THE CONTINUOUS SPECTRUM OF MHD WAVES IN 2D SOLAR LOOPS AND ARCADES. FIRST RESULTS ON POLOIDAL MODE COUPLING FOR POLOIDAL MAGNETIC FIELDS

S. POEDTS* and M. GOOSENS

Astronomisch Instituut, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, B-3030 Heverlee, Belgium

(Received 3 January; in revised form 25 February, 1987)

Abstract. A first attempt is made to study the continuous spectrum of linear ideal MHD for 2D solar loops and to understand how 2D effects change the continuum eigenfrequencies and continuum eigenfunctions. The continuous spectrum is computed for 2D solar loops with purely poloidal magnetic fields and it is investigated how non-circularity of the cross-sections of the poloidal magnetic surfaces and variations of density along the poloidal magnetic field lines change the continuous spectrum and induce poloidal wave number coupling in the eigenfunctions. Approximate analytical results and numerical results are obtained for the eigenfrequencies and the eigenfunctions and the poloidal wave number coupling is clearly illustrated. It is found that the continuum frequencies are substantially increased, that the ranges of the continuum frequencies are considerably enlarged and that the derivatives of the continuum frequencies normal to the magnetic surfaces are substantially increased. The eigenfunctions are strongly influenced by poloidal wave number coupling. Implications of these findings for the heating mechanisms of resonant absorption and phase mixing are briefly considered.

1. Introduction

The mechanisms for heating the chromosphere and the corona of the Sun have attracted renewed attention in recent years as it was recognized that the theory based solely on the dissipation of acoustic waves generated in the convection zone could not explain the observations. X-ray observations have shown that the solar corona is a highly complex structure consisting of myriades of magnetic loops and that the coronal heating is closely associated with the magnetic field structure in the corona. Estimates of upper limits to the acoustic energy flux in the chromosphere and the transition zone based on observations in UV lines showed that the acoustic energy flux is too low to produce the necessary heating in the solar corona. It is now widely believed that the heating of the solar corona is magnetic in nature. Resonant absorption and phase mixing of Alfvén waves are two candidates for magnetic heating of the solar corona which have attracted attention in recent years (Ionson, 1978; Heyvaerts and Priest, 1983; Nocera et al., 1984; Sakurai and Granik, 1984; Sakurai, 1985). They are based on the existence of a continuous part in the spectrum of normal modes of linear ideal MHD (see, e.g., Appert et al., 1974; Goedbloed, 1975, 1983; Poedts et al., 1985; Goossens et al., 1985). The normal modes that correspond to the continuum frequencies have spatial eigenfunctions with non-square integrable singularities on the magnetic surfaces, and the continuous

* Research Assistant of the Belgian National Fund for Scientific Research.
part of the spectrum gives rise to independent oscillations of the individual magnetic surfaces at their own continuum frequency.

Resonant absorption involves the excitation of a collective wave with a frequency in the continuous spectrum. The collective wave is the remnant of a surface wave or a fast magneto-acoustic wave. It transfers energy across the magnetic surfaces and couples to the Alfvén continuum wave at the resonant magnetic surface where the driving frequency equals the local Alfvén frequency. In ideal MHD the amplitude of the wave becomes unbounded on the resonant surface, but non-ideal effects limit the amplitude to finite but very large values and cause strong dissipation. Phase mixing arises when individual magnetic surfaces are excited in phase at their footpoints. The magnetic surfaces oscillate at their own local Alfvén frequency. The waves that are generated propagate along the magnetic surfaces at their own local speed, and as they propagate, waves on neighbouring magnetic surfaces become increasingly out of phase. Large spatial derivatives are built up in the direction normal to the magnetic surfaces and these spatial derivatives lead to strong dissipation. In closed loops the spatial derivatives are built up in time and are caused by the variation of the Alfvén continuum frequency normal to the magnetic surfaces. Resonant absorption and phase mixing involve the same physical mechanism but differ in the way the energy of the driver is coupled to the continuum Alfvén waves. In phase mixing there is direct coupling, while in resonant absorption a collective wave is needed to couple the energy of the driver to the Alfvén continuum wave.

The range of driving frequencies for resonant absorption and phase mixing, the position of the resonant magnetic surface for a given driving frequency, the spatial confinement of resonant absorption and the efficiency of phase mixing are determined by the continuous spectrum of linear MHD. It is therefore necessary to study the continuous spectrum of linear MHD in order to understand resonant absorption and phase mixing. In the context of the heating of the solar corona, the continuous spectrum, resonant absorption and phase mixing have been studied only for 1D plasmas. In a 1D plasma, the equilibrium quantities depend on one spatial coordinate and are constant on a resonant surface as the resonant and magnetic surfaces coincide with the coordinate surfaces of the non-ignorable coordinate. The perturbed quantities are Fourier-analyzed with respect to the two remaining spatial ignorable coordinates so that the spatial behaviour of the eigenfunction of a singular mode on the resonant surface is described by one pair of wave numbers. In a 2D plasma, the resonant surfaces are still the magnetic surfaces, but the equilibrium quantities now vary not only normal to the magnetic surfaces but also on the magnetic surfaces and there is only one ignorable spatial coordinate. The variations of the equilibrium quantities on the magnetic surfaces introduce changes in the continuous spectrum compared with the 1D plasma case. The range of the continuum frequencies, the position of the resonant surface for a given frequency, and the variation of the continuum frequency normal to the magnetic surfaces differ for a 2D plasma from the corresponding quantities in a 1D plasma. In addition, the spatial behaviour of the eigenfunction of a singular mode in the resonant surface is changed. The dependence of the eigenfunction on the ignorable coordiante can of course
be specified by one wave number, but the dependence on the other coordinate on the magnetic surface cannot be described by one wave number as in the 1D case. This phenomenon of wave number coupling is typical for the continuous spectrum of 2D plasmas. The changes in the continuous spectrum caused by 2D effects of the equilibrium plasma affect resonant absorption and phase mixing. Appert et al. (1981) have studied resonant absorption for axisymmetric toroidal plasmas in tokamaks and have found that 2D effects can substantially influence resonant absorption. Study of the continuous spectrum of 2D solar loops and arcades is obviously needed to see how 2D effects change the continuous spectrum and influence resonant absorption and phase mixing.

The present paper initiates the study of the continuous spectrum for 2D solar loops and arcades. It is concerned with loops and arcades that have purely poloidal magnetic fields and investigates how non-circularity of the cross-sections of the poloidal magnetic surfaces and non-homogeneity of density on the magnetic surfaces change the continuous spectrum and induce poloidal mode number coupling in the solutions. The plan of the paper is as follows. The equilibrium and the system of magnetic coordinates are discussed in Section 2. The equations that govern the continuous spectrum of a 2D plasma with a poloidal field are given in Section 3. Analytical and numerical results for the continuous spectrum of 2D solar loops are presented in Sections 4 to 6. The effects due to the non-circularity of the cross-sections and to the stratification of density on magnetic surfaces have been studied separately in Sections 4 and 5. The combined effect of non-circularity of the cross-sections and density stratification on magnetic surfaces is studied in Section 6.

2. The Equilibrium and Coordinates

The wave number coupling is studied for static 2D models of coronal loops and arcades with a purely poloidal magnetic field. We take a straight configuration with periodicity length $2\pi$. A Cartesian coordinate system $(x, y, z)$ is adopted with $y$ the ignorable coordinate so that the equilibrium quantities depend on $x$ and $z$ but not on $y$.

The poloidal magnetic field is described by a flux function $\psi(x, z)$:

$$B = -\nabla \psi(x, z) \times \mathbf{l}_y,$$

(1)

with the equilibrium current $\mathbf{J} = \nabla^2 \psi \mathbf{l}_y$. Inside the plasma

$$\psi(x, z) = \psi_c \psi^*(x, z),$$

(2)

where

$$\psi^*(x, z) = \frac{x^2}{a^2} + \frac{z^2}{b^2}, \quad \psi_c = \frac{J_y a^2 b^2}{2(a^2 + b^2)}.$$

$\psi_c$ is the value of the flux function at $x = a$, $z = 0$ and is related to the magnetic field.
strength $B_c$ at $x = a$, $z = 0$ as

$$\psi_c^2 = \frac{1}{4}a^2B_c^2.$$ 

$\psi^*(x, z)$ is a dimensionless flux function.

The cross-section of the magnetic surfaces with the coordinate plane $y = 0$ are similar ellipses when $a \neq b$ within the plasma (see Figure 1). In order to use $\psi$, which is the continuous label of the magnetic surfaces, as a coordinate we introduce coordinates $(\theta, \psi, y)$ within the plasma, defined as

$$\begin{align*}
x &= a \left( \frac{\psi}{\psi_c} \right)^{1/2} \cos \theta, \\
z &= b \left( \frac{\psi}{\psi_c} \right)^{1/2} \sin \theta, \\
y &= y.
\end{align*}$$

The coordinate system $(\theta, \psi, y)$ has the properties:

$$\nabla \psi \cdot \nabla \theta = -\frac{\psi_c}{a^2b^2} (b^2 - a^2) \sin 2\theta,$$  

$$\nabla \psi \cdot \nabla y = 0,$$  

$$\nabla \theta \cdot \nabla y = 0.$$
\[ |\nabla \psi|^2 = \frac{4\psi \psi_c}{a^2 b^2} \left[ \left( \frac{b^2 + a^2}{2} \right) + \left( \frac{b^2 - a^2}{2} \right) \cos 2\theta \right], \tag{7} \]

\[ |\nabla \theta|^2 = \frac{\psi_c}{a^2 b^2 \psi} \left[ \left( \frac{b^2 + a^2}{2} \right) - \left( \frac{b^2 - a^2}{2} \right) \cos 2\theta \right]. \tag{8} \]

The relations (4)–(8) imply that the system of magnetic coordinates \((\theta, \psi, y)\) is non-orthogonal as \(a \neq b\). We have decided to use the non-orthogonal \((\theta, \psi, y)\) coordinate system as it allows us to cast the equations for the continuous spectrum in a simple form with all the geometrical effects contained in the metric tensor elements, \(g_{ij}\), and not appearing explicitly in the equations of the continuous spectrum (see next section). In addition, the Jacobian is constant:

\[ J = (\nabla \theta \cdot \nabla \psi \times \nabla y)^{-1} = \frac{ab}{2\psi_c}, \tag{9} \]

which is in direct contrast to the case of the more intuitive coordinate system, orthogonal flux coordinates, where the Jacobian is rather difficult to compute.

The relations (4)–(8) also show that the metric elements contain terms with \(\sin 2\theta\) and \(\cos 2\theta\). Since the metric elements are equilibrium quantities, the terms with \(\sin 2\theta\) and \(\cos 2\theta\) will introduce a coupling of different Fourier modes in \(\theta\) for non-circular cross-sections with even and odd modes decoupled. This decoupling is a consequence of the symmetry property of the ellipse.

The pressure satisfies the equation of hydrostatic equilibrium and is a linear function of \(\psi\) alone:

\[ p(\psi) = p_0 - 2 \frac{\psi_c^2}{\mu_o} \frac{a^2 + b^2}{a^2 b^2} \psi* \]

\[ = p_c \left[ 1 + \frac{1}{\beta_c} \left( 1 + \frac{1}{c^2} \right) (1 - \psi*) \right], \tag{10} \]

where

\[ c = \frac{b}{a}, \quad \beta_c = 2\mu_o \frac{p_c}{B_c^2}; \]

\(p_0\) is a constant and \(p_c\) is the value of the pressure for \(x = a, z = 0\). \(\beta_c\) is the plasma beta for \(x = a, z = 0\).

The density does not occur in the equation of hydrostatic equilibrium. The density varies normal to magnetic surfaces and hence depends on \(\psi\). We further consider the two cases where the density is either constant on magnetic surfaces or varies with height.
z on the magnetic surfaces, so that

$$\rho = \rho_0(\psi)$$

or

$$\rho = \rho_0(\psi) \exp(-\varepsilon_2 z)$$

with $\varepsilon_2$ a positive constant. $\rho_0(\psi)$ specifies the variation of density across the magnetic loop for $z = 0$. In this paper we take $\rho_0(\psi)$ to vary linearly with the distance across the loop. Other density profiles $\rho_0(\psi)$ will be considered in further investigations. The main object of the present paper is to illustrate the phenomenon of poloidal wave number coupling in 2D solar loops and arcades. Let $\psi = \psi_c$ and $\psi = \psi_2$ be the inner and outer magnetic surfaces that bound the loop and $\rho_2$ the density for $\psi = \psi_2$ and $z = 0$. The linear variation of $\rho_0(\psi)$ with distance across the loop is then given by

$$\rho_0 = \rho_c \left[ 1 - \left( 1 - \frac{\rho_2}{\rho_c} \right) \frac{1 - (x/a)}{1 - (x_2/a)} \right] \text{ for } z = 0,$$

$$\rho_0 = \rho_c \left[ 1 - \left( 1 - \frac{\rho_2}{\rho_c} \right) \frac{1 - \sqrt{\psi^*}}{1 - \sqrt{\psi_2^*}} \right],$$

where $x_2$ is the $x$-coordinate of $\psi = \psi_2$ for $z = 0$, and $\psi_2^*$ is defined as $\psi_2 = \psi_c \psi_2^*$. The equilibrium of the loop is specified by the eight quantities $a, c, \psi_2^*, B_c, \rho_c, \rho_2, \varepsilon_2$. The quantities $a, c$, and $\psi_2^* = \sqrt{x_2/a}$ determine the geometrical structure of the loop. $B_c$ and $\rho_c$ determine the magnetic field and the gas pressure. Finally, $\rho_c, \rho_2,$ and $\varepsilon_2$ specify the variations of density. The relative variations of density across and along the loop are determined by $\rho_2/\rho_c$ and $\varepsilon_2$, respectively. The system of magnetic coordinates is schematically shown in Figure 1.

3. Equations

The continuous part of the spectrum of a 2D plasma is governed by two second order differential equations on the magnetic surfaces (Poedts et al., 1985; Goedbloed, 1975). For a purely poloidal magnetic field the continuous spectrum is degenerate with respect to the toroidal wave number and the two second-order differential equations that govern the continuous spectrum, are uncoupled. The continuous spectrum then consists of two uncoupled parts which can be referred to as an Alfvén continuum and a slow continuum since the solutions are polarized in the magnetic surfaces either perpendicular to the magnetic field lines (Alfvén continuum) or along the magnetic field lines (slow continuum) (Poedts et al., 1985; Goossens et al., 1985).

The equations are derived as in Poedts et al. (1985) and Goossens et al. (1985), but $(\theta, \psi, y)$ are non-orthogonal flux coordinates now. We assume the periodicity length in the ignorable $y$-direction is $2\pi$ and we Fourier-decompose the eigenfunctions in this
coordinate:

\[ \xi^i(\psi, \theta, \gamma) = \sum_{n=1}^{\infty} \xi^i_n(\psi, \theta) e^{in\gamma} \quad (\text{real part}). \] (13)

The different terms in \( n \) decouple and each toroidal wave number can be treated separately. We drop the subscript \( n \) in what follows. The perturbed quantities are Fourier-analyzed with respect to time \( t \) and put proportional to \( \exp (i \sigma t) \). After a lengthy calculation, which is omitted here, we obtain the following equations, where we use covariant and contravariant tensor notation, for the Alfvén continuum and the slow continuum, respectively:

\[ \lambda \rho(\psi, \theta) \xi^x(\psi, \theta) + \frac{\partial^2}{\partial \theta^2} \xi^x(\psi, \theta) = 0, \] (14)

\[ \lambda \rho(\psi, \theta) g_{\theta\theta}(\psi, \theta) \xi^\theta(\psi, \theta) + \frac{\partial}{\partial \theta} \left( g_{\theta\theta} - \frac{B_0 B_\theta}{B^*} \right) \frac{\partial}{\partial \theta} \xi^\theta(\psi, \theta) = 0, \] (15)

where

\[ \lambda = \frac{\mu_0}{(B^*)^2} \sigma^2, \quad g_{\theta\theta} = J^2 B^2, \quad B_0 = JB^2, \quad B^* = \mu_0 \gamma p + B^2. \]

The operators in Equations (14) and (15) are differential operators in \( \theta \) but algebraic in \( \psi \). This allows us to separate in the solutions the improper normal dependence from the proper tangential dependence. We now restrict the analysis to the neighbourhood of a particular magnetic surface \( \psi = \psi_0 \). On that magnetic surface Equations (14) and (15) can be written in compact form as

\[ L(\psi_0) V(\theta) = \lambda V(\theta), \] (16)

where

\[ V = [\xi^x, \xi^\theta]^t. \]

Equation (16) is a nonsingular eigenvalue problem when supplemented with proper boundary conditions. It can be shown that a solution \( V_0(\theta) \) to the nonsingular eigenvalue problem (16) corresponds to a solution of the form

\[ V(\psi, \theta) = \delta(\psi - \psi_0) V_0(\theta) \] (17)

of the singular eigenvalue problem (Goedbloed, 1983). The continuous spectrum is found as follows. We solve the nonsingular eigenvalue problem (16) for a given toroidal wave number \( n \) on a given magnetic surface and obtain two discrete sets of eigenvalues \( \{\lambda_A(\psi_0)\}_m \) and \( \{\lambda_c(\psi_0)\}_m \). The label \( m \) refers to the different oscillatory solutions that are found. When the flux surface \( \psi = \psi_0 \) is varied, the discrete sets of eigenvalues yield two continuous spectra for a given \( n \) and \( m \) \{\( \min \lambda_A(\psi) \), \( \max \lambda_A(\psi) \} \) and \( \{\min \lambda_c(\psi), \max \lambda_c(\psi)\}_m \).
Equations (14) and (15) both give rise to a Sturm–Liouville eigenvalue problem when provided with general homogeneous boundary conditions

\[ \alpha_1 \tilde{\xi}(\psi, \theta) + \beta_1 \frac{\partial \tilde{\xi}(\psi, \theta)}{\partial \theta} = 0 \quad \text{for} \quad \theta = 0 , \]

\[ \alpha_2 \tilde{\xi}(\psi, \theta) + \beta_2 \frac{\partial \tilde{\xi}(\psi, \theta)}{\partial \theta} = 0 \quad \text{for} \quad \theta = \pi , \quad i = \theta, y , \]

where \( \alpha_1, \alpha_2, \beta_1, \) and \( \beta_2 \) are four constants.

We take the particular homogeneous boundary conditions

\[ \tilde{\xi}(0) = \tilde{\xi}(\pi) = 0 \]

for \( i = \theta, y . \)

The eigensolutions obtained under these boundary conditions are mutually orthogonal and can be used to represent any function in \([0, \pi]\) after a simple change of variable (see, e.g., Section IV of Heyvaerts and Priest, 1983, for a similar situation).

In order to rewrite Equations (14) and (15) in dimensionless form we introduce dimensionless quantities \( x^*, z^*, \xi^*, p^*, \rho^*, \psi^* , \) and \( \mathbf{B}^* \) defined as

\[ x = ax^* , \quad z = az^* , \quad \xi = a\xi^* , \]

\[ p = p_c p^* , \quad \rho = \rho_c \rho^* , \quad \psi = \psi_c \psi^* , \quad \mathbf{B} = B_c \mathbf{B}^* . \]

In what follows we use only dimensionless quantities, and we shall omit the asterisk. The dimensionless equations for the Alfvén continuous spectrum and the slow continuous spectrum are

\[ \lambda \rho \xi_{\psi}^\prime + \frac{d^2 \xi_{\psi}}{d \theta^2} = 0 , \]

\[ \lambda \rho B^2 \xi_\theta + \frac{d}{d \theta} \left[ \frac{\gamma \beta_c p B^2}{\gamma \beta_c p + 2 B^2} \frac{d \xi_\theta}{d \theta} \right] = 0 . \]

It is convenient to collect the expressions for the dimensionless density, pressure and magnetic field here. From Equations (10) and (11) we obtain (21):

\[ \rho = \rho_0(\psi) \exp(-\varepsilon_2 z) , \]

\[ \rho_0(\psi) = 1 - \left( 1 - \frac{\rho_2}{\rho_c} \right) \left( \frac{\sqrt{\psi} - 1}{\sqrt{\psi} - 1} \right) , \]

\[ p = 1 + \frac{1}{\beta_c} \left( 1 + \frac{1}{c^2} \right) (1 - \psi) , \]

\[ B = \left( x^2 + \frac{z^2}{c^4} \right)^{1/2} = \sqrt{\psi} \left( \cos^2 \theta + \frac{1}{c^2} \sin^2 \theta \right)^{1/2} . \]

The Jacobian \( J \) is a constant and has been dropped from Equation (20).
4. Non-Circular Cross-Sections

This section is concerned with the poloidal wave number coupling due to non-circularity of the cross-sections. The effect of non-circularity of the cross-section is studied by taking elliptical cross-sections. Throughout this section, the density is constant along the poloidal magnetic field lines. The Alfvén modes are polarized in the $y$-direction, $\nabla y$ and $\nabla \theta$ are mutually orthogonal, and the curvature of the $y$-coordinate lines is independent of $\theta$ (as a matter of fact the $y$-coordinate lines are straight lines). As a consequence, the equation that describes the Alfvén continuous spectrum does not contain any geometrical effects. Non-circularity does not affect the Alfvén continuous spectrum and causes poloidal wave number coupling only for the slow continuous spectrum. The slow continuum is described by Equation (20) where $\rho$ is constant for a given $\psi$. For circular cross-sections, the coefficients of Equation (20) are constant, and we can Fourier analyse in $\theta$ and study each mode separately. For non-circular cross-sections, $B$ is a function of $\theta$ and the different Fourier modes in $\theta$ couple.

4.1. Weak-coupling limit

4.1.1. Perturbation Expansion

The poloidal wave number coupling of slow modes can be discussed analytically for small values of $\varepsilon_1$ with the use of a perturbation scheme. $\varepsilon_1$ is defined as

$$\varepsilon_1 = \frac{b^2 - a^2}{b^2 + a^2} = \frac{c^2 - 1}{c^2 + 1}$$

and is a measure of the ellipticity of the cross-sections. We have

$$|B| = |\nabla \psi| = \sqrt{\psi} \left( \cos^2 \theta + \frac{1}{c^2} \sin^2 \theta \right)^{1/2}$$

$$= B_0(\psi) \left( \frac{1 + c^2}{2c^2} \right)^{1/2} (1 + \varepsilon_1 \cos 2\theta)^{1/2},$$

(22)

with

$$B_0(\psi) = \sqrt{\psi}.$$

Correct to the first order in $\varepsilon_1$ the poloidal magnetic field is given by

$$B(\psi, \theta) = B_0(\psi) \left[ 1 + \frac{1}{2} \varepsilon_1 (\cos 2\theta - 1) \right].$$

(23)

Using the expansion given in Equation (23) the coefficient functions that appear in Equation (20) may be similarly separated into 'unperturbed' parts containing the 'circular' terms plus small 'perturbed' parts due to ellipticity. This decomposition can be written as

$$M(\theta) \equiv \rho_0 B^2 = M_0 + \varepsilon_1 M_1 + O(\varepsilon_1^2),$$

(24)
\[ N(\theta) = B^2 \frac{\gamma \beta_c p}{\gamma \beta_c p + 2B^2} = N_0 + \varepsilon_1 N_1 + O(\varepsilon_1^2). \]  

(25)

The operators \( M_0, N_0(\partial^2/\partial \theta^2), M_1 \) and \((\partial/\partial \theta)N_1(\partial/\partial \theta)\) are all Hermitian and \( \varepsilon_1 M_1 \) and \( \varepsilon_1(\partial/\partial \theta)N_1(\partial/\partial \theta) \) represent small corrections \((\varepsilon_1 \ll 1)\) to \( M_0 \) and \( N_0(\partial^2/\partial \theta^2) \), so we can use the formal perturbation techniques of quantum mechanics (Davydov, 1965). In this method, the eigenfunction \( \xi^\theta \) is expanded in terms of any complete set of functions. It is generally most convenient to choose the eigenfunctions, \( \phi^{0\theta} \), of the zeroth-order, or unperturbed, equation:

\[ \lambda^0 M_0 \phi^{0\theta}(\theta) + N_0 \frac{\partial^2}{\partial \theta^2} \phi^{0\theta}(\theta) = 0, \]  

(26)

where \( \lambda^0 \) is the eigenvalue associated with the reduced problem. Since \( M_0 \) and \( N_0 \) are constants, we can Fourier-analyze \( \phi^{0\theta}(\theta) \) with respect to \( \theta \) and solve for each Fourier component.

The solutions of Equation (26), \( \phi_m^{0\theta} \), form a complete set, and we can expand the eigenfunctions of the elliptical coronal loop or arcade, \( \xi^\theta \), in terms of the 'circular' eigenfunctions \( \phi_m^{0\theta} \) as follows:

\[ \xi^\theta = \sum_m a_m \phi_m^{0\theta}. \]  

(27)

4.1.2. Solutions of the Lowest-Order Equation

We now derive and discuss the properties of the basis functions by solving Equation (26). Since \( M_0 \) and \( N_0 \) are constants, Equation (26) is a one-dimensional Helmholtz equation with the boundary conditions

\[ \phi^{0\theta}(0) = \phi^{0\theta}(\pi) = 0. \]  

(28)

The solutions of this Helmholtz eigenvalue problem are then

\[ \phi_m^{0\theta} = \eta \sin m\theta, \]  

(29)

with

\[ \lambda^0_m = \frac{\gamma \beta_c P_0}{\gamma \beta_c P_0 + 2B_0^2} \frac{m^2}{\rho_0} \quad (m = 1, 2, \ldots), \]

where \( \eta \) is a normalization parameter.

The unperturbed eigenfunctions are mutually orthogonal with respect to \( M_0 \) and the normalization factor is chosen so that their norm equals one.

4.1.3. First-Order Corrections and Mode Coupling

The first- and second-order corrections to the \( \alpha \)-th unperturbed eigenvalue are given by

\[ \lambda^1_\alpha = -H'_{\alpha \alpha}, \]

\[ \lambda^2_\alpha = \sum_{\beta \neq \alpha} \left| \frac{H^0_{\alpha \beta}}{\lambda^0_\alpha - \lambda^0_\beta} \right|^2, \]  

(30)
where the prime signifies that \( \lambda \) has been replaced by \( \lambda_2 \) in the term (Davydov, 1965). Now

\[
H_{mm'} = \left\langle \phi_{m}^{0\theta} \right| \lambda M_1 + \frac{\partial}{\partial \theta} N_1 \frac{\partial}{\partial \theta} \left| \phi_{m'}^{0\theta} \right\rangle \\
= -[\lambda - \lambda_0 A] \delta(m - m') \delta(n - n') + \\
\frac{1}{2} \left[ \lambda - \lambda_0 A \pm 2 \frac{mA^2}{\rho_0} \right] \delta(m - m' \pm 2) \delta(n - n') ,
\]

(31)

with

\[
A = \frac{\gamma \beta_c p}{\gamma \beta_c p + 2B_0^2}.
\]

Thus the first-order correction to \( \lambda_0 \) is

\[
\lambda_1 \equiv \lambda_0 (1 - A).
\]

(32)

We now derive expressions for the coupling of basis eigenfunction \( \phi_M^{0\theta} \) to the other basis eigenfunctions in the first order. We truncate the Fourier series and take only \( 2L + 1 \) terms centered around \( \phi_M^{0\theta} \),

\[
\xi_\theta = \sum_{m = M - L}^{M + L} a_m \phi_m^{0\theta},
\]

and we solve the system of equations for \( a_m \).

In the first order we find

\[
\xi_\theta = a_{M - 2} \phi_{M - 2}^{0\theta} + a_M \phi_M^{0\theta} + a_{M + 2} \phi_{M + 2}^{0\theta},
\]

(33)

\[
a_{M - 2} = \frac{\left( \lambda_0^0 - \lambda_0 A + 2 \frac{MA^2}{\rho_0} \right)}{\lambda_0^M - \lambda_0^M - 2} \frac{a_M \varepsilon_1}{2},
\]

\[
a_{M + 2} = \frac{\left( \lambda_0^0 - \lambda_0 A - 2 \frac{MA^2}{\rho_0} \right)}{\lambda_0^M - \lambda_0^M + 2} \frac{a_M \varepsilon_1}{2}.
\]

(34)

All the other coefficients are zero or of higher-order in \( \varepsilon_1 \). Thus, in first order, \( \phi_M^{0\theta} \) couples to \( \phi_{M - 2}^{0\theta} \) and \( \phi_{M + 2}^{0\theta} \). The even and odd modes are decoupled for reasons of symmetry (\( \forall i : i \) is odd; \( a_{M + i} = 0 \)).

4.2. Strong Coupling

The perturbation scheme can only be applied for small values of \( \varepsilon_1 \) and indicates the coupling of poloidal wave numbers \( M - 2, M, M + 2 \). As the value of \( \varepsilon_1 \) is increased,
the wave number coupling will involve gradually more and more wave numbers. We now want to see how poloidal wave number coupling progresses as $\varepsilon_1$ is increased. Of course, Equation (20) can be solved by direct numerical methods but direct numerical solutions are not very instructive for the discussion of the number of wave numbers involved in the coupling. We, therefore, picture the progress in the poloidal wave number coupling by comparing approximate numerical solutions obtained with a Galerkin procedure to the ‘exact’ direct numerical solutions. The ‘exact’ solutions were obtained by means of the NAG-routine D02KEF which solves the Sturm–Liouville eigenvalue problem using a Pruefer transformation and a shooting method. The approximate numerical results were obtained with a Galerkin procedure (in the same way as the analytical result (4.1) but without the perturbation expansion). The eigenfunctions are Fourier-analyzed with respect to $\theta$, and this infinite series is truncated. Only $2L + 1$ terms are retained, centered around the zeroth-order eigenmode with the same number of nodes as the mode that is studied and $\xi^\theta$ is written as

$$\xi^\theta = \sum_{m = M - L}^{M + L} a_m \sin(m\theta).$$ (35)

The numerical results reported in this paper have been obtained for $\rho_2/\rho_c = \frac{1}{3}$, $\beta_c = 0.5$, and $1 \leq \psi \leq 1.21$. The range of $\psi$ implies $1 \leq x \leq 1.1$ so that for $e = 0$ the width of the loop is $\frac{1}{10}$ of the radius of the interior magnetic surface. Results for the slow continuum are shown in Figures 2 and 3. On Figure 2 we have depicted the slow continuum for a circular loop (zero-order slow continuum, indicated by $Z$) and the slow continuum for an elliptical loop with $b/a = 2$ obtained by direct numerical integration (exact solution, indicated by $E$). The slow continuum eigenvalues are increased by non-circularity and the range of frequencies is enlarged.

![Fig. 2. The slow continuum for circular ($Z$) and non-circular ($E$) cross-sections for the continuum mode with $M = 5$ (density is constant along the field lines, $b/a = 2$).](image-url)
Fig. 3. The eigenfunction of the fifth slow continuum mode \((M = 5)\) on the magnetic surface \(\psi = 1\) (density is constant along the field lines, \(b/a = 2\)).

The increase of the continuum frequencies and the enlargement of the range of continuum frequencies can be understood, at least qualitatively, in terms of 1D models. The non-circularity of the cross-sections of the magnetic surfaces makes the magnetic field strength, and by consequence the magnetic pressure, at the apex lower according to Equation (23), leading to the increase in the cusp frequency. Since this effect is different on different magnetic surfaces the range of continuum frequencies is enlarged.

On Figure 3 we have depicted the variation of the fifth slow continuum mode with respect to \(\theta\) on the magnetic surface \(\psi = 1\). The latter eigenfunction has to be compared with \(\sin 5\theta\) which is the eigensolution of the circular loop or the zero-order eigensolution. The difference between the actual eigenfunction and \(\sin 5\theta\) shows that the actual eigenfunction cannot be represented by one poloidal wave number, namely \(M = 5\), but that several poloidal wave numbers are required (in principle we need to have an infinite number of poloidal wave numbers, in practice a finite number suffices). We have compared the results obtained with the Galerkin method and with direct numerical integration and have found that the slow continuum and the eigenfunction can be recovered accurately with the Galerkin method when 13 terms are introduced in Equation (35). The components of the normalized eigenvector are:

\[
\begin{align*}
    a_1 &= 0.15872, & a_3 &= -0.37525, & a_5 &= 0.90482, \\
    a_7 &= 0.12220, & a_9 &= 0.01915, & a_{11} &= -0.00132, \\
    a_{13} &= 0.00053, & a_{2l} &= 0.
\end{align*}
\]

These numbers clearly illustrate the poloidal wave number coupling of the slow continuum modes. Ellipticity induces substantial poloidal wave number coupling in the
eigensolutions of the slow continuum and this poloidal wave number coupling acts to increase the eigenfunction at the top of the loop and to decrease it at the bottom of the loop.

5. Density Stratification

Since the density of most coronal loops and arcades varies along the loop or arcade, we are concerned in this Section with wave number coupling due to variation of density along the poloidal field lines. We consider a poloidal magnetic field with circular cross-sections \( a = b \) and a density profile that decreases exponentially with height \( z \) as given by Equation (11b). As the density appears in Equations (19) and (20), variation of density along the poloidal magnetic field lines induces poloidal wave number coupling in both the slow modes and the Alfvén modes. The Alfvén continuum is treated here explicitly. The slow continuum can be treated in a similar way.

For circular cross-section \( b = a \) the magnetic field strength is a flux function and the coordinate system \( (\theta, \psi, \gamma) \) is orthogonal. The wave number coupling is induced by the \( \theta \)-dependence of the coefficients of Equation (19), i.e., by the density stratification. The solutions of Equation (19) are discussed following the same lines as in Section 4.

5.1. Weak Coupling

5.1.1. The Perturbation Expansion

In the limit of small \( \varepsilon_2 \), the density varies as

\[
\rho(\psi, \theta) = \rho_0(\psi)(1 - \varepsilon_2 z) \quad (36)
\]

with

\[
z = \sqrt{\psi} \sin \theta.
\]

5.1.2. Solution of the Lowest-Order Equation

In the lowest order we again obtain a one-dimensional Helmholtz eigenvalue problem for the basis functions, \( \phi_m^{0\nu} \),

\[
\lambda_m^0 \rho_0 \phi_m^{0\nu}(\theta) + \frac{d^2}{d\theta^2} \phi_m^{0\nu}(\theta) = 0 , \quad (37)
\]

with boundary conditions

\[
\phi_m^{0\nu}(0) = \phi_m^{0\nu}(\pi) = 0 .
\]

The solutions of this eigenvalue problem are

\[
\phi_m^{0\nu} = \eta \sin m\theta ,
\]

\[
\lambda_m^0 = \frac{m^2}{\rho_0} \quad (m = 1, 2, 3, \ldots) . \quad (38)
\]
The normalization factor $\eta$ is chosen as $1/\pi \sqrt{\rho_0}$ so that the norm of the unperturbed eigenfunctions with respect to $\rho_0$ equals one.

5.1.3. First-Order Corrections and Mode Coupling

The inner products, $H_{mn'}$, are now given by

$$H_{mn'} = \langle \phi_{mn}^0 | \lambda \rho^1 | \phi_{m'n'}^0 \rangle$$

$$= -\frac{2}{\pi} \sqrt{\psi} \lambda \left( \int_0^\pi \sin \theta \sin m\theta \sin m' \theta \, d\theta \right) \delta(n - n').$$  \hspace{1cm} (39)

$H_{mn'}$ is non-zero if, and only if, $m$ and $m'$ have the same parity and consequently the modes with even and odd poloidal mode number again decouple.

However, as $H_{mn'} \neq 0$, the first-order correction on the eigenvalue is

$$\lambda_m^1 = -H_m^0 = \frac{2}{\pi} \lambda_m^0 \sqrt{\psi} \left( \frac{4m^2}{4m^2 - 1} \right).$$  \hspace{1cm} (40)

Equation (40) has to be compared with Equation (32). The relative shifts of the continuum frequencies, $\lambda^1/\lambda^0$ for density variations along field lines ($\lambda^1/\lambda^0 \approx 2/\pi$ for $\psi = 1$) and for non-circularity of the cross-sections ($\lambda^1/\lambda^0 = \frac{12}{17}$, for $\psi = 1$) ($\beta_c = \frac{1}{2}$, $\gamma = \frac{3}{2}$), are of the same magnitude.

In the case of elliptical cross-sections the inner product, $H_{mn'}$, has a term proportional to $\delta(m - m')$ and one proportional to $\delta(m - m' \pm 2)$ and by consequence the basis eigenfunction $\phi_M^0$ couples, in first order, to $\phi_{M-2}^0$ and $\phi_{M+2}^0$. Now $H_{mn'} \neq 0$, whenever $m$ and $m'$ have the same parity. Consequently, the basis function $\phi_M^0$ couples, in first order to all the other basis eigenfunctions with a poloidal mode number of the same parity. Variation of density along the poloidal magnetic field has a stronger impact than non-circularity of the cross-sections as far as poloidal wave number coupling is concerned.

5.2. Strong Coupling

The analytical results for small $\varepsilon_1$ already suggest that density variations along the poloidal magnetic field lines have a strong effect on the Alfvén continuous spectrum. The perturbation scheme can only be applied for small values of $\varepsilon_1$. Equation (19) has been solved by direct numerical integration. Again we have followed the progress in the poloidal wave number coupling by comparing approximate numerical solutions obtained with a Galerkin procedure to the ‘exact’ numerical results. Results for the Alfvén continuum of circular loops in which the density varies along the poloidal field lines, are shown on Figures 4(a), 4(b), and 5. As $\varepsilon_2 = 1.6$, the density varies by a factor of 5 from the bottom to the top of the loop on the inner magnetic surface. On Figure 4(a) we have depicted the Alfvén continuum for a circular loop with constant density along the field lines (indicated by $Z$) and the Alfvén continuum for a circular loop with $\varepsilon_2 = 1.6$ obtained by direct numerical integration (indicated by $E$) for the continuum mode with
Fig. 4a. The Alfvén continuum for circular cross-sections of the fifth Alfvén continuum mode \((M = 5)\). \(Z\) denotes the zeroth-order solution. \(E\) is the solution obtained by direct numerical integration for \(\varepsilon_2 = 1.6\).

\[\lambda\]

\[\psi/\psi_0\]

\[1.00\quad 1.05\quad 1.10\quad 1.15\quad 1.20\]

Fig. 4b. The successive Galerkin approximations to the Alfvén continuum of the fifth Alfvén continuum mode with 1, 5, and 9 Fourier terms \((a = b, \varepsilon_2 = 1.6)\).

\[\lambda\]

\[\psi\]

\[1.05\quad 1.06\quad 1.07\quad 1.08\quad 1.09\quad 1.10\]

\(M = 5\). Figure 4(a) shows that density variations along the field lines have a profound effect on the Alfvén continuum. The continuum eigenvalues are shifted to larger values but more importantly the range of continuum frequencies and also \(\frac{d\sigma_A(\psi)}{d\psi}\) is considerably increased and this can have substantial consequences for resonant absorption and phase mixing. Again these two effects of the mode coupling can be understood qualitatively in terms of 1D models. The Alfvén frequency increases due to the decrease
in density in higher loops when the density stratification is included. The density stratification depends explicitly on \( \psi \) according to Equations (3) and (11) and this dependence causes the enlargement of the range of the Alfvén continuum. On Figure 4(b) we have given the 'exact' Alfvén continuum and the approximations of the Alfvén continuum obtained with the Galerkin procedure when 1, 5, and 9 terms are used. The \( \psi \)-interval is limited to \( 1.055 \leq \psi \leq 1.1 \) for reasons of clarity. Figure 4(b) shows that the Galerkin approximation improves when more terms are used in the expansion and actually the Galerkin approximation with 13 terms to the Alfvén continuum coincides with the exact Alfvén continuum obtained by direct numerical integration (E). This illustrates the effect of poloidal wave number coupling on the Alfvén continuum. The variation of the eigenfunction of the Alfvén continuum with \( M = 5 \) on \( \psi = 1 \) is depicted on Figure 5. This eigenfunction has to be compared with \( \sin 5 \theta \) which is the eigensolution of the circular loop with constant density along the field lines. Again the difference between the actual eigenfunction and \( \sin 5 \theta \) illustrates the effect of poloidal wave number coupling on the eigenfunction. The eigenfunction can be recovered accurately when 13 terms are introduced in the expansion. The components of the normalized eigenvector are:

\[
\begin{align*}
  a_1 &= 0.06658, & a_3 &= -0.49920, & a_5 &= 0.80522, \\
  a_7 &= 0.29743, & a_9 &= 0.09224, & a_{11} &= 0.03004, \\
  a_{13} &= 0.01076, & a_{2i} &= 0.
\end{align*}
\]

The values of the coefficients \( a_m \) show that the variation of density along the poloidal
magnetic fields induces substantial poloidal mode coupling in the eigenfunctions of the Alfvén continuum modes. Here, also the poloidal wave number coupling acts as to increase the amplitude of the eigensolution at the top of the loop and to decrease it at the bottom of the loop. Similar results have been obtained for the slow continuous spectrum.

6. Combined Effect

In the previous sections the changes of the continuum frequencies and eigenfunctions caused by geometrical effects and density variations along the field lines have been studied separately. It was found that geometrical effects primarily affect the eigenfunctions (of the slow modes) and that density variations primarily affect the continuum frequencies. We now investigate how the combination of geometrical effects and density variations along the field lines change the continuum frequencies and eigenfunctions.

Fig. 6. The slow continuum and the Alfvén continuum for the fifth slow mode and the fifth Alfvén mode. $AZ$ and $SZ$ denote the zeroth-order solutions ($a = b, \varepsilon_2 = 0$). $AE$ and $SE$ are the solutions obtained by direct numerical integration ($\varepsilon_2 = 1.6, b/a = 2$).
Equations (19) and (20) have been solved for elliptical loops with a density stratification along the magnetic field lines. Some of our results for these loops are shown in Figures 6–8. These figures are for loops with $b/a = 2$ and $\varepsilon_2 = 1.6$. Figure 6 gives the continuous spectrum of the Alfvén modes ($AE$) and the slow modes ($SE$) for the
eigenfunction with $M = 5$. Also given are the continuous spectrum of the Alfvén modes ($AZ$) and the slow modes ($SZ$) for a circular loop with constant density along the field lines. Figures 7 and 8 give the variation of the fifth eigenfunction on the magnetic surface $\psi = 1$ respectively for the slow mode and the Alfvén mode. Figures 6–8 clearly show that the combination of non-circularity of the cross-sections and density variations along the field lines has a profound effect on both the continuum frequencies and eigenfunctions of both the Alfvén and the slow modes. The continuum frequencies are increased by a factor of about 6 compared with the 1D frequencies and the ranges of frequencies are enlarged by about the same factor. The shape of the eigenfunctions is also substantially changed as the eigenfunctions deviate strongly from the 1D eigenfunction, $\sin 5\theta$, and the ratio of the maximum of the eigenfunction at the top to the maximum near the base is about 2.1 for the Alfvén modes and 3.5 for the slow mode.

The results given in this section for the Alfvén modes differ from those given in Section 5. Figures 4 and 5 of Section 5 and Figures 6 and 8 of this section are all for the fifth Alfvén mode and allow direct comparison. The differences between Figures 4 and 6 for the continuum frequencies and Figures 5 and 8 for the continuum eigenfunctions do not contradict our statement in Section 4 that non-circularity does not influence the Alfvén modes. What we meant there, was that non-circularity on its own (i.e., in absence of density variations along the field lines) does not influence the Alfvén modes. However, if there are density variations along field lines then the non-circularity has an indirect effect on the Alfvén continuum. Indeed, an increase of $b$ for fixed $a$ (i.e., an increase of non-circularity) results in a larger variation of equilibrium density from the bottom to the top of the loop and induces changes in the Alfvén continuum.

7. Conclusions

The present paper has initiated the study of the continuous spectrum of linear ideal MHD of 2D solar loops and has made a first attempt to understand how 2D effects change the continuum frequencies and the continuum eigenfunctions. The continuous spectrum has been computed for 2D solar loops with purely poloidal magnetic fields and it has been investigated how non-circularity of the cross-sections of the poloidal magnetic surfaces and variations of density along the poloidal magnetic field change the continuous spectrum and induce poloidal wave number coupling of the continuum eigenfunctions. As the main objective of the paper was to illustrate the influence of 2D effects on the continuous spectrum, a simple variation of equilibrium density across the loop has been taken.

The changes in the continuum frequencies and the poloidal wave number coupling of the eigenfunctions due to non-circularity and density variations have been determined separately first, with an analytical approximate method, by direct numerical integration and with a Galerkin method. It was found that non-circularity has not any effect on the Alfvén modes and that non-circularity changes the continuum frequencies and induces substantial poloidal wave number coupling in the eigenfunctions of slow continuum modes while density variations change the continuum frequencies and induce poloidal
wave number coupling in the eigenfunctions of both the slow and the Alfvén modes. The geometric effects primarily influence the eigenfunctions while the most profound impact of density variations along field lines is the change of continuum frequencies.

The combination of non-circularity of the cross-sections and density variations along the poloidal magnetic field lines has a strong influence on both the continuum frequencies and the continuum eigenfunctions. The continuum frequencies are increased, their ranges are substantially enlarged and their derivatives normal to the magnetic surfaces $|d\sigma_A(\psi)/d\psi|$, $|d\sigma_c(\psi)/d\psi|$ are considerably increased. The eigenfunctions show considerable poloidal wave number coupling which acts to increase the absolute values of the extrema of the eigenfunctions from the bottom to the top of the loop. Let us now briefly touch upon possible consequences of the present findings for resonant absorption and phase mixing. We do not give quantitative statements about resonant absorption and phase mixing in 2D solar loops here, since that requires substantial further computation which is beyond the scope of the present paper. We feel it is relevant to first point out here that resonant absorption and phase mixing of slow continuum modes cannot be excluded a priori as heating mechanisms in favour of resonant absorption and phase mixing of Alfvén waves. In plasmas where $V_A^2/c^2 \leq 1$ resonant absorption and phase mixing of slow continuum waves should certainly be considered as heating mechanisms (all the more so in the limit $V_A^2/c^2 \to 0$) in addition to resonant absorption and phase mixing of Alfvén waves as the frequency ranges involved are comparable. 2D effects influence the continuous spectrum and as a consequence they also influence resonant absorption and phase mixing. As resonant absorption and phase mixing require driving at a continuum frequency, the present results indicate that the possible driving frequencies are increased and that the range of possible driving frequencies is substantially increased. The increase of $|d\sigma_A(\psi)/d\psi|$ affects the rate at which energy is absorbed at the resonant surface in resonant absorption. It also affects phase mixing. The efficiency of phase mixing of standing oscillations in a closed 1D loop of resistive and viscous plasmas is estimated by the phase mixing time, $\tau_{mix}$, which fulfils the following proportionality (Sakurai, 1985):

$$\tau_{mix} \sim (\nu + \eta)^{-1/3} \left| \frac{d\sigma_A(x)}{dx} \right|^{-2/3},$$

(41)

where $\nu$ is the viscosity, $\eta$ the magnetic diffusivity, and $x$ the coordinate normal to the planar magnetic surfaces. Expression (41) for $\tau_{mix}$ also follows from Equation (49) of Heyvaerts and Priest (1983). In a 2D plasma the derivative normal to the magnetic surfaces is $\partial/\partial\psi$. Our results then indicate that 2D effects reduce $\tau_{mix}$ and enhance the efficiency of phase mixing of Alfvén continuum waves (and also of slow continuum waves). The dissipation of wave-energy is also affected. In ideal MHD the magnetic surfaces oscillate independently at their own Alfvén (or slow) continuum frequency so that the displacement field for the Alfvén continuum is given by (see, e.g., Goedbloed, 1983)

$$A(\psi)\xi_y(\theta, \psi) \sin[\sigma_A(\psi)t + \varphi(\psi)],$$

(42)
where $A(\psi)$ and $\varphi(\psi)$ are determined by the boundary and initial conditions. If the field lines are displaced in phase with the same amplitude at $t = 0$, then $A$ and $\varphi$ do not depend on $\psi$. We have denoted the eigenfunction of the continuum Alfvén modes as $\xi_y(\theta, \psi)$ instead of $\xi_y(\theta)$ to stress that $\xi_y(\theta)$ varies differently on different magnetic surfaces. Dissipative effects introduce damping on the time-scale $\tau_{\text{mix}}$. For time $t \leq \tau_{\text{mix}}$, the wave field is adequately described by (49). The dissipation of wave-energy at position $(\theta, \psi)$ over a time-interval $\tau_{\text{mix}}$,

$$
\int_0^{\tau_{\text{mix}}} \rho(v + \eta) \left( \frac{\partial V}{\partial \psi} \right)^2 \, dt,
$$

is then proportional to

$$
[A^2(\psi)]^2 [\xi_y(\theta, \psi)]^2 \rho(\theta, \psi) \sigma_\alpha^2(\psi).
$$

The dissipation of wave energy then depends on the amplitude of the driving $A(\psi)$, and on the eigenfrequencies and eigensolutions of the continuous spectrum. The dissipation depends on $\psi$ and $\theta$. It varies from field line to field line, but on a given field line it varies with respect to $\theta$ as $\rho_x(\psi)$ depends on $\theta$. More quantitative statements about resonant absorption and phase mixing require further numerical computation and are beyond the scope of the present paper.

References