MAGNETOHYDRODYNAMIC MODES OF A PERIODIC MAGNETIC MEDIUM

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Abstract. The oscillations of a magnetic medium periodic in the x-direction with \( B \) parallel to \( z \), have been studied. The case with no gravity and a stepwise profile for \( B(x) \), allowing a normal mode analysis, has been examined and dispersion relations have been derived. The dispersion curves in the diagram \( k_z - \omega \) display two types of modes, kink and sausage, like in the isolated slab, but the profiles are different and depend on Bloch's number \( k_\theta \). Moreover, modes usually absent in the isolated slab (propagating and tunelling) appear here, connecting surface- or body-wave domains. The detectability of this characteristic structure of the diagnostic diagram on the observations is discussed, and prospects for a more realistic analysis including gravity are given.

1. Introduction

Observations of the solar surface reveal a spatial organization (supergranulation) of convective motions on a large scale (average supergranule size \( L \approx 32 \) Mm) which concentrates magnetic flux at the boundaries. How does the arrangement of the flux tubes on a large-scale \( L \) in a network pattern react on the oscillations of the atmosphere? Oscillation modes of isolated magnetic structures have been studied in various geometries (Parker, 1964, 1974; Uberti, 1972; Gram and Wilson, 1975; Defouw, 1976; Roberts and Webb, 1978, 1979; Webb and Roberts, 1978; Wentzel, 1979a, b; Wilson 1978a, b, 1979, 1980; Uberti and Somasundaram, 1980; Roberts, 1981a, b; Spruit, 1982; Edwin and Roberts, 1982, 1983; Bogdan, 1984) but little has yet been done about the effect of their large-scale organization at the surface of the Sun. When the horizontal wavelength \( \lambda \) is much smaller than \( L \), it is likely that the local wave properties will not be sensitive to the periodicity of the medium. On the other hand when \( \lambda \) is of the same order as \( L \), we may expect some influence of the periodicity. Waves in periodic media are well known in solid state physics (Elachi, 1975, 1976), and we take over some general methods from this field. Effects of periodicity, though it is not perfect on the Sun’s surface, could possibly be observationally detected on diagnostic diagrams \( k - \omega \) when the spatial period of the network \( (L \approx 32 \) Mm) lies in the range of wavelengths actually measured. The present lower bound in observational data is mainly due to the instrumental resolution \((\sim 1 \) arc sec) while the upper bound is fixed by the slit size (generally up to 40 arc sec).

Global oscillations observed with space resolution, on the other hand, explore a range of wavelengths between the full Sun-size, and the degree of the mode \( l, l = 40 \), say, which grossly corresponds to \( \lambda = 2\pi R_\odot / l \sim 100,000 \) km, a bit larger than the supergranular size. Attempts are being made (Foing, 1983; Damé et al., 1984) to observe this interesting wavelength domain.

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This paper is a first approach to the problem. A realistic study should take into account the radial structure of the atmosphere and sphericity effects, including a reasonable model of field distribution, with horizontal and vertical coordinates, exhibiting canopy-like structures and assuming some definite internal structure in the convection zone. We will study here only a very simplified problem of methodological interest, in which weak field regions alternate with strong field regions in a plasma which is subject to no gravity force.

Dispersion relations for waves in this one-dimensional periodic medium will be derived in Section 2. How these turn into homogeneous plasma waves or into isolated-tube waves in limit cases is discussed in Section 3. Results valid far from these limits are shown in Section 4 and discussed in Section 5.

2. Derivation of Dispersion Relations

We consider a medium composed of an ideal, perfectly conducting fluid (zero viscosity and infinite electrical conductivity) pervaded by a magnetic field of constant direction (along z) and periodic along x. In consistency with the equilibrium equations, the pressure, density, and magnetic field will be functions $p_0(x), \rho_0(x), B_0(x)\hat{z}$ of $x$ alone, where $\hat{z}$ denotes the upward unit vector along the z-direction. The variables $p_0$ and $B_0$ satisfy the equilibrium equation

$$
\frac{d}{dx} \left( p_0 + \frac{B_0^2}{2\mu} \right) = 0.
$$

(1)

It is known that normal modes are to be expected only in the case of sharply structured media, that is, when regions of field strengths $B_1$ and $B_2$ are separated from each other by a discontinuity. Otherwise, continuous profiles for $B(x)$ yield continuous wave spectra (Rae and Roberts, 1981). We, therefore, assume that homogeneous strong field slabs (strength $B_{0i}$, width $L_i$) periodically alternate with homogeneous weak field slabs (strength $B_{0e}$, width $L_e$) as shown in Figure 1. The periodicity of the medium is

![Fig. 1. Profile of the magnetic field $B_0$ in the equilibrium state. The pattern should be repeated at infinity in x, with periodicity $L = L_i + L_e$.](image-url)
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thus \( L = L_i + L_e \), and the pressure balance Equation (1) writes:

\[
\rho_i \left( \frac{c_{Si}^2}{\gamma} + \frac{c_{Ai}^2}{2} \right) = \rho_e \left( \frac{c_{Se}^2}{\gamma} + \frac{c_{Ae}^2}{2} \right),
\]

where \( c_S, c_A, \gamma \) denote, respectively, the sound speed, the Alfvén speed, and the adiabatic index. The subscripts \( i \) and \( e \) refer to the internal and external medium, respectively.

Then, writing the perturbation equations and looking for velocity perturbations \( v_x \) in the form

\[
v_x = \hat{v}_x(x) e^{i(\omega t - k_y y - k_z z)}
\]

and assuming \( \hat{v}_y = 0 \), we get the classical differential equation for \( \hat{v}_x \) (Roberts, 1981; Priest, 1982)

\[
\frac{d}{dx} \left[ \rho_0(x) \frac{k_x^2 c_A^2(x) - \omega^2}{k_x^2 + q^2(x)} \frac{d\hat{v}_x}{dx} \right] - \rho_0(x) [k_x^2 c_A^2(x) - \omega^2] \hat{v}_x = 0.
\]

Modes with \( \hat{v}_y \neq 0 \) are just trivial local Alfvén waves. The function \( q(x) \) is defined by

\[
q^2(x) = \frac{k_x^2 c_A^2(x) - \omega^2}{\left[ c_S^2(x) + c_A^2(x) \right]} [k_x^2 c_A^2(x) - \omega^2],
\]

where \( c_T \) is the usual tube velocity defined by

\[
c_T^2 = \frac{c_S^2 c_A^2}{c_S^2 + c_A^2}.
\]

\( q^2 \) is actually constant in each medium and need not be positive.

In each medium (strongly or weakly magnetized), Equation (4) reduces to

\[
[k_x^2 c_A^2(x)] \left[ \frac{d^2 \hat{v}_x}{dx^2} - \{k_x^2 + q^2(x)\} \hat{v}_x \right] = 0.
\]

We find the two Alfvén modes \( \omega = k_x c_A \) and \( \omega = k_x c_A e \), of course not simultaneous in the two kinds of media \( (c_A \neq c_A e) \), since the phase speed \( (\omega/k_x) \) is unique, and the non Alfvénic modes solutions of

\[
\frac{d^2 \hat{v}_x}{dx^2} - \{k_y^2 + q^2(x)\} \hat{v}_x = 0.
\]

These correspond in the isolated flux tube situation to the so-called propagating, body, surface, or tunnelling modes (Rae and Roberts, 1983) according to the sign of \( k_x^2 + q_i^2 \) and \( k_x^2 + q_e^2 \), as shown in Table I.

Dispersion relations will be given for any of these cases but the derivation procedure is identical and will be explained in detail for propagating waves only. The method described by Heyvaerts and Priest (1983) can be used, \( q(x) \) being an even function of \( x \).
The solutions for $\hat{v}_x$ from Equation (7) in the slabs ‘i’ and ‘e’ are the following (the $x$-subscript in $v_x$ will be dropped from now on):

$$\hat{v}_i(x) = \alpha_i e^{i\varphi_{i,x}} + \beta_i e^{-i\varphi_{i,x}},$$
$$\hat{v}_e(x) = \alpha_e e^{i\varphi_{e,x}} + \beta_e e^{-i\varphi_{e,x}}.$$  \hspace{1cm} (8)

The boundary conditions

$$\hat{v}_i = \hat{v}_e,$$  \hspace{1cm} (9)

will be written at $x = L_i/2$ and $x = L_i/2 + L_e$ (Figure 1). They express the continuity of the normal component of the velocity and of the total pressure, respectively, and yield four relations between the amplitudes $\alpha, \beta$ in slabs 1, 2, 3 (Figure 1). Eliminating the amplitudes in slab ‘2’, we can obtain the amplitudes in slab ‘3’ as functions of the amplitudes in slab ‘1’.

Now, Bloch’s theorem states that solutions of Equation (4) which remain bounded at infinity can be written

$$\hat{v}(x) = aF(x) e^{ik_0 x} + bF(-x) e^{-ik_0 x},$$  \hspace{1cm} (10)

where $F(x)$ is a periodic function of $x$ with the periodicity $L$ of the slabs, and $k_0$ is the so-called Bloch wavenumber, which plays here the same role as the wavenumber $k_x$ in a homogeneous medium. Then

$$\frac{\hat{v}(L)}{\hat{v}(0)} = \cos k_0 L.$$  \hspace{1cm} (11)

This constitutes a dispersion relation, relating $\omega, k_0, k_y,$ and $k_z$. Actually, inserting the solutions (8) in (11) we get the relation:

$$\cos k_0 L = \cos \theta_i \cos \theta_e - \frac{1}{2} \left( S + \frac{1}{S} \right) \sin \theta_i \sin \theta_e.$$  \hspace{1cm} (12)
where $\theta_i$, $\theta_e$, and $S$ are defined by:

$$\theta_i = \left| q_i^2 + k_y^2 \right|^{1/2} L_i; \quad \theta_e = \left| q_e^2 + k_y^2 \right|^{1/2} L_e,$$

$$S = \frac{\rho_i}{\rho_e} \frac{L_i}{L_e} \frac{\theta_e}{\theta_i} \frac{k_y^2 c_{\lambda i}^2}{k_y^2 c_{\lambda e}^2} - \frac{\omega^2}{\omega^2}.$$  \hfill (14)

Other methods used in solid state physics (Kronig and Penney, 1931; Leighton, 1959; Kittel, 1976) give the same result.

In the case of surface, body or tunnelling modes, the dispersion relations write, respectively

$$\cos k_0 L = \cosh \theta_i \cosh \theta_e + \frac{1}{2} \left( S + \frac{1}{S} \right) \sinh \theta_i \sinh \theta_e,$$

$$\cos k_0 L = \cos \theta_i \cosh \theta_e + \frac{1}{2} \left( S - \frac{1}{S} \right) \sin \theta_i \sinh \theta_e,$$

$$\cos k_0 L = \cosh \theta_i \cos \theta_e - \frac{1}{2} \left( S - \frac{1}{S} \right) \sinh \theta_i \sin \theta_e,$$

with $\theta_i$, $\theta_e$, and $S$ being still given by (13) and (14). The dispersion curves $\omega(k_z)$ will depend upon Bloch’s wavenumber $k_0$ as a parameter.

Before dealing with general cases, we shall first examine limit cases.

### 3. Limit Cases

#### 3.1. Homogeneous Medium

The homogeneous medium is recovered in the limit when $L_e$ goes to zero. Equation (12) then reduces to:

$$\cos k_0 L = \cos \theta_i,$$

or equivalently

$$k_0 = \pm \sqrt{k_y^2 + q^2} \pm 2\pi/n L,$$

where $n$ is an integer. According to (10), the $x$-wavenumber $k_x$ should be defined by

$$k_x = k_0 + n \pi/2 L.$$  \hfill (19)

Then

$$k_x = \pm \sqrt{k_y^2 + q^2},$$

and since $q^2 + k_y^2$ is here negative (see Table I, propagating mode)

$$-k_x^2 = q^2 + k_y^2,$$  \hfill (20)
which, making use of (5), can be cast into the well-known dispersion relation for homogeneous medium MHD waves
\[
\omega^4 - (k_x^2 + k_y^2 + k_z^2)(c_s^2 + c_A^2)\omega^2 + (k_x^2 + k_y^2 + k_z^2)c_s^2c_A^2 = 0.
\] (21)

Also, as expected, and as it can be checked from Equations (15), (16), (17), surface and tunnelling modes do not exist in the limit \(L_e \to 0\), and the body mode just reduces to a propagating mode.

3.2. ISOLATED-SLAB WAVES

The isolated-slab wave should be recovered by taking the limit \(L_e \to +\infty\). Then, \(\theta_e\) goes to infinity and Equation (15) reduces to
\[
\frac{1}{2}\left(S + \frac{1}{S}\right)\tanh \theta_i + 1 = 0
\]
which factors as
\[
(S + \tanh \frac{\theta_i}{2})\left(S + \coth \frac{\theta_i}{2}\right) = 0.
\] (22)

This is to be recognized as the dispersion relation of surface modes for the isolated slab (Roberts, 1981b; Edwin and Roberts, 1982), the two factors representing the sausage and the kink modes, respectively.

Similarly, Equation (16) reduces, in this limit, to
\[
\frac{1}{2}\left(S - \frac{1}{S}\right)\tanh \theta_i + 1 = 0,
\]
and factors as
\[
\left(S - \tan \frac{\theta_i}{2}\right)\left(S + \cotan \frac{\theta_i}{2}\right) = 0,
\] (23)
which is the dispersion relation of body modes for the isolated slab.

From Equations (12) and (17) it can be seen that propagating and tunnelling modes do not have particular behaviours as \(\theta_e\) tends to infinity.

4. Original Results

4.1. Case \(k_y = 0\) and \(k_z = 0\)

Let us consider the simple case of pure \(x\)-direction propagation. From (5),
\[
q^2(x) = -\omega^2/c^2(x)
\]
with
\[ c(x)^2 = c_S(x)^2 + c_A(x)^2, \]
and Equation (7) becomes
\[ \frac{d^2 \vartheta_x}{dx^2} + \frac{\omega^2}{c^2(x)} \vartheta_x = 0. \tag{24} \]

Only propagating waves are allowed in this case (Table I). The dispersion relation (12) reduces to
\[ \cos k_0 L = \cos \omega \frac{L_i}{c_i} \cos \omega \frac{L_e}{c_e} - \frac{1}{2} \left( S + \frac{1}{S} \right) \sin \omega \frac{L_i}{c_i} \sin \omega \frac{L_e}{c_e}, \tag{25} \]
and \( S \) simply becomes the constant \( \rho_i c_i / \rho_e c_e \), which is equal to unity for an homogeneous medium.

From (25) we get in that limit (\( L_i \to 0 \))
\[ k_0 + n \frac{2\pi}{L} = \pm \frac{\omega}{L} \left( \frac{L_i}{c_i} + \frac{L_e}{c_e} \right) \approx \pm \frac{\omega}{c_e}. \]

With the usual identification (19), we recognize here the traditional fast mode wave in perpendicular propagation.

In both limits \( S \to 0 \) and \( S \to +\infty \), Equation (25) reduces to
\[ \sin \omega \frac{L_i}{c_i} \sin \omega \frac{L_e}{c_e} = 0, \]
that is
\[ \omega = n \frac{\pi c_i}{L_i} \text{ or } \omega = n \frac{\pi c_e}{L_e}, \tag{26} \]
where \( n \) is a relative integer. Relations (26) mean that \( \omega \) should be an harmonic of one of the two fundamental frequencies
\[ \omega_i = \frac{\pi c_i}{L_i}, \quad \omega_e = \frac{\pi c_e}{L_e}, \tag{27} \]
irrespective of \( k_0 \). Such a situation corresponds to an infinite mismatch between the ‘MHD impedances’ \( \rho_i c_i \) and \( \rho_e c_e \) of the two slabs in contact. Due to this mismatch, waves are entirely trapped in either of the two media. Hence, they exhibit harmonic structures of an entirely trapped fundamental oscillation; \( k_0 \) just describes the phase detuning between two successive, independently oscillating slabs.

Since \( k_x \), defined from \( k_0 \) by (19), is the only wavenumber in play, the dispersion curves here represent \( \omega(k_0) \). When \( S \) is finite (\( S = 0 \) and \( S = 1 \)), a detailed analysis (see Appendix) of Equation (25) leads to dispersion curves as shown in Figure A2.
The important feature is the frequency jumps occurring for \( k_0 L = 0 \) and \( \pi \), a literal expression of which is given by (Equation (A15))

\[
\frac{\Delta \omega_n}{\omega_i} \approx \frac{1}{2} \left| S - \frac{1}{S} \right| \frac{\omega_i}{\omega_e} \frac{n}{1 - QR + \frac{n^2 Q^2 R^2}{16}},
\]

(28)

in the case \( \omega_i \ll \omega_e \). The more inhomogeneous the medium is (\( S \) getting away from unity), the larger the jumps \( \Delta \omega_n \) are. This behaviour is illustrated in Figure 2 for different values of the dimensionless parameters

\[
R_p = \frac{\rho_i}{\rho_e}, \quad R_c = \frac{c_i}{c_e}, \quad R_L = \frac{L_i}{L_e},
\]

(29)

![Graph showing solutions to the dispersion relation (25) for the propagating mode in the x-direction (\( k_x = 0 \)) in various circumstances showing the change as the inhomogeneity of the medium increases. (a) solid line: \( R_p = 0.8, R_c = 2.0, R_L = 1.25 \) (\( S = 1.6, Q = 1.1 \)); (b) dashed line: \( R_p = 0.5, R_c = 10, R_L = 1.25 \) (\( S = 5, Q = 2.6 \)); (c) dotted line: \( R_p = 0.5, R_c = 10, R_L = 5 \) (\( S = 5, Q = 2.6 \)). The dimensionless variables \( X \) and \( Z \) are defined by relations (30). As the medium becomes more homogeneous (case a), the dispersion curve comes closer to the line \( Z = X \) which is simply the dispersion curve of the perfect homogeneous medium. Note that even for a small departure from homogeneity, the jumps at \( \pi, 2\pi, 3\pi \ldots \), are not necessarily regular.

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with the dimensionless variables

\[ X = k_0 L, \]

\[ Z = \omega \left( \frac{L_i}{c_i} + \frac{L_e}{c_e} \right). \]  

(30)

The ratios \( R = \omega_i/\omega_e \), \( S \) and Equation (25) rewrite, respectively,

\[ R = R_c/R_L, \quad S = R_c R_{\rho} \]  

and

\[ \cos X = \cos \frac{Z}{1 + R} \cos \frac{RZ}{1 + R} - \frac{Q}{1 + R} \sin \frac{Z}{1 + R} \sin \frac{RZ}{1 + R}. \]  

(32)

As the inhomogeneity of the medium decreases, \( S \) and \( Q \) tend toward 1, and the dispersion curve consequently comes closer to the line \( X = Z \), that is, the dispersion curve of the perfect homogeneous medium.

Moreover, if we write in the variables \((X, Z)\) the slope of the dispersion curve at the origin given in the annex (Equation (A9)), noting that \( Y \) and \( Z \) are related by

\[ Z = (1 + R)Y. \]  

(33)

We get for the branch \( X > 0 \)

\[ \frac{dZ}{dX} = (1 + R) (1 + 2QR + R^2)^{-1/2}. \]  

(34)

Now, as the medium becomes more and more homogeneous, \( Q \) goes to 1 and from (34), \( dZ/dX \) as well. Therefore, the dispersion curve tends toward the diagonal \( X = Z \).

Note that reciprocally the condition \( S = 1 \) (or \( Q = 1 \)) alone does not mean necessarily that the medium is homogeneous. It simply means that the MHD impedances \( \rho_i c_i \) and \( \rho_e c_e \) are equal.

4.2. Case \( k_y = 0 \) and \( k_z \neq 0 \)

4.2.1. Dimensionless Ratios

When the propagation in the \( z \)-direction is taken into account, \( S \) is no longer a constant and becomes \( \omega \)-dependent (Equation (14)). Various situations are possible, according to the relative magnitude of the speeds \( c_S, c_A, \) and \( c_T \) in slabs ‘i’ and ‘e’, due to the signs of \( q_i^2 \) and \( q_e^2 \). Nevertheless, the choice is restricted by the balance equation (2) and the condition that \( B_i > B_e \), that is, the inner medium is more strongly magnetized, it will prove convenient to use dimensionless ratios:

\[ R_\rho = \frac{\rho_i}{\rho_e}, \quad R_T = \frac{T_i}{T_e}, \quad R_B = \frac{B_i}{B_e}, \quad R_L = \frac{L_i}{L_e}, \]  

(35)

where \( T \) denotes the temperature.
We, therefore, can write
\[
\frac{c_{S_l}^2}{c_{S_e}^2} = R_T; \quad \frac{c_{\Lambda_l}^2}{c_{\Lambda_e}^2} = \frac{R_B^2}{R_p^2}.
\]  
(36)

Equation (2) becomes
\[
\frac{c_{S_l}^2}{c_{\Lambda_l}^2} = \gamma \frac{R_p R_T}{2 R_B} \frac{R_B^2 - 1}{1 - R_p R_T},
\]
(37)
or equivalently
\[
\frac{c_{S_e}^2}{c_{\Lambda_e}^2} = \gamma \frac{R_B^2 - 1}{2 (1 - R_p R_T)}.
\]
(38)

The set of Equations (36), (37), and (38) shows that if we fix, say, \(c_{\Lambda_l}\) and choose \(R_p, R_T,\) and \(R_B (R_B > 1)\), then \(c_{S_l}, c_{\Lambda_e}, c_{S_e}\) are well defined, provided
\[
R_p R_T < 1
\]
(39)
is satisfied.

In order to make easier the comparison with Edwin and Roberts (hereafter referred to as ER), we first study, like them, the incompressible modes.

4.2.2. Incompressible Modes

The incompressible modes are obtained when \(c_S\) is infinite. Therefore, from (5) we get
\[
q^2 = k_z^2,
\]
(40)
and we see that only surface modes are possible. Equation (15) becomes
\[
\cos k_0 L = \cosh k_z L_i \cosh k_z L_e + \frac{1}{2} \left( S + \frac{1}{S} \right) \sinh k_z L_i \sinh k_z L_e.
\]
(41)
Let us then define the dimensionless arguments
\[
\theta = k_z L_i, \quad \phi = k_0 L
\]
(42)
and the phase velocity
\[
c = \frac{\omega}{k_z}.
\]
(43)

With use of (36), (41) is equivalent to
\[
\cos \phi = \cosh \theta \cosh \frac{\theta}{R_L} + \frac{1}{2} \left( S + \frac{1}{S} \right) \sinh \theta \sinh \frac{\theta}{R_L},
\]
(44)
where
\[ S = R_\rho \frac{c_{Ai}^2 - c^2}{c_{Ae}^2 - c^2}. \] (45)

The dispersion curve, analogous to ER's, will display the function \( c(\theta) \) satisfying (44) and (45), with \( \phi \) as parameter.

Inverting (45) yields
\[ c^2_\pm = \frac{R_\rho c_{Ai}^2 - S_\pm c_{Ae}^2}{R_\rho - S_\pm} \] (46)

with
\[ S_\pm = U \pm \sqrt{U^2 - 1}, \] (47)
\[ U = \frac{\cos \phi - \cosh \theta \cosh \frac{\theta}{R_L}}{\sinh \theta \sinh \frac{\theta}{R_L}}, \] (48)

provided that \( \theta \neq 0 \).

Expressions (46) and (47) show that there exist two modes like in ER's analysis, namely a modified sausage mode \( (c_+) \) and a modified kink mode \( (c_-) \).

Note also that, provided \( \phi \neq 0 \), we have
\[ \lim_{\theta \to 0} U = -\infty, \]
and
\[ \lim_{\theta \to 0} S_+ = 0, \]
\[ \lim_{\theta \to 0} S_- = -\infty \]
and, therefore,
\[ \lim_{\theta \to 0} c_- = c_{Ae} \quad \text{(kink)}, \]
\[ \lim_{\theta \to 0} c_+ = c_{Ai} \quad \text{(sausage)}, \]
that is, \( c(\theta) \) can be defined at \( \theta = 0 \) by
\[ c_+(0) = c_{Ai}; \quad c_-(0) = c_{Ae}. \] (49)

The homogeneous medium situation is recovered when \( R_L \to \infty \), which by means of (46), (47), and (48) leads to
\[ c_+ = c_{Ai}; \quad c_- = c_{Ae}. \]
The isolated-slab situation also can be quickly found by setting \( R_L = 0 \), that is
\[
U = - \coth \theta.
\]
Therefore,
\[
S_- = - \coth \frac{\theta}{2} \quad \text{(kink)},
\]
\[
S_+ = - \tanh \frac{\theta}{2} \quad \text{(sausage)}
\]
and
\[
c_-^2 = \frac{R_\rho c_{\Lambda i}^2 + c_{\Lambda e}^2 \coth \frac{\theta}{2}}{R_\rho + \coth \frac{\theta}{2}} \quad \text{(kink)},
\]
\[
c_+^2 = \frac{R_\rho c_{\Lambda i}^2 + c_{\Lambda e}^2 \tanh \frac{\theta}{2}}{R_\rho + \tanh \frac{\theta}{2}} \quad \text{(sausage)}
\]
which is actually identical with (12) in ER.

Now, let us turn to our more general situation with \( R_L \neq 0 \). The asymptotical behaviour as \( \theta \to + \infty \) follows simply from (48),
\[
U \sim - \coth \theta \coth \frac{\theta}{R_L},
\]
from (47),
\[
S_\pm \sim U,
\]
and eventually from (46),
\[
c_\pm^2 \approx \frac{R_\rho c_{\Lambda i}^2 + c_{\Lambda e}^2 \coth \theta \coth \frac{\theta}{R_L}}{R_\rho + \coth \theta \coth \frac{\theta}{R_L}}.
\]
As \( \theta \gg 1 \) (thick slab), the limit of (51) writes
\[
c_\infty^2 = \frac{R_\rho c_{\Lambda i}^2 + c_{\Lambda e}^2}{R_\rho + 1},
\]
which is perfectly identical with (14) in ER. Surprisingly this result does not depend on \( R_L \) or \( \phi \).

The behaviour of \( c(\theta) \) for \( \theta \ll 1 \) (slender slab) is also of particular interest inasmuch as it points out a sharp contrast with the isolated slab. Actually, from (48) we derive,
provided that $\phi \neq 0$,

$$U \sim -2R_L \theta^{-2} \sin^2 \frac{\phi}{2},$$

that is

$$\lim_{\theta \to 0} U = -\infty.$$  

From (47) we derive

$$S_+ \sim \frac{1}{2U},$$

$$S_- \sim 2U$$

and from (46)

$$c_+^2 \sim c_{Ai}^2 \left[ 1 + \frac{1}{R_p} \left( \frac{c_{Ai}^2}{c_{Ae}^2} - 1 \right) \frac{\theta^2}{4R_L \sin^2(\phi/2)} \right]$$

(53a)

for the sausage mode, and

$$c_-^2 \sim c_{Ae}^2 \left[ 1 + R_p \left( \frac{c_{Ai}^2}{c_{Ae}^2} - 1 \right) \frac{\theta^2}{4R_L \sin^2(\phi/2)} \right]$$

(53b)

for the kink mode. These expansions (53) are analogous to expressions (13) in ER, but with two important differences:

1. Our expansions are to second order, without first-order term.
2. The periodicity appears through the dividing factor $R_L \sin^2(\phi/2)$.

Now, if $\phi = 0$ then from (48)

$$U \sim U_0 + R_L \left( 1 - \frac{1}{R_L^2} \right)^2 \frac{\theta^2}{24},$$

$$U_0 = -\frac{1}{2} \left( R_L + \frac{1}{R_L} \right).$$

From (47)

$$S_{\pm} \sim S_{0\pm} + \left( 1 \pm \frac{1}{2\sqrt{U_0^2 - 1}} \right) U_0 R_L \left( 1 - \frac{1}{R_L^2} \right)^2 \frac{\theta^2}{12},$$

$$S_{0\pm} = U_0 \pm \sqrt{U_0^2 - 1}.$$
Then, from (46)
\[
c_+^2 \sim c_0^2 \left[ 1 + \frac{R_\rho}{(R_\rho - S_0^\pm)^2} \frac{c_{Ae}^2 - c_{Al}^2}{c_0^2} \times \right. \\
\left. \times \left( 1 \pm \frac{1}{2 \sqrt{U_0^2 - 1}} \right) U_0 R_L \left( 1 - \frac{1}{R_L^2} \right)^2 \theta^2 \right], \tag{54}
\]
\[
c_0^2 \pm = \frac{R_\rho c_{Al}^2 - S_0^\pm c_{Ae}^2}{R_\rho - S_0^\pm}.
\]

This result enables one to define \(c(0)\) as
\[
c_\pm(0) = c_0^\pm. \tag{55}
\]

It is obvious that \(c_\pm(0)\) as defined by (55) is different from either \(c_{Al}\) or \(c_{Ae}\), in opposition to (49) in the case \(\phi \neq 0\). More specifically, if \(R_L \gg 1\) or \(R_L \ll 1\), then (54) and (55) lead to
\[
c_+(0) \sim c_{Al}, \quad c_-(0) \sim c_{Ae},
\]
that is (49), whereas if \(R_L \simeq 1\), they lead to
\[
c_\pm^2(0) \sim \frac{R_\rho c_{Al}^2 + c_{Ae}^2}{R_\rho + 1},
\]
that is \(c_\infty\) (Equation (52)). Actually, in this latter case (\(\phi = 0\) and \(R_L = 1\)), it is easy to check from (46), (47), and (48) that \(c(\theta)\) is a constant function equal to \(c_\infty\).

These preliminary remarks put into evidence a striking difference in the behaviour of \(c(\theta)\) according as \(\phi \neq 0\) or \(\phi = 0\), which will appear on the dispersion curves to be drawn.

Following ER in their Figure 2a, we adopt
\[
\rho_i = \rho_e, \quad c_{Al} = 10c_{Ae},
\]
which, using (36), (37) and assuming \(\gamma = 5/3\), corresponds to
\[
R_\rho = 1, \quad R_T = 1, \quad R_B = 10. \tag{56}
\]

The results displayed on our Figure 3 show the changes in \(c(\theta)\) as \(\phi\) is fixed to one of the three values 0, \(\pi/2\), \(\pi\), and \(R_L\) is varied.

As expected, the case \(\phi = 0\) is quite peculiar, since the dispersion curves intersect the axis \(\theta = 0\) with \(c \neq c_{Al}\) and \(c \neq c_{Ae}\). Apart from that, when \(\phi \neq 0\) the following three points are remarkable:

1. The variations of \(c(\theta)\) with \(\phi\) are small. Actually, when \(\theta \ll 1\) the expansions (53) show that the curves are shifted along \(\theta\) by an amount \(\sin(\phi/2)\).

2. The curves at \(\theta = 0\) have a parabolic branch, and, therefore, intersect the curve corresponding to the isolated slab \((R_L = 0)\). The intersection abscissa is solution of the
equation

\[ \cos \phi = e^{-\theta/R_L} \cosh \theta. \]  

(57)

4.2.3. Compressible Modes

As compressibility comes into play, the problem becomes much more complex because:

1. not only surface, but also propagating, body, and tunnelling modes exist;
2. Equations (12), (15), (16), and (17) completed by (13) and (14) cannot be any longer solved explicitly to yield such a simple expression as (46).
Nevertheless, the slender-slab ($\theta \ll 1$) and thick-slab ($\theta \gg 1$) cases still remain of relevance since they make easier the comparison with ER. They will be examined first.

Extending the notation of the preceding subsection, we define $\theta$, $\phi$, and $c$ by (42) and (43) exactly in the same way, and set

$$q_t^* = \frac{(c_s^2 - c^2)(c_{A1}^2 - c^2)}{c_t^2(c_{T1}^2 - c^2)},$$

$$q_e^* = \frac{(c_s^2 - c^2)(c_{Ae}^2 - c^2)}{c_e^2(c_{Te}^2 - c^2)}.$$  

The dispersion relations (12), (15), (16), and (17), therefore, rewrite

$$\cos \phi = \cos q_t^* \theta \cos q_e^* \frac{\theta}{R_L} - \frac{1}{2} \left( S + \frac{1}{S} \right) \sin q_t^* \theta \sin q_e^* \frac{\theta}{R_L},$$  

(59)
\[
\cos \phi = \cosh q_t^* \theta \cosh q_e^* \frac{\theta}{R_L} + \frac{1}{2} \left( \frac{S}{S} + \frac{1}{S} \right) \sinh q_t^* \theta \sinh q_e^* \frac{\theta}{R_L},
\]

(60)

\[
\cos \phi = \cos q_t^* \theta \cos q_e^* \frac{\theta}{R_L} \frac{1}{2} \left( \frac{S}{S} - \frac{1}{S} \right) \sin q_t^* \theta \sin q_e^* \frac{\theta}{R_L},
\]

(61)

\[
\cos \phi = \cosh q_t^* \theta \cos q_e^* \frac{\theta}{R_L} - \frac{1}{2} \left( \frac{S}{S} - \frac{1}{S} \right) \sinh q_t^* \theta \sin q_e^* \frac{\theta}{R_L},
\]

(62)

with

\[
S = R \frac{q_e^*}{q_t^*} \frac{c_{\text{Ai}}^2 - c^2}{c_{\text{Ae}}^2 - c^2}.
\]

(63)
The slender slab corresponds to $\theta \ll 1$. In the case of surface waves we derive from (60)

$$-4 \sin^2 \frac{\phi}{2} = \left[ q_t^* q_\phi^* + \frac{q_\phi^* q_\phi^*}{R_L^2} + \left( S + \frac{1}{S} \right) \frac{q_t^* q_\phi^*}{R_L} \right] \theta^2.$$  

Fig. 4a–d. The phase-speed $c = \omega/k_\perp$ as a function of $\theta = k_\perp L$ for propagation in the $xz$-plane in a compressible medium. Phase-speeds are given in units of $c_{Si}$. We have adopted: $c_{Se} = 1.5$, $c_{AI} = 2.0$, $c_{Ae} = 0.5$ ($c_{P} = 2/\sqrt{5} \simeq 0.894$, $c_{TE} = 3/(2\sqrt{10}) \simeq 0.474$, $R_P = 59/104 \simeq 0.567$, $R_T = 4/9 \simeq 0.444$, $R_B = \sqrt{118/13} \simeq 3.013$) and (a) $\phi = \pi/2$, $R_L = 1$; (b) $\phi = \pi/2$, $R_L = 0.1$; (c) $\phi = \pi$, $R_L = 1$; (d) $\phi = \pi/4$, $R_L = 1$. In this figure and the following, the symbols $P$, $B$, $T$, $S$ written at the right of each band mean, respectively, propagating, body, tunneling and surface mode. The asymptotic values for surface waves are $c_{\infty} \simeq 0.882$ and $c_{\infty} \simeq 1.231$. The phase-speed $c_0$ defined by (71) is here ($R_L = 1$) $c_0 = 3.586$. This case corresponds to Figure 3 in ER.
With use of (63) this becomes

\[ (1 + \frac{1}{R_p R_L} \frac{c_{Ai}^2 - c_e^2}{c_{Te}^2 - c_e^2}) q_i^* q_e^* + \left(1 + R_p R_L \frac{c_{Ai}^2 - c_e^2}{c_{Te}^2 - c_e^2} \right) \frac{q_e^*}{R_L^2} = -4 \theta^{-2} \sin^2 \frac{\phi}{2}, \]

and if \( \phi \neq 0 \) has approximate solutions

\[ c^2 \sim c_{Ti}^2 \left[ 1 + \frac{1}{R_p} \left( \frac{c_{Ai}^2}{c_{Te}^2} + R_p R_L \frac{c_{Ai}^4}{c_e^2} \right) \frac{c_{Si}^2}{c_{Ai} c_i^2} \frac{\theta^2}{4 R_L \sin^2 (\phi/2)} \right], \quad (64a) \]

\[ c^2 \sim c_{Te}^2 \left[ 1 + R_p \left( \frac{c_{Ai}^2}{c_{Te}^2} + \frac{1}{R_p R_L} \frac{c_{Ai}^4}{c_e^2} \right) \frac{c_{Se}^2}{c_{Ai} c_e^2} \frac{\theta^2}{4 R_L \sin^2 (\phi/2)} \right]. \quad (64b) \]

It is easy to show that in the case of body waves, Equation (61) admits an approximate
solution equal to (64a) to the second order. The difference is due to higher order terms, not calculated here.

Expressions (64) are analogous to (16) and (18) in ER, but with more complex coefficients. The argument again is inversely proportional to \( R \sin^2(\phi/2) \) as for incompressible modes.

The thick slab corresponds to \( \theta \gg 1 \). The value \( c_{\infty} \) is given by the asymptotic form of (60) for surface waves, which, provided \( q^*_f \neq 0 \) and \( q^*_e \neq 0 \), simply reduces to

\[
S + 1 = 0 ,
\]

analogous to ER's expression (20), or

\[
(c_{Te}^2 - c^2)(c_{Ae}^2 - c^2)(c_{Si}^2 - c^2) = R^2 \frac{c_T^2}{c_e^2} (c_{Ti}^2 - c^2)(c_{Ai}^2 - c^2)(c_{Se}^2 - c^2) .
\]

(65)
Note that the solutions $c_\infty$ of this cubic equation do not depend on $\phi$ or $R_L$. The case $\phi = 0 \mod 2\pi$ can be studied by first factorizing (59) to (62) as

$$-\sin^2 \frac{\phi}{2} + \sin^2 \left(q_i^* - \frac{q_e^*}{R_L}\right) \frac{\theta}{2} = -\frac{(S + 1)^2}{4S} \sin q_i^* \theta \sin q_e^* \frac{\theta}{R_L},$$

$$-\sin^2 \frac{\phi}{2} - \sinh^2 \left(q_i^* - \frac{q_e^*}{R_L}\right) \frac{\theta}{2} = \frac{(S + 1)^2}{4S} \sinh q_i^* \theta \sinh q_e^* \frac{\theta}{R_L},$$

$$-\sin^2 \frac{\phi}{2} - \sinh^2 \left(iq_i^* - \frac{q_e^*}{R_L}\right) \frac{\theta}{2} = \frac{(S + i)^2}{4S} \sin q_i^* \theta \sin q_e^* \frac{\theta}{R_L},$$

$$-\sin^2 \frac{\phi}{2} - \sinh^2 \left(q_i^* + i \frac{q_e^*}{R_L}\right) \frac{\theta}{2} = -\frac{(S + i)^2}{4S} \sinh q_i^* \theta \sin q_e^* \frac{\theta}{R_L},$$

Fig. 4d.
and setting \( \phi = 0 [\text{mod} 2\pi] \), that is to say \( \sin \phi/2 = 0 \). The solutions of the resulting equations are

(1) for any phase-speed, \( \theta = 0 \);

Fig. 5a. The phase-speed \( c = \omega/k_z \) as a function of \( \theta = k_z L_z \) for propagation in the \( xz \)-plane in a compressible medium under the circumstances: (a) \( c_{S\infty} = 0.5 \), \( c_{AI} = 2.0 \), \( c_{AE} = 5.0 \), \( c_{TI} = 2/\sqrt{5} \approx 0.894 \), \( c_{TE} = 5/\sqrt{101} \approx 0.498 \), \( R_p = 253/52 \approx 4.865 \), \( R_T = 4 \), \( R_B = \sqrt{253/13} \approx 0.882 \), \( c_0 \approx 7.759 \). (b) \( c_{S\infty} = 1.5 \), \( c_{AI} = 0.6 \), \( c_{AE} = 0.95 \), \( c_{TI} = 3/\sqrt{34} \approx 0.514 \), \( c_{TE} = 57/(2 \sqrt{1261}) \approx 0.803 \), \( R_p = 1441/624 \approx 2.309 \), \( R_T = 4/9 \approx 0.444 \), \( R_B = (3/19) \sqrt{1441/624} \approx 2.026 \). The asymptotic value for surface waves is \( c_{S\infty} \approx 0.660 \), \( c_0 \approx 0.537 \). (c) \( c_{S\infty} = 0.75 \), \( c_{AI} = 0.5 \), \( c_{AE} = 1.5 \), \( c_{TI} = 1/\sqrt{5} \approx 0.447 \), \( c_{TE} = 3/(2 \sqrt{5}) \approx 0.671 \), \( R_p = 117/58 \approx 2.017 \), \( R_T = 16/9 \approx 1.778 \), \( R_B = \sqrt{13/58} \approx 0.473 \). The asymptotic values for surface waves are \( c_{S\infty} \approx 0.663 \) and \( c_0 \approx 0.876 \), \( c_0 \approx 0.955 \). (d) \( c_{S\infty} = 4.0 \), \( c_{AI} = 2.0 \), \( c_{AE} = 5.0 \), \( c_{TI} = 2/\sqrt{5} \approx 0.894 \), \( c_{TE} = 20/\sqrt{41} \approx 3.123 \), \( R_p = 17/2 \approx 8.5 \), \( R_T = 1/16 \approx 0.0625 \), \( R_B = (2/5) \sqrt{17/2} \approx 1.166 \), \( c_0 \approx 2.492 \). Throughout this figure \( \phi = \pi/2 \) and \( R_L = 1 \) and phase-speeds are given in units of \( c_{S\infty} \). The phase-speed \( c_0 \) defined by (71) is calculated for \( R_L = 1 \). The relevant cases correspond to Figures 4, 5, 6, and 7 in ER.
(2) for any $\theta$, $c$ equals $c_0$ solution of the system

$$q_e^* = R_L q_i^*, \quad S = -1,$$

which simply yields

$$c_0^2 = \frac{R_p R_L c_{Ai}^2 + c_{Ae}^2}{R_p R_L + 1}, \quad (70)$$

or else

$$c_0^2 = \frac{R_p^2 R_L + 1}{R_p R_L + 1} c_{Ae}^2. \quad (71)$$
Note that if $R_L = 1$, then $c_0$ equals $c_\infty$ defined by (52).

The reduction to Love's equation in the cold-plasma ($c_{Si} = c_{Se} = 0$) or non-magnetic ($c_{Ai} = c_{Ae} = 0$) approximations also studied by ER may be simply accounted for. Actually, Love's equation should be here replaced by (47) with $U$ defined by (48) and

\[
q_{\theta^2} = 1 - \frac{c^2}{c_{Ai}^2},
\]

(cold-plasma)

\[
q_{\phi^2} = 1 - \frac{c^2}{c_{Ae}^2}.
\]
or

\[
q_l^{*2} = 1 - \frac{c^2}{c^2_{Sl}}, \quad \text{(non-magnetic)}
\]

\[
q_e^{*2} = 1 - \frac{c^2}{c^2_{Se}}.
\]

Now, to illustrate the great variety of compressible configurations, we shall choose five sets of parameters adopted by ER.

For the first of these (our Figure 4, ER's Figure 3) we study the influence of \( \phi \) and \( R_L \). Once again the change in the \( \theta \)-scale is proportional to \( \sqrt{R_L} \sin \phi/2 \). The same modes as in ER are present but they extend somewhat out of their normal bands. For

![Fig. 5d.](image-url)
example the fast sausage surface mode (between \(c_{Si}\) and \(c_{Se}\)) passes through the tunnelling (between \(c_{Se}\) and \(c_{Ai}\)) and the propagating domains (above \(c_{Ai}\)), adjusting its character each time. Similarly, the slow kink surface mode (between \(c_{Ae}\) and \(c_{Ti}\)) passes through the tunnelling band (between \(c_{Ae}\) and \(c_{Te}\)) just beneath. As a consequence, the characteristic phase-speeds of the slender slab are always \(c_{Ti}\) and \(c_{Te}\) (see (64)) in contrast with the isolated slab, for which they may be \(c_{Ae}\) or \(c_{Se}\) (ER’s Figure 3).

It is easy to understand such a difference. Indeed, in the network of slabs the inside and outside media are strictly equivalent and commutable, because both have a finite width, whatever different their physical properties (\(R_p\), \(R_T\), and \(R_B\) \(\neq\) 1) may be. On the opposite, the isolated slab, of finite width, is quite singular in its infinite environment.

For the following four sets of parameters (our Figure 5a–d, corresponding to ER’s Figures 4, 5, 6, 7) we have computed only the situation with \(\phi = \pi/2\) and \(R_L = 1\), since we know from the preceding discussion that there is no big qualitative influence of \(\phi\) and \(R_L\). The same remarks as for Figure 4 hold here too.

5. Conclusion

The present model goes a step further than ER since it considers a distribution of tubes. Differences are found but they do not deeply modify the dispersion diagram. The modulation due to the network mainly results in a different \(\theta\)-scaling.

Nevertheless it does not include gravity forces, whose effects are a basic ingredient in the famous oscillations of the photosphere and chromosphere. These forces compete with the pressure forces so as to share the diagnostic diagram into the so-called gravity \(g\)- and acoustic \(p\)-modes, separated by the evanescent-wave domain. The next step in our study will attempt to include gravity into the model in order to see the consequences on the classical diagnostic diagram (Ulrich et al., 1979) and possibly the striped structure representing regions of existence and non-existence of the \(p\)- or \(g\)-modes. The natural approach we are thinking of uses the MHD energy principle (Zweibel, 1985).

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Appendix

The dispersion relation (25) can be cast into implicit form

\[
\cos X = F(Y),
\]

(A1)
with the following non-dimensional variables:

\[ X = k_0 L , \]

\[ Y = \pi \frac{\omega}{\omega_i} , \]  

(A2)

and parameters:

\[ Q = \frac{1}{2} \left( S + \frac{1}{S} \right) , \]  

(A3)

\[ R = \frac{\omega_i}{\omega_e} . \]

The expression for \( F(Y) \), therefore, writes

\[ F(Y) = \cos Y \cos RY - Q \sin Y \sin RY . \]  

(A4)

Let us notice from (A3) that \( Q \) is always larger than 1.

Now, our purpose is to determine the reciprocal \( Y = Y(X) \), i.e., to look for the roots of Equation (A1). Since these roots are not unique, the function \( Y(X) \) actually is multivalued, but periodical in \( X \), because of the \( 2\pi \)-periodicity of \( \cos X \).

We first determine the first and second derivatives \( dY/dX \) and \( d^2 Y/dX^2 \). By differentiating (A1), we get

\[ -\sin X \, dX = F'(Y) \, dY , \]

that is

\[ \frac{dY}{dX} = -\frac{\sin X}{F'(Y)} , \]  

(A5)

and then

\[ \frac{d^2 Y}{dX^2} = -\frac{\cos X}{F'(Y)} - \frac{\sin^2 X}{F'(Y)^3} F''(Y) \]  

(A6)

with

\[ F'(Y) = - (1 + QR) \sin Y \cos RY + (Q + R) \cos Y \sin RY , \]

\[ F''(Y) = - (1 + 2QR + R) \cos Y \cos RY - (Q + 2R + QR) \sin Y \sin RY . \]  

(A7)

It is more convenient to understand the curves of \( Y(X) \) in Figure A2 by first drawing the curve of \( F(Y) \) and taking the geometrical solution of Equation (A1). This equation will admit a solution only if \( |F(Y)| \leq 1 \). Intervals where this condition is not satisfied will lead to a jump of \( Y(X) \).
Values of particular interest are the locations of extrema, where \( \frac{dY}{dX} = 0 \), i.e., \( \sin X = 0 \), and, therefore, \( X = 0 \mod 2\pi \) (maximum) or \( X = \pi \mod 2\pi \) (minimum).

At those points, \( \frac{d^2 Y}{dX^2} = -\cos X/F'(Y) \). From Figure 6, we can check that:

1. If \( X = 0 \mod 2\pi \), then \( \cos X = 1 \); the lower root \( Y_e \) corresponds to \( F'(Y_e) > 0 \) and, therefore, \( \frac{d^2 Y}{dX^2} < 0 \); it is a maximum; the upper root \( Y_u \) corresponds to \( F'(Y_u) < 0 \), therefore, \( \frac{d^2 Y}{dX^2} > 0 \); it is a minimum.

2. If \( X = \pi \mod 2\pi \), then \( \cos X = -1 \); the opposite is true.

![Fig. 6. Representative curve of the function \( F(Y) = \cos Y \cos RY - Q \sin Y \sin RY \), with \( Q, R, Y \) defined by relations (A2) and (A3). A graphic resolution of the equation \( \cos X = F(Y) \) is explained.](image)

To be complete, it is useful to factor Equation (A1) for \( X = 0, \pi/2, \) and \( \pi \mod 2\pi \), respectively, as

\[
\left( \tan \frac{Y}{2} + S \tan \frac{RY}{2} \right) \left( \tan \frac{Y}{2} + \frac{1}{S} \tan \frac{RY}{2} \right) = 0,
\]

\[
\cotan Y \cotan RY = Q, \tag{A8}
\]

\[
\left( \tan \frac{Y}{2} \tan \frac{RY}{2} - S \right) \left( \tan \frac{Y}{2} \tan \frac{RY}{2} - \frac{1}{S} \right) = 0.
\]

The points \( X = 0 \mod 2\pi \) and \( Y = 0 \) are singular since \( F'(Y) = 0 \). It can be shown that the derivatives on the left and the right at such points are opposite,

\[
\frac{dY}{dX} = \pm (1 + 2Q R + R^2)^{-1/2}, \tag{A9}
\]

and yield cusp points.

An important question which arises is whether the neighbourhood of any extremum \( Y_e \) (except the first one \( Y_e = 0 \)) in the curve \( F(Y) \) (Figure 6) yields a jump of \( Y \) in the
Fig. 7. Representative curves of the function $Y(X)$, solution of the equation $\cos X = F(Y)$. These curves are symmetrical with respect to $\pi$; therefore, the interval $[0, \pi]$ (the so-called first Brillouin zone in solid state physics) is fully descriptive of the mode system.

curve $Y(X)$ (Figure 7). In other words, do we have

$$|F(Y_e)| > 1,$$  \hspace{1cm} (A10)

for any $Y_e$? Let us show that this is true.

Since $F'(Y_e) = 0$, from (A7)

$$\tan Y_e \cotan R Y_e = - \frac{Q + R}{1 + QR}.$$  \hspace{1cm} (A11)

Inserting (A11) into (A4) gives

$$F(Y_e) = - \frac{\sin R Y_e}{\sin Y_e} \frac{Q + R + (Q^2 - 1)R \sin^2 Y_e}{1 + QR},$$  \hspace{1cm} (A12)

or equivalently

$$F(Y_e) = - \frac{\sin Y_e}{\sin R Y_e} \frac{1 + QR + (Q^2 - 1) \sin^2 R Y_e}{Q + R}.$$  \hspace{1cm} (A13)

From (A11) it can be shown that

$$\frac{(Q + R)^2}{\sin^2 Y_e} = \frac{(1 + QR)^2}{\sin^2 R Y_e} + (Q^2 - 1)(1 - R^2).$$  \hspace{1cm} (A14)
When \( R < 1 \), (A14) leads to
\[
\left| \frac{\sin RY_e}{\sin Y_e} \right| > \frac{1 + QR}{Q + R}
\]
and, therefore, (A12) yields the inequality
\[
| F(Y_e) | > 1 + \frac{Q^2 - 1}{Q + R} R \sin^2 Y_e,
\]
which also confirms (A10) since \( Q^2 > 1 \).

On the other hand, when \( R > 1 \), (A14) leads to
\[
\left| \frac{\sin Y_e}{\sin RY_e} \right| > \frac{Q + R}{1 + QR}
\]
and (A13) yields the inequality
\[
| F(Y_e) | > 1 + \frac{Q^2 - 1}{1 + QR} \sin^2 RY_e,
\]
which also confirms (A10) since \( Q^2 > 1 \).

It is possible to derive an asymptotic expression for the jump \( \Delta Y \) at \( X = 0 \) or \( \pi [\text{mod} \ 2\pi] \) in the limit \( R \to 0 \)
\[
\Delta Y_n \sim R \left| S - \frac{1}{S} \right| \frac{\pi}{2} \frac{n}{n^2 - \frac{\pi^2 Q^2 R^2}{16}}.
\]
(A15)

The jump goes to zero as \( S \) approaches 1 (homogeneous medium). Moreover, in that limit \( Q \to 1 \) the slope at the origin \( (X \geq 0) \) from (A9) becomes
\[
\frac{dY}{dX} = (1 + R)^{-1}.
\]
(A16)

In other words, the dispersion curve gets closer and closer to the line defined by
\[
X = (1 + R)Y,
\]
(A17)
which characterizes the homogeneous medium.

It is worthwhile to determine the intersections of the dispersion curve with this limit line, by simply solving the system of Equations (A1), (A4), (A17). The solutions write
\[
X = n(1 + R)\pi \quad \text{or} \quad X = n(1 + R)\pi,
\]
(A18)
where \( n \) is a relative integer.
References