(20)

where the product over \( \alpha_{j} \) is set to unity when \( k = m \).

Again we note the formal relationship to radiative transfer in a smooth atmosphere, for which the analog to equation (20) is

\[
I = I_0 e^{-\tau} + \int_{0}^{\tau} S e^{-\tau} d\tau .
\]

(21)

If we define an effective optical thickness \( \tau_{m} \) from cell \( i \) to cell \( m \) as

\[
\tau_{m} \equiv \ln \left( \prod_{i=1}^{m} \alpha_{i} \right) = \sum_{i=1}^{m} (\ln \alpha_{i}) ,
\]

(22)

and an effective function \( \mathcal{F} \) via equation (19), then equations (20) and (21) become essentially equivalent. Seen from this perspective, therefore, even a structured atmosphere can be considered smooth in a certain statistical sense.

c) The Variance of the Intensity

The variance of the radiation intensity is, next to the mean, the most fundamental statistical radiation characteristic entering into our problem and, as such, merits a separate discussion. In an inhomogeneous medium the variance will be nonzero, in contrast to the case of a smooth (homogeneous) atmosphere where it, and all moments other than the mean, are essentially zero. Correspondingly, while the evolution of the mean value of the intensity through the medium can be described by the conceptually useful idea of an "equivalent smooth atmosphere," we shall find no such parallel for the variance.

For the more general case discussed immediately above we may write the variance of the intensity emerging from the \( m \)th cell in the form:

\[
\sigma_{m}^{2} = \sum_{j=1}^{2^{m}} \sigma^{2}_{m}(j)I_{j}^{(m)}(j) - \langle F_{m}^{0} \rangle^{2} .
\]

(23)

Following procedures similar to those used in deriving equation (16), we find readily enough that the expression (23) can be recast in the form:

\[
\sigma_{m}^{2} = \langle e^{-2\tau_{m}} \rangle \sigma_{m-1}^{2} + \mathcal{F}^{2}_{m-1} [\langle S_{m}(1 - e^{-\tau_{m}}) \rangle] + 2 \langle F_{m-1}^{0} \rangle \mathcal{F} \langle e^{-\tau_{m}} \rangle - \langle S_{m}(1 - e^{-\tau_{m}}) \rangle \langle e^{-\tau_{m}} \rangle .
\]

(24)

The expression (24) brings out the dependence of the variance of the intensity on the statistical characteristics of the radiation parameters, \( \tau_{i}, S_{i} \) and shows clearly that the variance disappears for a homogeneous atmosphere.

The recurrence relationship (24) may be expressed in the form:

\[
\sigma_{m}^{2} = a_{m} \sigma_{m-1}^{2} + b_{m} ,
\]

(25)

where \( a_{m} = \langle e^{-2\tau_{m}} \rangle \) and \( b_{m} \) is the sum of the last three terms of equation (24). Since the expression (25) has the same form as equation (17), we find immediately that the variance of the intensity emerging from the \( m \)th cell is:

\[
\sigma_{m}^{2} = \sum_{k=1}^{m} b_{k} \prod_{j=k+1}^{m} a_{j} ,
\]

(26)

where the product over \( a_{j} \) is set to unity when \( k = m \) and we note that \( \sigma_{0}^{2} = 0 \) since the incident intensity \( I_0 \) is a specified (rather than a statistical) quantity.

IV. MORE GENERAL PROBABILITY DISTRIBUTIONS

Because the cases discussed above allow only two possible values for the optical depth in a given cell, and two associated values for the source functions, the distributions obtained for the random variable \( I \) are discrete. For practical applications this represents a significant limitation even when the values of the radiation parameters are allowed to change from cell to cell along the line of sight. Thus, for example, if we wish to study

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the transfer of radiation through a gas consisting of cylindrical structures, or of structures with variable inclinations to the line of sight, then the possible optical path length which the line of sight encounters in traversing a structural element will range over a continuous distribution from zero up to some maximum value so that the random variable \( T \) will also cover a continuous range of values. In this section we discuss the changes to the preceding formulation which are needed to accommodate such continuous distributions. We shall continue to adopt a model which has only two components statistically distributed along the line of sight, and suppose that the emission and absorption coefficients are uniquely determined by the local conditions.

The statistical variation of the radiation parameters may arise from the random occupancy of the cells and from statistical variations in the length within a given cell which is occupied by material of a given type. Only the former source of statistical variation was considered in the preceding sections.

The generalization of equation (3) which is needed to accommodate the case where the optical thickness in a cell can vary continuously over a range is readily obtained in the form:

\[
F^m = F^{m-1} \exp \left[ - \int_0^{s_0} \kappa^{(m)} dl \right] + \int_0^{s_0} S^{(m)}(l) \exp \left( - \int_l^{s_0} \kappa^{(m)} dl \right) \kappa^{(m)} dl.
\]

In this equation \( s_0 \) is the slant length along the ray path through a cell and \( \kappa^{(m)} \) and \( S^{(m)} \) are the random variables representing the absorption coefficient and source function in the \( m \)th cell. The generalization of equation (6) for the expected value of \( F^m \) is obtained by taking the expected value of both sides of equation (29). Since, again, the statistical properties of \( F^{m-1} \) are independent of those of the attenuation term in the \( m \)th cell, the expected value of the optical depth is just the product of their expectation values. We therefore recover the basic form of equation (8) for the expected value of the intensity, namely

\[
\langle F^m \rangle = \langle F^{m-1} \rangle + (1 - \alpha_m) S_m,
\]

where now the physical quantities \( x \) and \( S \) are defined in a more general manner as

\[
\alpha_m = \langle \exp \left[ - \int_0^{s_0} \kappa^{(m)} dl \right] \rangle,
\]

\[
(1 - \alpha_m) S_m = \langle \int_0^{s_0} S^{(m)}(l) \exp \left( - \int_l^{s_0} \kappa^{(m)} dl \right) \kappa^{(m)} dl \rangle.
\]

The coefficients in the recurrence relationship (25) for the variance are to be generalized in an analogous manner. The averages in the two quantities \( x \) and \( S \) are taken over the probability distribution of the optical path length of each component in a cell. To compute this probability distribution we need a model, and we shall briefly consider one case in this section; a different case is discussed in more detail in the following section.

Suppose that the distribution \( f(\tau) d\tau \) represents the probability that the (cellular) optical depth encountered by a ray passing through a structure lies between \( \tau \) and \( \tau + d\tau \). If we neglect the residual ambient opacity in any cell which contains, but is not fully occupied by, a structure (an artificial situation but useful for illustration), then the probability distribution for the optical depth in a cell is

\[
\phi(\tau) = p_1 f(\tau) + p_2 \delta(\tau - \tau_2),
\]

where we have suppressed the index \( m \) for clarity. The expectation value \( \langle F^m \rangle \) then becomes (see eq. [17]),

\[
\langle F^m \rangle = \langle F^{m-1} \rangle \left[ p_1 \int_0^{\tau_1} e^{-\gamma_1} f(\tau) d\tau + p_2 e^{-\tau_2} \right] + \int_0^{s_0} S^{(m)}(l) \exp \left( - \int_l^{s_0} \kappa^{(m)} dl \right) \kappa^{(m)} dl.
\]

As an example, consider a case where the structures are circular cylinders of radius \( a \), with axes normal to the direction of propagation. If we let \( x \) be the closest distance which the optical path comes to the axis, and assume \( x \) to be distributed uniformly between \(-a\) and \(+a\), then it is readily seen that the optical depth will be distributed as

\[
\phi(\tau) = \frac{\tau/4ka}{\sqrt{(ka)^2 - \tau^2/4}}, \quad 0 \leq \tau \leq 2ka,
\]

with \( k \) being the absorption coefficient. Equation (31) then becomes, for this case,

\[
\langle F^m \rangle = \langle F^{m-1} \rangle \left[ p_1 \psi(ka) + p_2 e^{-\tau_2} \right] + S_1 p_1 \left[ 1 - \psi(ka) \right] + p_2 S_2 (1 - e^{-\tau_2}),
\]

with

\[
\psi(ka) = \frac{\int_0^{2ka} e^{-\tau} d\tau}{4ka \sqrt{(ka)^2 - \tau^2/4}}.
\]

Expressions for the expected values of the random variables entering into the expression for the variance of \( F^m \) can be obtained, for this case, in an analogous manner.

V. A NUMERICAL EXAMPLE

To illustrate the application of the formalism presented here we consider the problem arising in the analysis of eclipse observations in the infrared. We shall choose, correspondingly, numerical values of the absorption and emission coefficients which are representative of solar conditions at these wavelengths and a model with statistical characteristics similar to those observed in visible wavelengths. We wish here simply to exemplify the approach developed in this paper, rather than to seek an explicit inhomogeneous model of the Sun which is deferred to a subsequent paper.

We consider, then, the transfer of radiation through an inhomogeneous atmosphere consisting of radial structures ("spicules") immersed in a coronal ambion the whole being underlain by a spherical homogeneous (photospheric) gas. The radiation incident from this photosphere on the overlying inhomogeneous gas has an intensity \( I(\mu) \) at an angle \( \cos^{-1} \mu \) to the radial direction. We suppose that the (spicular) structures have an exponential height distribution such that the probability that the top of one lies within the height range \( h \) to \( h + dh \) is given by

\[
p_1(h) dh = p_1 \exp (-\gamma_1 h) dh,
\]

where we have suppressed the index \( m \) for clarity. The expectation value \( \langle F^m \rangle \) then becomes (see eq. [17]),

\[
\langle F^m \rangle = \langle F^{m-1} \rangle \left[ p_1 \int_0^{\tau_1} e^{-\gamma_1} f(\tau) d\tau + p_2 e^{-\tau_2} \right] + \int_0^{s_0} S^{(m)}(l) \exp \left( - \int_l^{s_0} \kappa^{(m)} dl \right) \kappa^{(m)} dl.
\]
TABLE 2

PARAMETER VALUES ADOPTED FOR SAMPLE CALCULATIONS

<table>
<thead>
<tr>
<th>Physical Variable</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spicule source function</td>
<td>( S_1 )</td>
<td>1.2*</td>
</tr>
<tr>
<td>Coronal source function</td>
<td>( S_2 )</td>
<td>150*</td>
</tr>
<tr>
<td>Spicule absorption coefficients ( \kappa_1 )</td>
<td></td>
<td>((2 \times 10^{-4}, 5 \times 10^{-4}, 10^{-3}) ) km(^{-1})</td>
</tr>
<tr>
<td>Absorption coefficient ratio ( \kappa_2/\kappa_1 )</td>
<td></td>
<td>10(^{-4})</td>
</tr>
<tr>
<td>Spicule scale height</td>
<td>( y_s )</td>
<td>(5 \times 10^{-4}) km(^{-1})</td>
</tr>
<tr>
<td>Coronal height factor</td>
<td>( y_c )</td>
<td>(2.5 \times 10^{-8}) km(^{-1})</td>
</tr>
</tbody>
</table>

* Units of photospheric intensity at the disk center.

Notes.—The source functions in the far-IR will be proportional to temperature. A spicular temperature of \(~6000\) K (see Beckers 1972) corresponds to a source function somewhat larger than that of the photosphere. The corona at \(10^6\) K gives a value of \(S_2\sim150\) times greater.

The absorption coefficients are representative of spicular and coronal densities and temperatures at wavelengths from a few tens of microns to a millimeter or so.

The scale height factors \(y_1\) and \(y_2\) are representative—no attempt is made to relate them in detail to solar conditions.

where \(h\) is the radial height above the top of the homogeneous (photospheric) layer. Similarly, the probability that a cell will be occupied by coronal ambium is assumed to fall off with height according to the law:

\[
p_2(h) = [1 - p_1(h)] \exp ( -y_2 h^2 ) .
\]  

(36)

The remaining volume, which occurs with probability \(1 - p_1(h) - p_2(h)\), is taken simply to be inert and so not to contribute either to the emission or the absorption. This characterization is convenient and a sufficient representation of the (itself highly inhomogeneous) corona as we need for our expository purpose. Finally, in order to limit the number of radial cells included in the analysis, we terminate the atmosphere after it reaches a certain radial height which is chosen to make both \(p_1(h)\) and \(p_2(h)\) much less than unity there.

Parameter values, adopted to be more or less representative of solar conditions, are set out in Table 2.

If a ray encounters a structural element on entering a particular cell, then there is a certain probability that it will pass out from the top of that structure prior to completing its passage through the cell. The probability distribution \(P(s)\) for the length \(s\) of the path which lies in the structure (given that a structure is present in the cell) is readily seen to be given by

\[
P(s)ds = \left[ y \sin \theta \exp ( -y_2 \sin \theta ) + \exp ( -y h_0 ) \delta(s - s_0) \right] ds ,
\]  

(37)

where (see Fig. 2*) \(s\) is the slant length along the ray, \(s_0\) is the corresponding length through the cell, and \(h_0\) is the height difference between the entry and exit points of the ray.

From the probability distribution (37) we may readily obtain the expected values of the extinction and emission terms in the basic recurrence relationship (28). Explicit expressions for these terms are derived in the Appendix, and, using these, we may compute the expected value of the radiation leaving the atmosphere using equation (20). The statistical moments required to compute the variance, using equation (26), are also found by straightforward integration as indicated in the Appendix.

With the parameter values of Table 2 we have computed the

\[ h_{m-1} \]

\[ \text{Top of Structure} \]

\[ h_m \]

Fig. 2.—Geometry of passage of a ray through a structure in the \(m\)th cell. The height \(h_0\) is the difference \(h_m - h_{m-1}\). For simplicity the figure is drawn for plane-parallel geometry rather than spherical geometry.
Fig. 3.—Expected value of the emergent intensity across the limb, for the inhomogeneous model defined in the text, for three values of the structure absorption coefficient. The abscissa is the ratio of the intercept point of the ray, or the impact parameter, to the radius at the top of the homogeneous layer.

Fig. 4.—Expected value of the intensity and the 1σ limits to either side for the model defined in the text and for an absorption coefficient of $10^{-3}$ km$^{-1}$. See legend to Fig. 3 for further details.
mean and the standard deviation of the intensity of the emergent radiation at a series of positions on the disk and above the limb. Means are shown in Figure 3 for three values of the absorption coefficient $\kappa_1$ (the ratio $\kappa_1/\kappa_2$ being kept constant at $10^{-4}$), and the mean, with the standard deviation above and below, is shown for two cases in Figures 4 and 5. These calculated limb profiles are similar to those obtained from our eclipse observations (see Lindsey et al. 1986). However, whether or not a detailed comparison of computations with observational data will allow the inference of a reliable inhomogeneous model remains to be seen from future studies.

VI. SUMMARY

In this paper we have derived a basic transfer equation from which the expected value and the variance of the radiation at a point in an inhomogeneous atmosphere may be calculated given the statistical relations governing the emission and absorption in the medium. The example given in § V illustrates an application of the theory; further cases specifically directed to the modeling of the solar atmosphere will be discussed in subsequent publications.

The theory provides us with a basic equation for the evolution of the random variable, $I(r)$, and from that we may find equations governing the moments of the distribution. As we have seen, this can be done readily enough for the cases where the emission and extinction terms are given either by a discrete or a continuous distribution. However, the present approach does not seem a profitable way to compute the distribution function itself. We can certainly do this for the discrete case through direct application of equation (3), although with $2^n$ terms to calculate this soon becomes a prohibitive task. For a continuous distribution of the emission and extinction terms one might use a Monte Carlo method for selecting a value of $r_m$ from its distribution, and solve equation (3) using the discrete values thus obtained—however, again, that is cumbersome, at best.

A better approach is found in working directly with the distribution function $f(l)$ and developing an equation for the change which this function experiences in progressing an elementary distance through the gas. Such an approach can be developed; it leads to an equation, analogous to the Boltzmann equation, which describes $f(l)$ in terms of its partial derivatives with respect to position and intensity, and the physical variables governing emission and absorption. This approach will be the subject of a subsequent paper in this series.

APPENDIX

We outline below the procedures for calculating the expectation values required to determine the mean and variance of the emergent intensity for the model defined in § V.

I. EXPECTATION VALUES INVOLVING THE ABSORPTION TERM

When a ray encounters a structure on entering a cell, its optical path length in that cell may be expressed

$$\tau_s = \tau_1(s) + \tau_2(s),$$

Fig. 5.—As for Fig. 4, but with an absorption coefficient of $2 \times 10^{-4} \text{ km}^{-1}$. 

One of us (J. T. J.) would like to express his thanks for the kind hospitality extended by the staff of the Institute for Astronomy, University of Hawaii, and particularly by its Director, Donald N. B. Hall.
with

\[ \tau_1(s) = \kappa_1 s, \]  
\[ \tau_2(s) = \kappa_2 (s_0 - s), \]  

where (see Fig. 2) \( s_0 \) is the total slant length through the cell and \( s \) is the path length of the ray in the structure. The statistical variable \( s \) in our problem is distributed as in equation (37), i.e., as

\[ P(s) = \frac{\gamma \sin \theta e^{-\gamma \sin \theta s} + \delta(s - s_0) e^{-\gamma \sin \theta s_0}}{\gamma \sin \theta + \delta(k_1 - k_2)}. \]

Clearly equation (A1) assumes that ambium lies above the spicular structure (rather than inert material).

The component of the expected value of the attenuation factor which arises from the case when a structure is present in the cell can then be written

\[ T_1 = \int_0^{s_0} P(s) \exp \left[ - (\kappa_1 s + \kappa_2 (s_0 - s)) \right] ds = \frac{\gamma \sin \theta e^{-k_2 s_0} + (k_1 - k_2) e^{-\gamma \sin \theta s_0}}{\gamma \sin \theta + k_1 - k_2}. \]

If the ray encounters only ambient material in the cell, then the expected value of the attenuation factor is simply

\[ T_2 = e^{-k_2 s_0}. \]

Finally, if the cell is empty (or occupied by inert matter), the attenuation factor is unity.

The expected value of \( e^{-\tau} \) is then

\[ \langle e^{-\tau} \rangle = p_1^{(m)} T_1 + p_2^{(m)} T_2 + 1 - p_1^{(m)} - p_2^{(m)}, \]

where, for our problem,

\[ p_1^{(m)} = p_1 \exp (-\gamma_2 h_m), \]
\[ p_2^{(m)} = [1 - p_1(h)] \exp (-\gamma_1 h_m^2), \]

and \( h_m \) is the (radial) height at the point where the ray enters the cell.

II. EXPECTATION VALUES INVOLVING THE EMISSION TERM

For the geometry of Figure 2 the emission contribution for the case when the ray entering cell \( m \) encounters a structure is

\[ S'(s) = S_1 (1 - e^{-\gamma_1}) e^{-\gamma_2} + S_2 (1 - e^{-\gamma_2}), \]

where \( \tau_1 \) and \( \tau_2 \) are defined in equation (A1), and we have dropped the cell identification \( m \) for convenience.

The component of the expected value of \( S \) which arises from the case when the entering ray encounters a structure is then

\[ R_1 = \int_0^{s_0} P(s) S(s) ds. \]

The term arising when the entering ray encounters ambium is simply

\[ R_2 = S_2 (1 - e^{-k_2 s_0}). \]

Finally, since inert material contributes nothing to the emission, we find

\[ \langle S_m \rangle = p_1(h_m) R_1 + p_2(h_m) R_2. \]

The terms required to compute the variance are found again simply by averaging them over the distribution \( P(s) \) when a structure is encountered or the distribution \( \delta(s - s_0) \) otherwise and forming the weighted sum as in equations (A4) and (A9).

III. RAYS TANGENT TO THE SPHERICAL ATMOSPHERE

For rays which pass through the entire spherical atmosphere (as at an eclipse or for observations above the solar limb) a slight modification to equation (A6) is needed in calculating the moments of those radiative parameters which involve the emission contribution. Thus, for radial cells which lie in the back half of the path (i.e., those more distant from the observer) it is straightforward to show that the emission factor corresponding to the case when a structure is present in the cell is

\[ S'(s) = S_1 (1 - e^{-\gamma_1}) + S_2 (1 - e^{-\gamma_2}) e^{-\gamma_1}, \]

where, again, \( s \) is the distance traversed in the structure. Here, however, the structure will be encountered at the exit of the ray from the mth cell, and the height \( h_m \) entering into the probability factors in equation (A9) will correspondingly be that at the exit.

For cells lying in the half of the ray path which is closer to the observer the discussion of § II of this Appendix applies. Also, no distinction is needed in calculating the attenuation factors for these tangent rays, since the factors are the same for each half of the ray path.
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