INERTIAL OSCILLATIONS IN THE SOLAR CONVECTION ZONE. III. A CYLINDRICAL MODEL FOR NONAXISYMMETRIC OSCILLATIONS IN A SUPERADIABATIC GRADIENT

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ABSTRACT

We examine the effects of a superadiabatic gradient and differential rotation upon inertial oscillations that are nonaxisymmetric about the axis of rotation as a generalization of earlier calculations published by Gilman and Guenther in 1985 for a cylindrical model of a stellar convection zone. We show that the frequencies of these oscillations are sensitive to both superadiabatic gradient and differential rotation. If such oscillations can be detected on the Sun, then they could be used to measure the degree of superadiabaticity of the convection zone, and also to distinguish fine differences in the radial gradient of angular velocity, for example, between rotation independent of depth, and one constant on cylinders parallel to the rotation axis, matched to the surface profile of latitudinal differential rotation, such as predicted for the Sun by various global convection models.

Subject headings: Sun: interior — Sun: oscillations — Sun: rotation

I. INTRODUCTION

In the first two papers in this series (Guenther and Gilman 1985, hereafter Paper I; Gilman and Guenther 1985, hereafter Paper II) we have shown how inertial oscillations, that arise from the Coriolis forces due to rotation, could measure such quantities as the depth of the solar convection zone and the differential rotation within it. Paper I presented a spherical shell model for these oscillations, while Paper II displayed a simple model in cylindrical geometry meant to apply to equatorial regions.

In both cases, the inertial modes studied were axisymmetric about the rotation axis and the medium was treated as being strictly adiabatic. But nonaxisymmetric inertial oscillations are also possible, and, of course, real stellar convection zones are generally somewhat superadiabatic. This paper explores how the frequencies and structures of inertial oscillations are changed by the presence of a superadiabatic gradient, what other families of frequency and wavenumber diagrams can be generated for nonaxisymmetric modes, and particularly how differential rotation affects the frequencies of different nonaxisymmetric oscillations. What we find is that a superadiabatic temperature gradient, such as that typically predicted by mixing-length theory, gives rise to a destabilizing buoyancy force that reduces all the frequencies of the inertial oscillations relative to the case of an adiabatic stratification. The buoyancy force is strong enough near the top of the convection zone to overpower the restoring Coriolis force completely, leading to the attenuation of the oscillations near the top relative to their amplitude near the bottom. This raises the question of how such neutral oscillations would then be seen at the top of the convection zone. This situation is rather analogous to that of gravity modes excited in the deep solar interior that attenuate in the convection zone where the buoyancy force changes from a restoring force to a destabilizing force.

But if these oscillations can be detected, and if we accept the estimates of convection zone depth from other sources, e.g., Christensen-Dalsgaard et al. (1985), then these modes can be used to estimate just how superadiabatic the convection zone is (D. O. Gough, private communication; Dziembowski, Kosovichev, and Kozlowski 1986).

We also find that nonaxisymmetric inertial oscillations are frequency-shifted by a given differential rotation much more than their axisymmetric counterparts. This is because a nonaxisymmetric wave pattern tends to be swept along by the differential rotation. As a consequence, they are a very sensitive measure of differential rotation, and if they can be found, should be able to distinguish between, for example, an angular velocity independent of depth, and one that is constant on cylinders and matched to the observed surface differential rotation profile, such as has been predicted by Glatzmaier (1984, 1985) and Gilman and Miller (1986).

II. BASIC EQUATIONS

As in Paper II, the convection zone is treated as a cylindrical annulus. The equations governing inertial oscillations in this annulus are therefore written in cylindrical polar coordinates \((r, \lambda, z)\) in a frame rotating with angular velocity \(\Omega\), taken to be that of the outer surface of the cylinder. The variable \(r\) denotes the time. \(U\) is the velocity of the unperturbed flow relative to this rotating frame, and is in the azimuthal direction.

The governing equations are generalized in two respects over those solved in Paper II. First, the convection zone now contains a mean specific entropy gradient \(\dot{s}_0/\dot{r}\) (< 0 for a superadiabatic convection zone) so that entropy perturbations are possible, which are described by a perturbation thermodynamic equation in the form

\[
\frac{\partial s}{\partial t} + \frac{U}{r} \frac{\partial s}{\partial \lambda} + w \frac{\partial s_0}{\partial r} = 0
\]

in which \(w\) is the radial velocity. The equations of perturbation motion, given below, now allow for azimuthal variations, as well as a perturbation buoyancy force written in terms of the
perturbation specific entropy,

$$\frac{\partial \delta u}{\partial t} + \frac{U}{r} \frac{\partial \delta u}{\partial \lambda} = -\frac{1}{r} \frac{\partial}{\partial r} \left[ 2\Omega + \left( \frac{U}{r} \right) + \left( \frac{dU}{dr} \right) \right] \delta w$$  \hspace{1cm} (2)

$$\frac{\partial \delta v}{\partial t} + \frac{U}{r} \frac{\partial \delta v}{\partial \lambda} = -\frac{\partial \delta \pi}{\partial \xi}$$  \hspace{1cm} (3)

$$\frac{\partial \delta w}{\partial t} + \frac{U}{r} \frac{\partial \delta w}{\partial \lambda} = -\frac{\partial \delta \pi}{\partial \xi} + \frac{\partial}{\partial r} \frac{c_p}{r} s + 2 \left[ \Omega + \left( \frac{U}{r} \right) \right] \delta u$$  \hspace{1cm} (4)

$$\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} (\delta \rho \delta w) \right) = 0 .$$  \hspace{1cm} (5)

In the above equation, \(u\) is the perturbation azimuthal velocity, \(\pi\) is the perturbation pressure divided by the reference state density, \(v\) is the axial (\(z\)) velocity, \(g\) is gravity, and \(c_p\) is the specific heat at constant pressure.

We use the same boundary conditions as explored in Paper II, namely that the radial velocity \(w\) vanish at the inner and outer boundaries of the cylinder. The inner boundary condition is consistent with placing the inner boundary of the cylindrical annulus at the bottom of the convection zone. The outer boundary condition is much more arbitrary, but is should be noted that in the presence of a superadiabatic gradient, the perturbation amplitudes of inertial modes become extremely small at a point interior to this boundary, so that our results should be essentially independent of the upper boundary condition.

In the axisymmetric case, we solved the governing equations by introducing a stream function. Here that will not work, but the equations can be simplified by introducing mass-weighted variables that are periodic in time \(t\), longitude \(\lambda\), and axial coordinate \(z\), as follows:

$$\tilde{\rho} r(u, v, w, s) = (u_m, v_m, w_m, s_m) \text{e}^{i(kx + lx - \omega t)} ;$$  \hspace{1cm} (6)

then by substitution, equations (1)-(5) may be written as

$$i \left( \frac{U}{r} \frac{k - \omega}{r} \right) u_m = -i \tilde{\rho} k \pi \left[ 2\Omega + \left( \frac{U}{r} \right) + \left( \frac{dU}{dr} \right) \right] w_m$$  \hspace{1cm} (7)

$$i \left( \frac{U}{r} \frac{k - \omega}{r} \right) v_m = -i \tilde{\rho} \pi \frac{\partial \pi}{\partial r}$$  \hspace{1cm} (8)

$$i \left( \frac{U}{r} \frac{k - \omega}{r} \right) w_m = -i \left( \frac{U}{r} \frac{k - \omega}{r} \right) s_m + 2 \left[ \Omega + \left( \frac{U}{r} \right) \right] u_m$$  \hspace{1cm} (9)

$$i \left( \frac{U}{r} \frac{k - \omega}{r} \right) s_m + w_m \frac{d\tilde{s}_0}{dr} = 0$$  \hspace{1cm} (10)

If we then eliminate all other dependent variables in favor of \(w_m\), we obtain a second-order differential equation of the form.

$$r^2(k^2 + l^2r^2)(Uk - \omega r) \frac{d^2 w_m}{dr^2} + r(Uk - \omega r)^2 \left[ (k^2 + l^2r^2) \left( 1 - \frac{r}{\rho} \frac{\partial \rho}{\partial r} \right) - 2l^2r^2 \right] \frac{dw_m}{dr}$$

$$+ \left[ k^2 + l^2r^2 \right] \frac{g}{c_p} \frac{ds_0}{dr} \left( r^2 + 2(\Omega r + U) \right)$$

$$\times \left[ 2\Omega r + U + \left( \frac{rdU}{dr} \right) \right] - (Uk - \omega r)^2 \right]$$

$$- kr(Uk - \omega r)(k^2 + l^2r^2) \left[ 2\Omega + \left( \frac{2dU}{dr} \right) + \left( \frac{rd^2U}{dr^2} \right) \right]$$

$$+ k(Uk - \omega r) \left[ 2\Omega r + U + \left( \frac{rdU}{dr} \right) \right]$$

$$\times \left[ (k^2 + l^2r^2) \left( 1 + \frac{r}{\rho} \frac{\partial \rho}{\partial r} \right) + 2l^2r^2 \right]$$

$$- 2k^2 l^2 (k^2 + l^2r^2)(\Omega r + U) \left[ 2\Omega r + U + \left( \frac{rdU}{dr} \right) \right] \frac{dw_m}{dr} = 0 .$$  \hspace{1cm} (13)

There are several limiting forms of equation (13) that relate to other calculations done previously. Of course, the limit \(k \to 0\) corresponds to the axisymmetric model solved in Paper II, with the addition of the superadiabatic terms. In addition, the limit \(l \to 0\), \(U \to 0\) gives rise to compressible vorticity waves that are independent of the axial coordinate. These have been previously studied for the case of an adiabatic gradient by Glatzmaier and Gilman (1981). The limit \(l \to 0\), \(U \to 0\) but \(U \neq 0\) gives us a linearized version of the equations solved by Van Ballegooijen (1986). His application was to the question of why giant cells predicted to occur in the solar convection zone apparently have such low amplitude near the photosphere. We have not attempted to find \(\omega = 0\) solutions to equation (13) here, because by definition their frequencies are not shifted by changing convection zone depth or differential rotation, and therefore cannot be used as diagnostics of interior properties in the way developed in Papers I and II.

We will attempt solutions to equation (13) only for the case when there are no singularities in the domain of integration, that is, \(Uk - \omega r\) does not change sign across the convecting layer. This generally means that mode frequencies must be sufficiently large that the second term dominates. Clearly, the higher the azimuthal wavenumber \(k\), the higher this frequency must be. Such higher frequency modes are probably also of greater relevance observationally, since they could be detected to a given level of accuracy in a shorter observational record than could lower frequency modes.

### III. Scaled Equations for a Specific Profile

As in Paper II, it is convenient to render equation (13) dimensionless, introducing a new independent variable \(x = r/r_0\), where \(r_0\) is the outer radius of the cylinder, and scaling \(\omega\) by \(\Omega\). We also scale \(l\) by \(1/r_0\), as before. If we denote the depth of the convection zone in cylindrical geometry by \(d\), and define \(e = d/2r_0\), then the range of \(x\) is \(1 - 2e < x \leq 1\).

For continuity with Paper II, we introduce the same differential rotation profile

$$U = \frac{U_0}{L} x(1 - x^2)$$

in which \(L = 4e(1 - e)(1 - 2e)\), so that \(U = 0\) at the outer boundary (corresponding to the solar equator) and \(U_0 > 0\) means rotation increases inward, \(U_0 < 0\) means rotation decreases inward. We then define a dimensionless measure of
the differential rotation with a Rossby number Ro = Uo/Ωd. Finally we let R = 2ε Ro/L = Ro/(2(1 − ε)(1 − 2ε)).

To illustrate the effect of a superadiabatic gradient on inertial oscillations, we choose a polynomial profile that represents a much steeper gradient near the top of the layer than near the bottom. In particular, we let the dimensionless buoyancy term in the coefficient of \( w_m \) in equation (13) be represented by

\[ \frac{g}{cp} \frac{1}{r} \left( \frac{dr}{\Omega^2} \right) = B x^8 \]

in which \( B \) is a positive number.

Using all of the above substitutions, equation (13) may then be written in compact form as

\[
\begin{align*}
   x^4(c_2 + d_2 x + \cdots + i_2 x^6 + j_2 x^7) & \frac{d^2 w_m}{dx^2} \\
   + x(b_1 + c_1 x + \cdots + h_1 x^6 + i_1 x^7) & \frac{dw_m}{dx} \\
   + (a_0 + b_0 x + \cdots + m_0 x^{12} + n_0 x^{13})w_m = 0 .
\end{align*}
\]

The definitions of all the alphabetical coefficients are given in the Appendix.

IV. SOLUTION BY THE METHOD OF FROBENIUS

The method of solution of equation (14) is the same as that in Paper II, provided there is no singularity in the domain of integration. In the present case, this restricts us to modes of sufficiently large frequency magnitude \( |\omega| \) that the factor \( Uk - cor \) does not change sign within the convection zone, as discussed in § III above. As in Paper II, we rewrite equation (14) in standard form, using the transformation \( x = \epsilon y + 1 - \epsilon \) (dropping the subscript \( m \) and using \( W \) for the dependent variable) to get

\[
R(y) \frac{d^2 W}{dy^2} + \frac{1}{y} P(y) \frac{dW}{dy} + \frac{1}{y^2} Q(y) W = 0
\]

in which

\[
R(y) = \sum_{r=0}^{q} R_r y^r,
\]

\[
P(y) = \sum_{p=0}^{q} P_p y^p,
\]

\[
Q(y) = \sum_{q=2}^{15} Q_q y^q .
\]

The expressions for each of the coefficient \( R_r, P_p, Q_q \) in terms of the previously defined parameters are given in the Appendix.

Following Paper II, the general solution of Equation (15) is of the form

\[
W = \sum_{n=0}^{\infty} W_n y^n
\]

Equation (15), which takes the form

\[
W_{n+2} = -\frac{1}{(n+2)(n+1)R_0} \left\{ (n+1)(nR_1 + P_1)W_{n+1} + \sum_{m=0}^{7} (n-m)(n-m-1)R_{m+2} + (n-m)P_{m+2} + Q_{m+2} \right\} W_{n-m} + \sum_{m=0}^{13} Q_{m+2} W_{n-m}
\]

in which terms with negative subscripts are set equal to zero.

Then the complete solution is written as

\[
W = A \sum_{n=0}^{\infty} W_{An} y^n + B \sum_{n=0}^{\infty} W_{Bn} y^n \equiv AW_A + BW_B
\]

The "A" solution is generated from equation (18) by setting \( W_0 = 0, W_0 = 1 \); the "B" solution by setting \( W_1 = 1, W_0 = 0 \). As in Paper II, we use simple boundary conditions, namely \( W(\pm 1) = 0 \), which requires we find roots of the equation

\[
W_{k}(-1)W_{k}(1) - W_{k}(1)W_{k}(1) = 0
\]

As before, we simply evaluate the left hand side of equation (20) for an array of values of \( l \) and \( \omega \), now for different \( k, Ro, \epsilon \), and plot the zero contour of this function, suppressing all other contours.

In practice, of course, we must truncate the series in expressions (19). The computer code to evaluate these series calculated terms in sequences of 10, out to the point where the absolute value of the latest group of 10 was no larger than \( 10^{-14} \) of the absolute value of the accumulated sum of lower terms. Convergence was faster for high frequencies than low when \( U = 0 \), and was faster the further the factor \( Uk - cor \) was from changing sign in the domain, as we should expect. With fast convergence, typically ~60 terms were needed, and with slow convergence, three or four hundred terms were commonly needed.

As a check on the solution technique, we reproduced some of the frequency-wavenumber diagrams shown in Paper II in the limiting case \( k = 0 \).

The method of Frobenius will work provided the factor \( Uk - cor \) does not change sign in the domain. For \( Ro < 0 \) (angular velocity decreasing inwards) this means that all modes with frequency \( \omega > 0 \) can be found, but only modes of negative frequency \( < -2 Ro \epsilon/(1 - 2\epsilon) \) can be calculated this way.

There probably are additional solutions to our equations in the domain where \( Uk - cor \) does change sign. They could be additional inertial modes, or they could be unstable modes. To find them requires a different solution technique that takes account of the singularity, which we have not attempted, given the algebraic complexity of equation (13). But, in any case, such additional solutions will not affect the conclusions we have drawn from the solutions we have found.

V. RESULTS

To limit the range of parameters, we assume a convection zone depth of \( 2 \times 10^{10} \) cm, corresponding to our \( \epsilon = 0.153 \), which we studied extensively in Paper II. We calculate frequency-axial wave number diagrams for the adiabatic stratification case (the parameter \( B = 0 \)), and a "solar" superadiabatic stratification case for which \( B = 10 \); in each of these cases we consider \( Ro = 0 \) (no differential rotation) and \( Ro = -0.35 \) (angular velocity decreasing with depth at a rate
consistent with the observed surface solar differential rotation, assuming the rotation rate below the surface is constant on cylinders concentric with the rotation axis). All cases are computed for a range of (integer) longitudinal wave numbers $k$ up through $k = 10$.

**a) Effects of a Superadiabatic Gradient**

Figure 1 displays six frequency wave number diagrams for the case with no differential rotation, with adiabatic gradient (upper three panels), and superadiabatic gradient (lower three panels), for representative longitudinal wavenumbers $k = 0, 4, 8$. As in Paper II, the curves with largest unsigned frequency magnitude represent the lowest radial order (zero interior nodes), the next curves the next lowest radial order (one interior node), etc.

The dominant difference between the adiabatic cases and the superadiabatic cases for all $k$ is that the frequency is significantly lowered for all radial orders, and the frequency spacing between orders is increased. The frequency lowering is produced by the fact that the buoyancy force in the superadiabatic case always pushes fluid elements further away from equilibrium, so it opposes the restoring Coriolis force that determines the frequency. The larger spacing between frequencies comes from the shrinking of the resonant cavity in which the inertial oscillation is confined. The shrinking occurs at the top, where the buoyancy force is sufficiently large to overpower the Coriolis force, so no neutral oscillation is possible there. Figure 2 shows the normalized amplitude of the lowest radial order eigenfunction for $k = 0, 8$, and $B = 0, 10$ with axial wavenumber $l = 30$. We see there is a region of zero amplitude near the top for both cases with $B = 10$. (In this region, solutions with complex frequency, corresponding to exponentially growing convective modes, would be present—we have excluded them from our solutions by keeping $\omega$ real.)

To understand the effect of shrinking the cavity, we need to note that for a given convection zone thickness, the frequency of oscillation is determined by the aspect ratio of the perturbation. The longer is the radial (or longitudinal) displacement compared to the axial displacement, the higher the frequency (up to the dimensional limit $2Q$), since axial displacements are not acted upon by Coriolis forces. To illustrate how shrinking the cavity spreads the frequency of adjacent radial orders, consider two convection zones, one with half the depth of the other. In the shallow zone the lowest radial order will have roughly the same aspect ratio, for given $l$, as the next to lowest radial order in the deep zone—but the second to lowest radial order in the shallow layer would correspond to the fourth from lowest radial order in the deep layer. In other words, a frequency wave number diagram for the thin layer could be constructed approximately by erasing every other radial order from the diagram for the thicker layer—hence the greater spacing. This effect was also discussed in Paper II. It is a common property of oscillations in resonant cavities, such as musical instruments.

A few other features on Figure 1 are worth noting. In particular, above about $l = 10$, the frequencies for corresponding radial orders for $k = 0$ and 4 differ by only a small fraction of the spacing between radial orders; $k = 8$ shows somewhat more variation from $k = 0$. The close coincidence of frequencies for low $k$ occurs because all fluid displacements in...
longitude that are long compared to the axial or radial displacement tend to conserve angular momentum. Longitudinal pressure torques are too weak to matter. Thus if there were little differential rotation with depth present in the convection zone of the Sun, the power in frequency wavenumber space for modes of different but low longitudinal wavenumber should nearly coincide for successive radial orders, presumably increasing the likelihood of the detection of such ridges of power.

There is also an asymmetry between positive and negative frequencies for \( k \neq 0 \) at low \( l \), with the corresponding positive frequencies being higher, and actually having a value for the limit of \( l = 0 \) when \( B = 0 \). These modes have been previously studied in Glatzmaier and Gilman (1981) and take the form of so-called potential vorticity waves, in which a fluid element conserves the absolute vorticity divided by the fluid density. Since density decreases outward, these waves always move in a prograde direction relative to the basic rotation. We can understand this point as follows. Outward velocity leads to negative vorticity relative to the rotating frame because the fluid column expands in the lower density environment. But in a periodic pattern of velocity and vorticity, positive relative vorticity leads this outward velocity in phase (in the direction of rotation), and negative vorticity lags. Therefore the leading edge of the negative vorticity pattern is moved prograde, and the trailing edge of the positive vorticity pattern is eaten away, so the whole pattern moves prograde.

In these modes, there is a near balance between Coriolis forces and pressure gradients, leading to their low frequency, such that angular momentum is not conserved for a given fluid element as it moves. The frequency is larger for more rapid density variation with depth. The asymmetry is much smaller in the superadiabatically stratified case, because the region of most rapid density variation, near the top, is not felt by the perturbations because their amplitude vanishes there (see Fig. 2).

**b) Effects of Differential Rotation**

Figure 3 shows the effect of differential rotation on representative frequency wavenumber diagrams, for \( k = 0, 4, 8 \). We show only the superadiabatic (\( B = 10 \)) case since that is more realistic for a stellar convection zone. Comparing the plots for \( k = 0 \), we see that the frequency for each radial order is reduced in magnitude by the presence of an angular velocity decreasing inward. As explained in Paper II this is due to the lower average rotation rate in the convection zone compared to that of the reference frame that is set to the surface value \( (\Omega = 0 \textrm{ there}) \), resulting in lower than average Coriolis force. But more importantly, we see in Figure 3 for \( k > 0 \) an increasing shift of all the frequency curves toward more negative frequencies, together with the opening of a gap between \( \omega = 0 \) and a negative value. This value is given by \(-2 \mathcal{R} \kappa (1 - 2e)\), as described at the end of § V above. The gap represents the region in which the method of Frobenius fails. The shift toward...
more negative frequencies for increasing $k$ is simply a result of the differential rotation sweeping the mode patterns along in a direction counter to the basic rotation. Thus modes with positive $\omega$, that propagate faster than the basic rotation rate, are slowed down (lower positive frequency for the same longitudinal wave number), while modes with negative frequency are speeded up in their retrograde movement. The result of this is that the asymmetry in frequency shifts between positive and negative frequency modes of the same radial order is equivalent to a frequency displacement across several radial orders.

Figure 4 shows a plot of the frequency of lowest radial order for even longitudinal wave numbers with differential rotation (dashed curves) and without (solid curves). We see that without differential rotation the spread in frequencies between $k = 0$ and 10 is quite small. But with differential rotation, the spread is of the same order as the average frequency itself. Modes with higher longitudinal wavenumber $k$ suffer a greater frequency change because in a given time more wavelengths of the mode are swept past any fixed point. For wave number 10, a differential rotation of 10%-15% of the average rotation would be enough to reduce the frequency of the prograde mode nearly to zero and roughly double the frequency of the retrograde mode. This is clearly evident from Figure 4.

Consequently, if inertial oscillations in this frequency range can be detected at all, it should be possible to distinguish clearly between angular velocity independent of depth, and angular velocity constant on cylinders and matched to the observed surface profile, as well as finer variations in between.

VI. COMMENTS ON THE SOLAR PROBLEM

We have shown that the frequencies of inertial oscillations are quite sensitive to how superadiabatic the stratification of the convection zone is, and that the frequencies of non-axisymmetric inertial oscillations are very sensitive to the amount of differential rotation present. These two properties can in principle be exploited to determine superadiabaticity and differential rotation in the solar convection zone, provided inertial oscillations can actually be observed. (If they are observed, we would of course, want to use a spherical shell model, such as analyzed in Paper I, to compare with observations, rather than the cylindrical model we have used here.) We note that acoustic modes, being of much higher frequency, are relatively little affected by the small departures from the adiabatic gradient. To distinguish between rotation independent of depth and rotation constant on cylinders in the solar convection zone appears to require much longer acoustic oscillation records than have been used so far (Brown and Morrow 1987).

To be seen as distinct ridges of power in a frequency wave number diagram, inertial oscillations must (a) have significant amplitude at the solar surface and (b) have lifetimes of a few years. We have shown that the very superadiabatic top part of the convection zone at the solar surface severely attenuates the amplitude of the oscillation there. Thus, some additional mechanism must be invoked to allow the oscillation to "poke through" to the photosphere. Since the velocity pattern associated with the oscillation changes on a time scale of weeks, we can imagine that the much shorter time scale granulation and supergranulation could bring the pattern to the surface. Supergranulation patterns could rotate at different rates at different latitudes, reflecting an underlying inertial oscillation velocity pattern, or granulation and supergranulation could turbulent-ly diffuse the associated momentum to the surface. This latter effect should work best for those inertial oscillations whose latitudinal wavelength is not small compared to their depth below the photosphere.

However, global solar convection modes (Glatzmaier 1984, 1985; Gilman and Miller 1986) predict giant cell convection somewhat below the solar surface much larger than observed at the surface that derives approximately the correct amplitude and profile of latitudinal differential rotation, so the transmission of such patterns to the surface by small-scale convection appears to be not very effective (for unknown reasons).

The lifetime of inertial oscillations is intimately tied up with mechanisms for their excitation and dissipation. We presume that such modes are excited by the convection itself. Indeed, how do inertial modes differ from global convection? We see the convection part of the global motion spectrum as being that part least inhibited by rotation, and so most unstable due to the superadiabatic gradient. In our model, we could find these modes by setting $l = 0$ and looking for complex $\omega$. The convection would take the form of rolls with axis parallel to the axis of rotation. Then, we see the inertial oscillations part of the spectrum as being that part most inhibited by rotation—these modes are not convectively unstable, so they must be excited by nonlinear forcing from the convectively unstable modes.
The damping of these inertial oscillations must occur due to some combination of nonlinear cascading of energy to smaller scales, as well as feeding energy back into convective modes other than those that drive the oscillations. For the inertial oscillations to form distinct ridges of power in a wavenumber frequency diagram, then, these nonlinear interactions must be weak enough that mode lifetimes of a few years are possible. This could be a particularly serious limitation to finding such modes, as pointed out by T. Bogdan (private communication).

Fortunately, it is not necessary to design an observing program especially to see these inertial oscillations. An observing program that would be used to measure persistent global flows of any kind, such as planned in the GONG (Global Oscillations Network Group) program, should allow such a search to be made as would measurements with the HAO-NSO Fourier tachometer at NSO-Tucson, once that instrument is capable of stable Doppler shift measurements of persistent velocities.

I wish to acknowledge useful discussions with Thomas Bogdan, Douglas Gough, and Timothy Brown on various aspects of this problem. Jack Miller programmed the computations reported here.

APPENDIX

The coefficients in the polynomial expressions in equation (14) are expressed in terms of parameters defined in the text as follows:

\[ c_2 = (\gamma - 1)k^2\left(1 - \frac{Rk}{\omega}\right)^2, \]
\[ d_2 = \left(\frac{1}{\beta} + 1 - \gamma\right)k^2\left(1 - \frac{Rk}{\omega}\right)^2, \]
\[ e_2 = (\gamma - 1)\left(1 - \frac{Rk}{\omega}\right)\left[2k^2 \frac{Rk}{\omega} + l^2\left(1 - \frac{Rk}{\omega}\right)\right], \]
\[ f_2 = (\frac{1}{\beta} + 1 - \gamma)\left(1 - \frac{Rk}{\omega}\right)\left[2k^2 \frac{Rk}{\omega} + l^2\left(1 - \frac{Rk}{\omega}\right)\right], \]
\[ g_2 = (\gamma - 1)\frac{Rk}{\omega}\left[2l^2\left(1 - \frac{Rk}{\omega}\right) + k^2\left(\frac{Rk}{\omega}\right)\right], \]
\[ h_2 = (\frac{1}{\beta} + 1 - \gamma)\frac{Rk}{\omega}\left[2l^2\left(1 - \frac{Rk}{\omega}\right) + k^2\left(\frac{Rk}{\omega}\right)\right], \]
\[ i_2 = (\gamma - 1)l^2\left(\frac{Rk}{\omega}\right)^2, \]
\[ j_2 = (\frac{1}{\beta} + 1 - \gamma)l^2\left(\frac{Rk}{\omega}\right)^2, \]
\[ b_1 = \gamma k^2\left(1 - \frac{Rk}{\omega}\right)^2, \]
\[ c_1 = (\frac{1}{\beta} + 1 - \gamma)\left(1 - \frac{Rk}{\omega}\right)^2, \]
\[ d_1 = \left(1 - \frac{Rk}{\omega}\right)\left[2\gamma Rk^3\omega + (2 - \gamma)l^2\left(1 - \frac{Rk}{\omega}\right)\right], \]
\[ e_1 = (\frac{1}{\beta} + 1 - \gamma)\left(1 - \frac{Rk}{\omega}\right)\left[2Rk^3\omega + l^2\left(1 - \frac{Rk}{\omega}\right)\right], \]
\[ f_1 = \frac{Rk}{\omega}\left[2(2 - \gamma)l^2\left(1 - \frac{Rk}{\omega}\right) + \gamma Rk^3\omega\right], \]
\[ g_1 = 2(1 - \frac{Rk}{\omega})\frac{Rk}{\omega}\left(\frac{1}{\beta} + 1 - \gamma\right)l^2, \]
\[ h_1 = (2 - \gamma)l^2\left(\frac{Rk}{\omega}\right)^2, \]
The coefficients $R_r$, $P_r$, $Q_r$ in the power series equations (16) are defined in terms of the above expressions as follows:

- $R_0 = c_2(1 - \epsilon)^2 + d_2(1 - \epsilon)^3 + e_2(1 - \epsilon)^4 + f_2(1 - \epsilon)^5 + g_2(1 - \epsilon)^6 + h_2(1 - \epsilon)^7 + i_2(1 - \epsilon)^8 + j_2(1 - \epsilon)^9$
- $R_r = \epsilon^2[c_2 + 3d_2(1 - \epsilon)^2 + 4e_2(1 - \epsilon)^3 + 5f_2(1 - \epsilon)^4 + 6g_2(1 - \epsilon)^5 + 7h_2(1 - \epsilon)^6 + 8i_2(1 - \epsilon)^7 + 9j_2(1 - \epsilon)^8]$
- $R_3 = \epsilon^3[d_2 + 4e_2(1 - \epsilon)^2 + 10f_2(1 - \epsilon)^3 + 20g_2(1 - \epsilon)^4 + 35h_2(1 - \epsilon)^5 + 56i_2(1 - \epsilon)^6 + 84j_2(1 - \epsilon)^7]$
- $R_4 = \epsilon^4[e_2 + 5f_2(1 - \epsilon) + 15g_2(1 - \epsilon)^2 + 35h_2(1 - \epsilon)^3 + 56i_2(1 - \epsilon)^4 + 84j_2(1 - \epsilon)^5 + 126j_2(1 - \epsilon)^6]$

where $\epsilon$ is a small parameter and $\omega$ is the angular frequency. The coefficients $i_0, j_0, k_0, l_0, m_0, n_0$ are defined in terms of $i, j, k, l, m, n$ as follows:

- $i_0 = \left(\frac{\gamma - 1}{\omega^2}\right)^2 [i^4 R^2(8 - k^2) - k^4 B]$
- $j_0 = \left(\frac{\gamma - 1}{\omega^2}\right)^2 [j^4 R^2(8 - k^2) - k^4 B]$
- $k_0 = -2\left(\frac{\gamma - 1}{\omega^2}\right)^2 k^2 l^2 B$
- $l_0 = -2\left(\frac{\gamma - 1}{\omega^2}\right)^2 k^2 l^2 B$
- $m_0 = \left(\frac{\gamma - 1}{\omega^2}\right)^2 [m^4 R^2(8 - k^2) - k^4 B]$
- $n_0 = \left(\frac{\gamma - 1}{\omega^2}\right)^2 [n^4 R^2(8 - k^2) - k^4 B]$

The coefficients $R_r, P_r, Q_r$ in the power series equations (16) are defined in terms of the above expressions as follows:

The coefficients $R_r, P_r, Q_r$ in the power series equations (16) are defined in terms of the above expressions as follows:
\[
R_5 = e^6[f_2 + 6g_2(1 - e) + 21h_2(1 - e)^2 + 56i_2(1 - e)^3 + 126j_2(1 - e)^4],
\]
\[
R_6 = e^6[g_2 + 7h_2(1 - e) + 28i_2(1 - e)^2 + 84j_2(1 - e)^3],
\]
\[
R_7 = e^7[h_2 + 8i_2(1 - e) + 36j_2(1 - e)^3],
\]
\[
R_8 = e^8[i_2 + 9j_2(1 - e)],
\]
\[
R_9 = j_2 e^0.
\]

\[
P_1 = e^6[b_1(1 - e) + c_1(1 - e)^2 + d_1(1 - e)^3 + e_1(1 - e)^4 + f_1(1 - e)^5 + g_1(1 - e)^6 + h_1(1 - e)^7 + i_1(1 - e)^8],
\]
\[
P_2 = e^8[b_1 + 2c_1(1 - e) + 3d_1(1 - e)^2 + 4e_1(1 - e)^3 + 5f_1(1 - e)^4 + 6g_1(1 - e)^5 + 7h_1(1 - e)^6 + 8i_1(1 - e)^7],
\]
\[
P_3 = e^8[c_1 + 3d_1(1 - e) + 6e_1(1 - e)^2 + 10f_1(1 - e)^3 + 15g_1(1 - e)^4 + 21h_1(1 - e)^5 + 28i_1(1 - e)^6],
\]
\[
P_4 = e^6[d_1 + 4e_1(1 - e) + 10f_1(1 - e)^2 + 20g_1(1 - e)^3 + 35h_1(1 - e)^4 + 56i_1(1 - e)^5],
\]
\[
P_5 = e^6[f_1 + 5g_1(1 - e)^2 + 15h_1(1 - e)^3 + 35i_1(1 - e)^4],
\]
\[
P_6 = e^8[g_1 + 7h_1(1 - e)^2 + 28i_1(1 - e)^3],
\]
\[
P_7 = e^8[h_1 + 8i_1(1 - e)],
\]
\[
P_8 = e^{10}[i_1 + 9j_1(1 - e)],
\]
\[
P_9 = i_1 e^0.
\]

The limiting forms of all the coefficients given above, when we separately take \(l \to 0, k \to 0, U \to 0\) have been checked against the corresponding formulas derived previously assuming, respectively, \(l = 0, k = 0,\) and \(U = 0\) from the outset.

\section*{REFERENCES}


