HEATING OF THE SOLAR CORONA BY THE RESONANT ABSORPTION OF ALFVÉN WAVES

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Received 1986 June 26; accepted 1986 November 11

ABSTRACT

In this paper an improved method for calculating the resonance absorption heating rate is discussed and the results are compared with observations in the solar corona. To accomplish this, the wave equation for a dissipative, compressible plasma is derived from the linearized magnetohydrodynamic equations for a plasma with transverse Alfvén speed gradients. For parameters representative of the solar corona, it is found that a two-scale description of the wave motion is appropriate. The large-scale motion, which can be approximated as nearly ideal, has a scale which is on the order of the width of the loop. The small-scale wave, however, has a transverse scale much smaller than the width of the loop, with a width ~0.3–250 km, and is highly dissipative. These two wave motions are coupled in a narrow resonance region in the loop where the global wave frequency equals the local Alfvén wave frequency. Formally this coupling comes about from using the method of matched asymptotic expansions to match the inner and outer (small and large scale) solutions. The resultant heating rate can be calculated from either of these solutions. A formula derived using the outer (ideal) solution is presented, and shown to be consistent with observations of heating and line broadening in the solar corona.

Subject headings: hydromagnetics — plasmas — radiative transfer — Sun: corona

I. INTRODUCTION

Resonance absorption was proposed as a mechanism for the heating of fusion plasmas nearly 20 years ago. McPherson and Pridmore-Brown (1966) presented one of the early papers reporting experimental results which confirmed the basic concepts of the theory of low-frequency wave propagation and damping in an inhomogeneous plasma. Chen and Hasegawa (1974) showed that the heating rate in an inhomogeneous tokamak plasma can be calculated, under rather general assumptions, from the ideal MHD solution. This calculation demonstrated that the heating rate could in some regimes be independent of the detailed dissipation physics, a concept which has sparked considerable controversy in the solar physics literature (Lee 1980). In a series of papers Tataronis et al. on resistive modes was subsequently extended to include the effects of viscous dissipation in the plasma (Mok and Einaudi 1985; Einaudi and Mok 1985; Bertin, Einaudi, and Pegoraro 1986). Ionson (1978, 1982, 1983) pointed out, using a heuristic “LRC equivalent circuit” approach, the potentially important role of density and magnetic field gradients for the dissipation of energy in the solar corona. The qualitative features of Ionson’s calculation were essentially confirmed by Hollweg (1984) using a similarly heuristic “dissipation length” formalism. Ionson and Hollweg both concluded that resonance absorption could explain the observed heating in the solar corona.

The purpose of this paper is to calculate the heating rate for a coronal magnetic loop using the more rigorous two-scale Fourier transform technique reported in the fusion literature by the authors referenced above. The results of this calculation demonstrate that, to the accuracy of the observations, resonance absorption is consistent with the observed heating rate necessary to explain the soft X-ray emission from active region loops, and therefore must be considered a viable mechanism for the heating of the coronae of the Sun and other late-type stars.

The fundamental concepts of this coronal heating theory can be summarized as follows. The corona is composed of discrete magnetic loops which act as high-quality resonance cavities for hydromagnetic waves. Each loop, driven by turbulent motions at its base, has a large-scale frequency response sharply peaked at the global resonant frequency of the loop. These large-scale motions are very nearly dissipationless, and therefore do not significantly heat the corona directly. However, they can resonantly excite small-scale waves in the loop whenever the global wave frequency equals the local Alfvén wave frequency through the process of resonance absorption. Dissipation of these small-scale waves then provides the energy necessary to heat the coronal plasma. The essential feature of the theory is the existence of the global mode and its subsequent coupling to the small-scale dissipative waves. Since the bandwidth of the large-scale oscillation is narrow, the loop extracts power from the stochastic driving spectrum only in a narrow band of frequencies centered on \( \omega_0 \approx 2\pi v_c(x)/L \), where \( v_c \) is the local Alfvén speed and \( L \) is the length of the coronal loop. The loop can then be treated as a driven high-quality resonator, and the dissipation occurs in a very narrow, single-resonance layer where the local Alfvén speed is equal to the single frequency of the large-scale ideal MHD oscillation. The heating...
The viscosity coefficients, $\eta$, and the electrical conductivity, $\sigma$, are dissipation coefficients with magnitudes given in Braginskii (1965). The specific forms for the dissipation terms are not as given by Braginskii, however. By neglecting derivatives of the dissipation coefficients in equations (1) and (2), it is implicitly assumed that dissipation processes are important only in a narrow layer. If this is the case, and it is easily verified a posteriori, then within this layer terms which are proportional to gradients of the dissipation coefficients can be neglected compared to terms proportional to gradients of the velocity amplitude. It is further assumed that $B_0 = B_0 \hat{z}$, and that all variations in $B_0$ and $\rho_0$ are transverse to the field and in the $x$-direction, so that $v_A = v_A(x)$ only (Fig. 1). The plasma is assumed to have $\beta_p \ll 1$. Because of this, the transverse gradients of $B_0$ and gradients in the thermal pressure can be neglected since they are proportional to $\beta_p$, and the total pressure $p$ can be approximated as $p = B_0 B_0 / 4\pi$.

Then equations (1) and (2) can be written as

$$\frac{\partial v}{\partial t} - \beta^2 \nabla^2 v - \gamma^2 \nabla (\nabla \cdot v) = -\frac{1}{\rho_0} \nabla p + \frac{B_0}{4\pi \rho_0} \frac{\partial B}{\partial z},$$

and

$$\frac{\partial B}{\partial t} - \alpha^2 \nabla^2 B = B_0 \frac{\partial v}{\partial z} - z B_0 (\nabla \cdot v).$$

These two equations can then be combined to obtain a single equation for the velocity in the following manner. Take the time derivative of the momentum equation and use the induc-
tion equation to eliminate all derivatives of \( B \). Since dissipation is assumed to be small, one can neglect terms which are proportional to products of the dissipation coefficients. The resulting equation describing wave motion and dissipation in the corona is

\[
\left( \frac{\partial}{\partial t} - \alpha^2 \frac{1}{\rho_0} \nabla^2 \rho_0 \right) \left[ \frac{\partial \mathbf{v}}{\partial t} - \beta^2 \nabla^2 - \gamma^2 \nabla (\mathbf{v} \cdot \mathbf{v}) \right] - v_x^2 \frac{\partial^2 v_x}{\partial z^2} = v_x^2 \nabla (\mathbf{v} \cdot \mathbf{v}_x). \tag{5}
\]

To describe the resonance absorption heating process in a low \( \beta_p \) plasma properly (\( \beta_p \) is the ratio of thermal to magnetic pressure), one cannot simply assume that the plasma is incompressible, i.e., \( \mathbf{v} \cdot \mathbf{v} = 0 \). In fact, the assumption of incompressibility artificially restricts the response of a plasma like the solar corona which is dominated by magnetic pressure. For example, consider the equation for \( v_x \),

\[
\frac{\partial^2 v_x}{\partial t^2} - \alpha^2 \frac{1}{\rho_0} \nabla^2 \rho_0 \frac{\partial v_x}{\partial t} - \beta^2 \nabla^2 \mathbf{v} - \gamma^2 \frac{\partial^2 v_x}{\partial z^2} = \gamma^2 \frac{\partial (\mathbf{v} \cdot \mathbf{v}_x)}{\partial z}. \tag{6}
\]

From this we can see that in a low \( \beta_p \) plasma, the \( z \)-component of velocity is coupled to the transverse wave only through the compressive viscous dissipation term. For most cases, including the solar corona, this coupling is extremely weak since it is proportional to the dissipation coefficient, \( \gamma^2 \). This means that in a low \( \beta_p \) plasma, the parallel component of wave motion is for all practical purposes independent of the transverse components. Formally, whenever compressible viscosity can be neglected (\( \gamma^2 \approx 0 \), or the plasma is nearly incompressible \( \mathbf{v} \cdot \mathbf{v} \approx 0 \), the component of velocity along the field is independent of the transverse velocity components. This is the proper approximation for the solar corona. Analyses which assume \( \mathbf{v} \cdot \mathbf{v} = 0 \) and a velocity of the form \( \mathbf{v} = (v_x, 0, v_z) \) such as Steinolfson (1984), Einaudi and Mok (1985), Mok and Einaudi (1985), Steinolfson et al. (1986), Lee and Roberts (1986), and others, do not properly describe the physics of wave propagation in the solar corona, because one is not free to arbitrarily choose \( v_y = 0 \) as these authors have done. As demonstrated below, the transverse components of the wave amplitude are strongly coupled in a low \( \beta_p \) plasma. The wave amplitude, \( v_x \), must be found by obtaining self-consistent solutions of the coupled equations for the \( x \) and \( y \) wave amplitudes. It is the parallel component, \( v_z \), which may be chosen arbitrarily. Fortunately, however, it turns out that the wave equation derived in the low \( \beta_p \) approximation, although physically different, has the same mathematical form as the one derived using the incompressible approximation with \( v = (v_x, 0, v_z) \). It is this similarity which allows much of the mathematical analysis contained in these earlier papers to be almost directly applied in the low \( \beta_p \) plasma case, although the physical interpretation of the results must be modified. This point was briefly mentioned in the paper by Lee and Roberts (1986). In this paper, it will be assumed that the coupling between the transverse and parallel velocity components is sufficiently weak that they may be considered as independent. This seems to be a reasonable approximation for the solar corona since dissipation is weak.

With this in mind, let us consider for the moment wave motion in a plasma in which dissipation is small. Equation (5) then reduces to

\[
\frac{\partial^2 v_x}{\partial t^2} - v_x^2 \frac{\partial^2 v_x}{\partial z^2} = v_x^2 (\nabla \cdot \mathbf{v}_x). \tag{7}
\]

Except for the term on the right-hand side, we see that this equation simply describes the propagation of fast mode and shear Alfvén waves of ideal MHD in a low \( \beta_p \) plasma. In a homogeneous, incompressible plasma \( (\mathbf{v} \cdot \mathbf{v} = 0) \) these modes propagate independently. In a compressible plasma \( (\mathbf{v} \cdot \mathbf{v} \neq 0) \) with a gradient in the Alfvén speed of the background medium, the term on the right acts to couple these normally independent modes, so that the "normal modes" of the coupled system are combinations of these two basic wave motions. If one arbitrarily assumes \( \mathbf{v} \cdot \mathbf{v} = 0 \), the physics of the coupling between the transverse wave components is missed entirely. Wentzel (1979) reached the same conclusion regarding compressibility using an entirely different, although apparently equivalent argument.

If the Alfvén speed varies discontinuously, the coupling between the fast and shear modes result in the propagation of surface waves along the interface. To show this, consider a coronal loop driven at \( z = 0 \) with a perfectly reflecting boundary at \( z = L \). Fourier transform equation (5) using

\[
v_x(x, y, z, t) = \frac{2}{L} \sum_{n=0}^{\infty} \cos (k_n z) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} v_x(x, y, \omega). \tag{8}
\]

This form automatically satisfies the boundary condition at \( z = L \) as long as \( k_n = (2n + 1)\pi/(2L) \), with \( n = 0, 1, 2, \ldots \). Define the following dimensionless parameters: \( V_x^2(x) = v_x^2(x)/v_{10}^2 \), \( R_{||} = \beta_p^2/\alpha^2 \), and \( \gamma_{||} = \gamma^2/\alpha^2 \), where \( v_{10} \) is some typical value of the Alfvén speed within the loop, \( R \) is the magnetic Reynolds number, and \( \gamma_{||} \) and \( \gamma_p \) the magnetic Prandtl numbers associated with shear and compressional viscosity. There has also been some controversy surrounding the assumption of perfect reflectivity (e.g., Hollweg 1984). It should be emphasized that this is an approximation only. Whenever the leakage time due to imperfect reflection is much longer than the resulting dissipation time, it does not lead to significant error. In this paper we shall consider only this case. With these definitions, all lengths measured in units of \( d \) the transverse scale of the loop, and \( \kappa^2(x) = \omega^2 d^2/v_{10}^2 - k_n^2 \), the equation for the velocity can be written in the dimensionless form

\[
V^2(V \cdot v_{||}) + \kappa^2 v_{||} = \frac{i}{R V^2(x)} \left[ \frac{L}{\rho_0} \nabla v_{||} + \gamma_{||} \nabla^2 + \gamma^2 V (V \cdot v_{||}) \right]. \tag{9}
\]

Combine the coupled equations for \( v_x \) and \( v_y \), ignoring the dissipative terms on the right, to obtain a single equation for the wave amplitude \( v_x \),

\[
d \frac{\kappa^2}{dx} \frac{dv_x}{dx} + \kappa^2 v_x = 0 . \tag{10}
\]

Solutions of the ideal equation have been discussed previously by Jonson (1978) and Wentzel (1979) for a low \( \beta_p \) plasma. These authors demonstrated that in an inhomogeneous plasma, this equation describes the propagation of MHD surface waves and guided wave modes. They showed that when the Alfvén speed varies discontinuously, this equation describes the propagation of surface waves along the interface, and that when the Alfvén speed varies continuously, energy can be transferred across the field and presumably dissipated in a thin resonance layer.
In a series of papers Roberts and coauthors (Rae and Roberts 1981; Lee and Roberts 1986) considered in detail solutions of an equation of the same form as equation (10), but derived from different physical approximations. These solutions basically confirmed the results of Tataronis et al., namely that the plasma response to an arbitrary excitation consists of a continuous spectrum of waves and a discrete spectrum, and that energy is transferred continuously between the large-scale, discrete spectrum waves and the small-scale continuous spectrum waves. The work of Roberts et al. did not include dissipation. Nevertheless, they did confirm the existence of the large-scale, global response of the loop. In a later paper, Rae and Roberts (1982) briefly discussed wave propagation in a low $\beta$ inhomogeneous plasma. Solutions of the ideal equation have also been discussed by Davila (1985) in the context of acceleration of the solar wind in coronal holes.

The terms of the right side of equation (9) are all due to dissipation and are therefore nonideal MHD terms. In the solar corona these terms are “small” since they are all multiplied by the inverse of the magnetic Reynolds number, $R$, which is on the order of $10^{14}$ for typical solar parameters. In spite of this “smallness” these terms are crucially important at $x = x_A$ where $k^2(x_A) = 0$. At this location the first term on the left can only be balanced by the dissipative terms on the right, even though they are small. Sufficiently far from $x_A$, in the outer region, the ideal MHD solution is an excellent approximation to the actual solution. This deal or outer solution is the so-called global or large-scale mode, because the transverse scale of this wave motion is generally on the order of the diameter of the loop itself. However, near $x_A$, in the inner region, the character of the solution changes (the equation changes from second to fourth order), and the assumption of ideal MHD is not valid. In this region one must have the dissipative, small-scale inner solution. This is a classic example of a singular perturbation, similar in many ways to boundary layer problems of ordinary hydrodynamics.

Heyvaerts and Priest (1983) considered the problem of wave propagation in the corona with dissipation. They found the small-scale inner solution discussed above, but not the global large-scale solution. The reason for this apparent contradiction can be easily understood if one considers in detail the assumptions of the Heyvaerts and Priest analysis. They assumed a velocity of the form $\mathbf{v} = [0, v_y, v_x]$, and an Alfvén speed $v_A = v_A(x)$ to obtain, using equation (5) above,

$$\frac{\partial^2 v_y}{\partial t^2} - \frac{\partial^2 v_y}{\partial x^2} = (x^2 + \beta^2) \frac{\partial^2 v_x}{\partial x^2}. \tag{11}$$

Note that this expression is the same as equation (11) in Heyvaerts and Priest (1983). Consider equation (11) in the limit of no dissipation. One can easily see that it does not describe the propagation of surface waves or any of the other wave modes guided by the inhomogeneities in the plasma. Since these modes are the necessary to describe the large-scale (ideal) global mode of the loop, and for determining the narrow band frequency response of the loop, the Heyvaerts and Priest formulation cannot describe this behavior. One might attempt to counter this argument by saying that, yes, the particular wave polarization discussed by Heyvaerts and Priest is not the most general possible motion; nevertheless, this wave mode should be available to the plasma, and for this motion there is no global mode. The reason that this is not correct is very simple: the form of the velocity assumed by Heyvaerts and Priest, $\mathbf{v} = (0, v_y, 0)$ is not likely to be excited in the solar corona.

To see this, consider the components of equation (5) in the limit of small dissipation:

$$\frac{d^2 v_y}{dx^2} + k^2 v_x = -ik_x v_y, \tag{12}$$

$$(k^2 - k_x^2)v_y = -ik_x \frac{dv_y}{dx}. \tag{13}$$

These equations describe the large-scale, ideal response of the loop. Notice that if $k_x = 0$, the $x$ and $y$ velocity components are decoupled, and therefore they can be determined independently, as assumed by Heyvaerts and Priest. However, to excite modes with $k_x \approx 0$ the driver must have a correlation length, $L_y$, in the $y$-direction (i.e., in a direction transverse to the Alfvén speed gradient) which is much longer than the gradient scale of the loop, $d$. This condition is not likely to be satisfied because for heating one is considering magnetic loops which are roughly cylindrical, so that the typical transverse coherence length over which waves are excited by the driver is of order $2n_d$. Using this, one can estimate $k_x \approx 1$ in the normalized units of this paper. Therefore, the $x$ and $y$ velocity components are relatively strongly coupled together. Because of this, even if one imposes the (artificial) condition that the driving motion is confined to only the $y$-direction at the source surface (at say $z = 0$ in the loop), since $v_y \neq 0$, equation (12) shows that motion in the $x$-direction will be generated throughout the entire loop. So that over most of the volume $v_x \approx v_y \neq 0$, and the Heyvaerts and Priest assumption for the form of the velocity cannot be satisfied.

Physically, if one neglects compressibility, the plasma can only communicate across the field on the dissipation length scale, and so only the continuous spectrum, small-scale dissipative modes were found. The global modes, similar to surface waves, were completely, and artificially excluded in the Heyvaerts and Priest solutions. This lead Heyvaerts and Priest to conclude that there should be a continuum of resonance layers across the coronal loop. This is not the case when one accounts for the compressibility of the plasma.

In the following paragraphs singular perturbation theory and the method of matched asymptotic expansions will be used to obtain a solution which is valid both outside and inside the resonance layer. To do this one must first reduce the coupled equations for the vector components of the velocity to a single equation for one velocity component, say $v_y$.

III. SOLUTION OF THE OUTER EQUATION

To illustrate the basic idea of the resonance absorption layer and to demonstrate the matched asymptotic solution method, let us assume that the dominant dissipation mechanism is either shear viscosity or electrical resistivity. This assumption cannot be completely justified at this point, so it must be regarded simply as an Ansatz. An investigation which incorporates ohmic, compressive viscous and shear viscous dissipation self-consistently in a numerical solution is currently underway. The results of this investigation will be published when they become available. The wave equation can be written

$$\nabla \cdot (\mathbf{V} \cdot \mathbf{v}_\perp) + \kappa^2 \mathbf{v}_\perp = -i \epsilon \nabla^2 \mathbf{v}_\parallel, \tag{14}$$

where $\epsilon = (1 + \text{Pr}_x)/(R \lambda^2)$. These can be combined to obtain an equation for $v_{\parallel, n}$ correct to first order in $\epsilon$ for the appropriate...
ordering $1 \gg k^2 \gg x^2 \gg \epsilon(k_x^2 + k_z^2)$:
\[
\frac{d}{dx} \left( \frac{d^2 v_{x\infty}}{dx^2} - k^2 v_{x\infty} \right) = i e \frac{d^2 v_{x\infty}}{dx^2}.
\] (15)

The outer solution is obtained by expanding the velocity, $v_{x\infty}$, as a power series in a small parameter $\epsilon$. The lowest order term must then satisfy the ideal MHD equation,
\[
\frac{d}{dx} \left( \frac{d^2 v_{x\infty}^{(0)}}{dx^2} - k^2 v_{x\infty}^{(0)} \right) = 0.
\] (16)

Detailed solutions of this equation have been obtained before (e.g., Tatarko and Grossmann 1973). They have found that for appropriate choices of the parameters, the steady state solution of this equation can be represented by oscillations at a single (or at most a few) discrete frequencies. When the width of the loop, $d$, is small compared to the wavelength of the disturbance, this frequency approaches the surface wave frequency obtained for a discontinuous interface.

For our purposes it is only necessary to obtain the solution near the resonance layer, i.e., where $x \rightarrow x_a$, with $x_a$ defined by the resonance condition $k^2(x_a) = 0$. Expand $k^2$ in a Taylor series around $x_a$ and keep only the linear term. In this region, the first term dominates, and the solution can be approximated as
\[
v_{x\infty}^{(0)} \approx A \ln (x - x_a),
\] (17)

where $A$ is the wave amplitude determined by matching boundary conditions at the driver ($z = 0$). It should be emphasized that the solution obtained in equation (17) is valid near the resonance layer where $k^2 \propto x - x_a$; no matter what the $x$-dependence of $k^2$ is elsewhere. If $k^2$ is linear everywhere inside the loop and constant outside, the solution can be represented as a series of Bessel functions. These can be seen to reduce to equation (17) as $x \rightarrow x_a$.

IV. SOLUTION OF THE INNER EQUATION

The inner solution can be obtained by considering the scale stretching transformation given by $z = (x - x_a)/a$, where $a$ is the small-scale parameter to be determined in the problem. Using this transformation, and expanding $k^2$ near the resonance layer as
\[
k^2(x) = (x - x_a) \frac{d[k^2]}{dx} + \cdots
\]
\[
\approx (x - x_a) k_x^2 \left( \frac{1}{\rho_0} \frac{d\rho_0}{dx} \right) + \cdots,
\] (18)

equation (14) can be written as
\[
\frac{d}{dx} \left( \frac{d^2 v_{x\infty}}{dx^2} - i \frac{k_x^2}{k_y^2} \right) \frac{dv_{x\infty}}{dx} = -i k_y^2 a^2 \varepsilon v_{x\infty}.
\] (19)

The typical scale of the resonance layer, $a$, is given by $a^2 = \epsilon/\lambda$, where $\lambda = -k_y^2 \ln (\rho_0/d_x)$. This choice for $a$ is not an arbitrary one. To obtain a physical solution, one must choose the so-called distinguished limit (Nayfeh 1981, p. 270). This limiting process determines the dependence of the inner scale, $a$, on the dissipation parameter $\epsilon$.

To solve equation (19) assume a power-law expansion of $v_{x\infty}$ with $a^2$, assumed to be small, as the expansion parameter. Then the solution of the zeroth-order equation is
\[
v_{x\infty}^{(0)}(z) = C_1 - C_2 \int_0^z dp \exp \left( - \frac{p^2}{3} \right) \left[ 1 - \exp \left( \frac{-ip\lambda}{ip} \right) \right].
\] (20)

The integral portion of equation (20) is well known as a generalization of the Airy function (Olver 1974, p. 429), and its values have been tabulated by Nosova and Tumarkin (1965).

Equation (20) gives the velocity amplitude throughout the resonance layer; therefore, it only remains to show that this inner solution matches the outer solution obtained above. To this end, we will be interested in the inner solution: in the limit as $z \rightarrow \infty$ it can be shown that
\[
v_{x\infty}^{(0)}(z) \approx C_1 - i C_2 \ln z.
\] (21)

This shows that by proper choice of the constants, namely $C_1 = i A$, $C_2 = A \ln a$, the inner and outer solutions match, as is physically required. With these constants determined, the inner solution becomes
\[
v_{x\infty}^{(0)}(z) = A \left\{ \ln a + i \int_0^z dp \exp \left( - \frac{p^2}{3} \right) \left[ 1 - \exp \left( \frac{-ip\lambda}{ip} \right) \right] \right\}.
\] (22)

The inner solution is nonsingular. A graph of $v_x$ and $v_y$ within the resonance layer is shown in Figures 2 and 3. These were obtained by numerical integration of equation (22) using electrical conductivity $\sigma = 5 \times 10^8$, Alfvén speed $v_A = 2 \times 10^8$, and a wave period $P = 300$ s. Authors who have considered only the outer (ideal) solution have assumed that the solution is not singular; however, by explicitly obtaining the inner solution it has been demonstrated more directly. This means that even though the ideal solution is nonintegrable and therefore strictly speaking nonnormalizable, when the proper account is taken of the resonance layer, where the approach of ideal behavior breaks down, the complete solution is clearly integrable, and can therefore be easily normalized.

The maximum of the velocity occurs at $x = x_a$, as anticipated in earlier papers. For typical solar parameters, and resistive dissipation, the maximum in $v_x$ is $\sim 10$ times the wave amplitude outside the resonance layer. This value is, of course, a function of the dissipation mechanism, but not a strong one since $v_{x\infty} \approx \ln R^{1/3}$. The $y$-component of the wave amplitude can be much larger, however. To lowest order in $a$, $v_y$ is related to $v_x$ by
\[
v_{y\infty} = - \frac{1}{a} \frac{k_x}{k_y^2} \frac{dv_{x\infty}}{dx}.
\] (23)

For the parameter regime considered here, $k_x^2 > k_y^2$, this expression can be approximated as $v_{y\infty} \approx (k_x/a)^3$. To obtain this approximation we used the fact that the $(dv_{x\infty}/dx)_{\max} \approx 1$ (see Fig. 3). For resistive dissipation within the resonance layer, $a \approx 6.3 \times 10^5$ in units of the loop width, $d$, which means that $v_y$ is of order $10^5$ times larger than the velocity amplitude outside the resonance layer. Since by observation the amplitude in a typical active region loop is on the order of $0.01 v_A$, the amplitude inside the resonance layer would be much larger than the Alfvén speed. If the assumption of linear wave motion is violated within the resonance layer, one would conclude that before the wave amplitude within the resonance layer is large enough for linear resistive dissipation to be important, nonlinear wave effects could become important.

If instead of electrical resistivity and shear viscosity, one considered dissipation by compressive viscosity, the inner scale is changed drastically from $a = 6.3 \times 10^5$ to $a = 5 \times 10^7$, again in units of $d$. Combining this with a typical estimate of $k_x \approx 0.3$, one obtains the result that $v_{y\infty} \approx$ of order $60$ times the amplitude outside the resonance layer. Again using the fact that turbulence measurements indicate that $v_{rms} = 0.01 v_A$, one
would find that the amplitude of the wave within the resonance layer is marginally linear. It is clear from this discussion that detailed solutions incorporating compressive viscosity must be obtained before the question of linearity of the wave motion within the resonance layer can be addressed with any confidence.

Using the values for the inner scale \( a \) calculated above and \( d = 5 \times 10^8 \) cm one can estimate the thickness of the resonance layer as 0.3 km. This is, of course, far below the resolution of any solar instrument in the foreseeable future. This means that we should not expect to observe resonance layers in the near future if resistivity or shear viscosity are the dominant dissipation mechanisms. There is, however, another possibility. If, instead, one assumed that the dissipation rate is dominated by the largest viscosity coefficient available, compressible viscosity, one would obtain an estimate of the thickness of the resonance layer of \( a \approx 250 \) km. If, however, this estimate proves to be correct, these sheets could be observed with instruments with spatial resolution on the order of 0.1 if they have sufficient sensitivity (\( \int n^2 dl \approx 10^{26} \)). Detailed solutions incorporating compressible viscosity are currently underway, and results will be reported in a future paper.
V. ENERGETICS: CALCULATION OF THE HEATING RATE

Equation (22) gives the solution for velocity amplitude throughout the resonance layer. If one desired, this result could be used to calculate the volumetric heating rate throughout the resonance layer from the usual formulae (Braginskii 1965). It could then be integrated over the volume of the heated region to obtain the total heating rate of the coronal loop. In this paper we show that the heating rate is more conveniently, and more generally, calculated by integrating the Poynting flux over the surface of the resonance layer (Cheng and Hasegawa 1974). Since the “surface” of the resonance layer is, by definition, located in the region where both the inner and outer solutions are valid, this procedure allows one to calculate the heating rate without considering the inner solution at all. This emphasises the point, made by previous authors, that although in the steady state the details of the velocity profile within the resonance layer do depend on the dissipation mechanism (e.g., see eq. [22]), the heating rate does not. Therefore even if shear viscosity is not the dominant dissipation mechanism, the heating rate calculated here will still apply as long as the resonance layer assumptions are appropriate.

Consider the resonance layer geometry shown in Figure 4. Conservation of energy gives

\[ \frac{\partial u}{\partial t} + \nabla \cdot S = -h, \]  

(24)

where \( u \) is the energy density, \( S \) is the Poynting flux, and \( h \) is the steady state volumetric plasma heating rate. The energy balance implicity in equation (24) assumes that all electromagnetic energy which flows into the resonance layer eventually end up as thermal energy in the plasma. Integrate equation (23) over the volume of the resonance layer and use Gauss's theorem to convert the volume integral of \( \nabla \cdot S \) to a surface integral. The result is

\[ H = -A_s S_x(x) \bigg|_{x_1}^{x_2}, \]

\[ \text{where } A_s \text{ is the total surface area of the resonance layer and } H \text{ is the total heating rate for a single resonance layer within a coronal loop. It is worthwhile at this point to emphasize that equation (24) represents the integrated heating rate within the resonance layer, not the surface wave “damping rate” calculated previously (Jonson 1978; Mok and Einaudi 1985; Lee and Roberts 1986); it is, however, similar to the damping rate calculated by Bertin, Einaudi, and Pegoraro (1986). Use causality to obtain the analytic continuation of the outer solution (eq. [17]) across the resonance point (Kapraff and Tataronis 1977):

\[ \ln (x - x_A) = \begin{cases} \ln |x + x_A| & \text{for } x - x_A > 0, \\ \ln |x - x_A| - i\pi & \text{for } x - x_A < 0. \end{cases} \]  

(26)

Fig. 4.—Integration of the Poynting flux over the surface of the resonance layer. The layer has a characteristic thickness of order \( \approx 0.3 - 250 \text{ km.} \)
When equations (18) and (25) are substituted into equation (24), the resulting expression for the heating rate is

$$H = \frac{B_0^2}{2\pi \rho_0} \left[ \frac{1}{\rho_0(x_A)} \right] \pi |A|^2. \quad (27)$$

To determine whether this heating rate is consistent with observational constraints, let us simply equate it to the observed radiation rate in soft X-rays $[L_s = \Lambda(T)A]$, to obtain an expression for the rms velocity amplitude required to explain the observed emission:

$$V_{rms} = \left\{ \frac{\Lambda(T)2\pi \rho_0}{1} \right\}^{1/2} \left[ \pi (B_0^2/8\pi) \rho_0^{1/2} \rho_0^{1/2} \right]. \quad (28)$$

Using typical values of the parameters, $B_0 = 100 \mathrm{G}$, $d = 5 \times 10^8 \mathrm{cm}$, $\Lambda(T) = 10^{-22}$ (Rosner, Tucker, and Vaiana 1978), $P = 300 \mathrm{s}$, $k_y = 2\pi r$, $\rho_0^{1/2} \rho_0^{1/2} \approx -1$, and $\int n_e^2 \, dl = 10^{-28} - 10^{-29}$ (Webb et al. 1986), one obtains an estimate of $V_{rms} = -2 - 6 \mathrm{~km\,s^{-1}}$. This is comparable to the observed value of $10 - 20 \mathrm{~km\,s^{-1}}$ regularly seen from observations of nonthermal line broadening in the corona (Cheng, Doschek, and Feldman 1979).

VI. POSSIBLE LIMITATIONS OF THE PRESENT ANALYSIS

The primary conclusion to be drawn from these calculations is that the level of the approximation adopted here, the observations of the heating rate and nonthermal line broadening in the solar corona are consistent with heating by the resonance absorption mechanism. This basic agreement is encouraging; however, several complications remain which, although not considered in the present analysis, still could have some influence on the efficiency and hence the viability of the resonance absorption process for the heating of stellar coronae. The purpose of this section is to discuss some of these and consider their likely effect on the heating rate calculated in this paper.

1. The plane symmetry assumed here is highly idealized. It has been shown in the plasma physics literature that for the tokamak problem introducing cylindrical symmetry has presented no new physics. Nevertheless, when considering the heating rate to an accuracy of, say, factors of 2–5, the geometry factors probably must be properly accounted for.

2. Observations of the turbulent power spectrum at the base of the corona are badly needed as input for the theory. These observations should be carried out in ions which are present at or above the transition region temperatures. It seems that EUV observations would be the most appropriate.

3. Other sources of dissipation must be considered. For although the heating rate is independent of the dissipation mechanism for any reasonable value of the coefficients, the amplitude of the velocity inside the resonance layer and the width of the layer both depend on the magnitude of the dissipation coefficient. High-resolution instruments, such as POE, may be able to observe velocities within the narrow resonance regions in the reasonably near future. Therefore, it is worthwhile to consider theoretically the observational consequences of various dissipation mechanisms now.

4. In the calculation presented here, shear viscosity and resistive dissipation were assumed to be the dominant energy-loss mechanisms. This is not necessarily the case for solar conditions. Compressive viscosity may actually be the dominant dissipation mechanism (Hollweg 1985). Detailed calculations are underway to clarify this point. I emphasize here once again that this does not affect the heating rate calculated above; it does, however, affect the details of the potentially observable velocity field within the resonance layer.

VII. CONCLUSION

The calculation presented here demonstrates that it may be possible to heat the corona by the resonant absorption of Alfvén waves, and, although additional work is needed, it is reasonable to conclude that resonance absorption is a viable mechanism for heating the corona of the Sun and other late-type stars.

The author would like to thank J. A. Johnson, G. D. Holman, and D. S. Spicer for many helpful suggestions during the course of this research. I would also like to thank an anonymous referee for his help in clarifying several important points in the manuscript. This work was supported in part by NASA grant RTOP 188-38-53.

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