Low Frequency Oscillations in Slowly Rotating Stars. I.
General Properties

by

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ABSTRACT

The case of general nonuniform rotation is considered. Some of the results, however, are applicable only to spherical rotation. Partial differential equations for adiabatic oscillations are reduced to a system of ordinary equations by means of a truncated spherical-harmonic expansion. Asymptotic solution valid in regions where angular velocity of rotation, \( \Omega \), is much smaller than the Brunt-Vaisala frequency, \( N \), are obtained and used in the discussion of mode properties and classification. These solutions are also employed in the numerical method of a quantitative treatment of the problem. It is pointed out that approximations used in previous studies of quasi-toroidal modes are not generally valid in whole stellar models, and this refers also to the case of the uniform rotation.

1. Introduction

We will investigate in this series the theoretical properties of oscillations in the frequency range where the inertial forces due to rotation play an important role. There is no clear evidence that such oscillations have ever been observed in stars. However, excitation of such low frequency modes was suggested as a possible explanation of variability in \( \beta \) CMa and other hot main sequence stars (Papaloizou and Pringle 1978; Ando 1981) in ZZ Cet stars (Kepler et al. 1981; Saio 1982) and the 160 min variability in the Sun (Ando 1981).

Four distinct types of low frequency oscillations have been discussed in the past

1. gravity — existing in nonrotating stars, but strongly affected by rotation,
2. convective modes that become oscillatory in rotating stars,
3. r- or quasi-toroidal modes which are trivial without rotation,

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(4) inertial modes which do not exist in nonrotating stars.

Effects of rotation on low-frequency gravity modes were investigated by Papaloizou and Pringle (1978), and Berthomieu et al. (1978). Although our main interest is in toroidal and inertial modes we received significant guidance from the latter paper. Toroidal modes have been studied by Papaloizou and Pringle (1978), Provost et al. (1981), Smeyers et al. (1981), Saio (1982), and by Berthomieu and Provost (1983). In the last two papers nonadiabatic effects were included. Smeyers et al. and Saio took into account perturbations caused by oscillations in the gravitational potential.

There are still, however, unsolved problems on more elementary level, which may cast some doubts on very existence of r-modes. They are connected with occurrence of region in stellar interiors where, even with very slow rotation rate, \( \Omega \), the ratio of \( \Omega/|N| \), where \( N \) is the Brunt-Vaisala frequency cannot be regarded as a small quantity. Moreover, all these investigations were restricted to the case of uniform rotation. We will show that the departure from uniformity causes qualitative changes in toroidal mode properties.

Present paper introduces basic formalism and discusses the problem of mode classification. Our main assumptions are: slow rotation, Cowling approximation and adiabaticity of oscillations. We are quite aware that the last assumption is not justified in application to real stars. We believe, however, that this should not invalidate our conclusions concerning dynamical properties of the modes.

In Section 2 we derive the basic set of ordinary differential equations describing the problem. The assumption \( \Omega \ll |N| \) results in a considerable simplification. An asymptotic solution valid in this case is derived in Section 3. Section 4 is devoted to the problems of r-modes. In Section 5 we present an algorithm for determining eigen-solutions for modes in whole low-frequency range of stellar oscillation spectra.

2. General Equations

Linearized equations for adiabatic oscillation of a rotating star neglecting perturbation of the gravitational potential may be written in the form

\[
Z + \frac{1}{\varrho} \nabla \rho' - \frac{\varrho'}{\varrho^2} \nabla \rho = 0, \tag{2.1}
\]

\[
\varrho' + \text{div}(\varrho \xi) = 0, \tag{2.2}
\]

\[
\frac{\varrho'}{\varrho} = \frac{1}{\Gamma} \frac{p'}{p} + \xi \cdot A, \tag{2.3}
\]

where \( \cdot \) denotes the Eulerian perturbation, \( \xi \) is the displacement vector, \( \varrho \) and \( p \) are the unperturbed density and pressure, \( \Gamma = (d \log p / d \log \varrho) \).
\[ Z = \frac{\partial v'}{\partial t} + (v' \cdot \nabla) v + (v \cdot \nabla) v', \quad (2.4) \]
\[ A = \frac{1}{\Gamma} V \ln p - V \ln \rho, \quad (2.5) \]

and
\[ v = \Omega(r, \theta) r \sin \theta e_\theta. \quad (2.6) \]

Since we consider oscillations about a steady and axisymmetric equilibrium we can assume the \( \exp [i(\omega t + m \phi)] \) dependence for perturbations. Then, following \( e.g. \) Cox (1980), we get
\[ Z + V \left( \frac{p'}{q} \right) - \frac{p'}{q} A - \frac{V p}{q} (\xi \cdot A) = 0, \quad (2.7) \]
\[ \text{div} \xi + \frac{1}{\Gamma} (\xi \cdot V \ln p) + \frac{p'}{p \Gamma} = 0, \quad (2.8) \]
\[ Z = -\psi^2 \xi + 2i\Omega \psi e_z \times \xi + (e, \sin^2 \theta + e, \sin \theta \cos \theta) r \xi \cdot V \Omega^2, \quad (2.9) \]
where \( \psi = \omega + m \Omega. \)

We now make an explicit use of the assumption of slow rotation. More specifically, we consider the case when \( \varepsilon = \frac{\Omega^2 r}{\alpha} \ll 1 \), but not necessarily \( \frac{\Omega^2}{|N|^2} = \frac{\varepsilon}{\alpha} \ll 1 \), where \( \Omega_0 \) is some typical value of the angular velocity of rotation, which will be subsequently used as a frequency unit, \( g \) is the local gravitational acceleration, and \( \alpha = r A_r \). For the purpose of the forthcoming derivation we assume additionally that \( \varepsilon |r| \ll 1 \) and \( V_g \varepsilon = \Omega^2 r^2 / \nu_s^2 \ll 1 \), where \( \nu_s \) is the adiabatic sound speed. This inequalities are valid everywhere in the sun but cannot be satisfied for instance near the surface in polytropic models. We use them to simplify the analysis but we will explain afterwards how the equations should be modified to be valid regardless of this assumption.

We will use four scalar unknowns \( a, b, c, \) and \( u \) defined by the following relations.
\[ u = \frac{p \epsilon^{\text{irot}}}{q \Omega_0^2 r}, \quad \xi = (e_r a + V H b + i e_r \times V H c) \epsilon^{\text{irot}}. \quad (2.10) \]

Keeping only leading terms in the coefficient at each variable we get from Eq. (2.8)
\[ r \frac{\partial a}{\partial r} + (2 - V_g) a + \varepsilon V_g u + V_H^2 b + m \nu_g \mu h_6 c = 0, \quad (2.11) \]
where \( V_g = gr / \nu_s^2, \mu = \cos \theta, h_6 \) will be given below.
The following equations are derived from Eq. (2.7) by considering its $r$-component, $\text{div}_H$, and $\text{curl}_r$, respectively

$$\left( \frac{N}{\Omega_0} \right)^2 - \omega^2 - h_1 + (1 - \mu^2) r \frac{\partial \Omega^2}{\partial r} + r \frac{\partial \mu}{\partial r} + (1 - \alpha) u \right) + \left( h_8 \mu \sin \vartheta \frac{\partial}{\partial \vartheta} + m h_8 \right) b + \left( h_2 \mu \sin \vartheta \frac{\partial}{\partial \vartheta} + m \mu h_8 \right) c = 0, \quad (2.12)$$

$$\left\{ h_8 \mu \sin \vartheta \frac{\partial}{\partial \vartheta} + \frac{\partial}{\partial \mu} \left[ (\mu^3 - \mu) h_8 \right] - mh_2 \right\} a + \nabla_H^2 u$$

$$+ \left\{ - \left[ \omega^2 + h_1 + \mu (1 - \mu^2) \frac{\partial \Omega^2}{\partial \mu} \right] V_H^2 + h_4 \mu \sin \vartheta \frac{\partial}{\partial \vartheta} - m \left( h_5 + 2 \mu \frac{\partial \Omega^2}{\partial \mu} \right) \right\} b$$

$$+ \left[ h_2 \mu V_H^2 \left( h_5 + 2 \mu \frac{\partial \Omega^2}{\partial \mu} \right) \sin \vartheta \frac{\partial}{\partial \vartheta} + m h_4 \mu \right] c = 0, \quad (2.13)$$

$$\left\{ h_2 \sin \vartheta \frac{\partial}{\partial \vartheta} + \left[ 2 h_2 - m h_8 + (1 - \mu^2) \left( \frac{\partial \mu}{\partial \vartheta} + \left( h_3 + \frac{m \partial \Omega^2}{\mu} \right) \frac{\partial}{\partial \mu} \right) \right] \right\} a$$

$$+ me \mu h_7 u + (h_2 \mu V_H^2 - h_5 \sin \vartheta \frac{\partial}{\partial \vartheta} + m^2 h_3 \mu) b$$

$$+ \left[ - (\omega^2 + h_1) V_H^2 + mh_3 \mu \sin \vartheta \frac{\partial}{\partial \vartheta} - mh_5 + \varepsilon m^2 \mu^2 h_6 h_7 \right] c = 0, \quad (2.14)$$

where \((N/\Omega_0)^2 = \alpha / \varepsilon\),

\[
\begin{align*}
 h_1 &= 2 m \omega \Omega + m^2 \Omega^2, \\
 h_2 &= 2 (\omega \Omega + m \Omega^2), \\
 h_3 &= 2 \frac{\omega + m \Omega \partial \Omega}{\mu} \\
 h_4 &= mh_3 + \frac{1 - 3 \mu^2}{\mu} \frac{\partial \Omega^2}{\partial \mu} + (1 - \mu^2) \frac{\partial^2 \Omega^2}{\partial \mu^2}, \\
 h_5 &= h_2 + h_3 \mu^2, \\
 h_6 &= - \frac{1}{\mu \partial \mu \Omega_0^2 r^2}, \\
 h_7 &= \frac{A_9 \mu}{\mu \sin \vartheta \Omega_0^2}, \\
 h_8 &= h_7 + \frac{\mu^2 - 1 \partial \Omega^2}{\mu \partial \mu}. \quad (2.15)
\end{align*}
\]

It should be noted that we did not make use in Eq. (2.12) of \((N/\Omega_0)^2 \sim \varepsilon^{-1}\) because \(\alpha\) vanishes at some points in stellar models and it is very small in the regions of efficient convection. In Eq. (2.14) we retain the term \(\sim \varepsilon\) in the coefficient at \(c\) because for quasi-toroidal modes it is essential for determination of their radial structure.

To proceed further we have to specify the \(\mu\)-dependence in the rotation law. Here we assume

$$\Omega = \sum_{k=0}^{k_T} \Omega_k^{(1)} \mu^{2k} \quad (2.16)$$
Thus

$$\Omega^2 = \sum_{k=0}^{2k_T} \Omega_k^{(2)} \mu^{2k},$$  \hspace{1cm} (2.17)

where \( \Omega_k^{(2)} = \sum_{j = \max(0,k-k_T)}^{k} \Omega_k^{(1)} \Omega_j^{(1)} \).

With these formulae it is easy to obtain corresponding expansions for the first five \( h \) elements. For the remaining ones we need to know the expansion coefficients for \( p \) and \( q \). From the equation of mechanical equilibrium we get for \( k \geq 1 \)

$$p_k = -\Omega_0^2 r^2 q \left( \frac{\Omega_k^{(2)}}{2k} + \chi_k \right),$$  \hspace{1cm} (2.18)

$$q_k = \frac{d\varrho}{dp} p_k + \varrho \varrho_k,$$  \hspace{1cm} (2.19)

where

$$\chi_k = \frac{1}{\Omega_0^2 r^2} \sum_{j=1}^{k} \phi_j T_{jk},$$

$$T_{jk} = (-1)^{j-k} \frac{(2j+2k)!}{2^j(j-k)!(j+k)!(2k)!},$$

$$Q_k = \Omega_k^{(2)} + \Omega_k^{(2)} \left( \frac{1}{k} - 1 \right) + \frac{1}{2k} \frac{d\Omega_{k-1}^{(2)}}{d \ln r}$$

and \( \phi_j \) is the coefficient in the Legendre polynomial expansion of the gravitational potential. It may be determined from the following equation

$$\frac{d}{dr} \left( r^2 \frac{d\phi_j}{dr} \right) - \left[ 2j(2j+1) + \frac{d\varrho}{dr} \frac{4\pi G}{gr} \right] \phi_j = 4\pi G \varrho \sum_{k=1}^{j} \tilde{T}_{jk}^{-1} \left( Q_k \varrho + \frac{\Omega_k^{(2)} \varrho}{2k} \frac{d\varrho}{dr} \right).$$  \hspace{1cm} (2.20)

Elements of matrix \( \tilde{T}^{-1} \) are given by the following formula

$$\tilde{T}_{jk}^{-1} = \frac{2^{2j}(4j+1)(k+j)!(2k)!}{(2k+2j+1)(k-j)!}.$$  \hspace{1cm} (2.21)

All unsubscripted quantities in Eqs. (2.18)-(2.20) are to be calculated from the unperturbed model.
The expansion coefficients for all \( h \)-elements may now be readily calculated to give

\[
\begin{align*}
    h_{1,k} &= 2m\Omega_k^{(1)} + m^2\Omega_k^{(2)}, \\
    h_{2,k} &= 2\omega\Omega_k^{(1)} + 2m\Omega_k^{(2)}, \\
    h_{3,k} &= (2k+2)(2\omega\Omega_k^{(1)} + m\Omega_k^{(2)}), \\
    h_{4,k} &= mh_{3,k} + 4(k+1)[(k+1)\Omega_k^{(2)} - k\Omega_k^{(2)}], \\
    h_{5,k} &= h_{2,k} + h_{3,k-1}, \\
    h_{6,k} &= \Omega_k^{(2)} + (2k+2)\chi_{k+1}, \\
    h_{7,k} &= h_{8,k} + (2k+2)\Omega_k^{(2)} - 2k\Omega_k^{(2)}, \\
    h_{8,k} &= r \frac{d\Omega_k^{(2)}}{dr} - \alpha h_{6,k}.
\end{align*}
\]

(2.22)

Let us consider now the case when \( \varepsilon \ll 1 \) but \( \varepsilon |\alpha| \) and \( \varepsilon V_\phi \) are not small quantities. This may happen even in the atmospheres of realistic stellar models because \( \alpha \) and \( V_\phi \) are of the order of \( 10^3 \) for main sequence stars and \( 10^6 \) for white dwarfs. In such cases we do not have \( |p_k|/p \ll 1 \) and \( |\varphi_k|/\varphi \ll 1 \). A perturbation approach is still valid provided that one uses a different independent variable, \( \tilde{r} \), defined by the following equation

\[
r = \tilde{r}(1 - \varepsilon \int h_6 \mu d\mu).
\]

Then all \( p_k \) coefficients become zero and \( \varphi_k/\varphi = \varepsilon \varphi_k \ll 1 \), unless the radial derivative of \( \Omega \) is very large. Eq. (2.20) which is needed to calculate \( h_6 \) remains valid because the problem occurs only in the outer layers playing a negligible role in the gravitational potential. One may see that there is no singularity in this equation at the surface of polytropic models. In Eqs. (2.11)-(2.14) in addition to replacing \( r \) with \( \tilde{r} \) one has to restore certain terms of the order of \( \varepsilon \alpha \) and \( \varepsilon V_\phi \). This is done by replacing \( a \) with

\[
\tilde{a} = a - \varepsilon h_6 \sin \theta \frac{\partial b}{\partial \theta}
\]

in the term \( aV_\phi \) of Eq. (2.11) and in term \( ax \) of Eqs. (2.13) and (2.14) where it occur in \( ah_8 \).

When the expansions of the coefficients are used in Eqs. (2.11)-(2.14) it may be seen that the solution may be looked for in the form of the following series

\[
a = \sum_{n=1}^{n_T} a_n(r) Y_{|m|+2n-2}^{m} (\theta, \varphi),
\]

\[
c = \sum_{n=1}^{n_T} c_n(r) Y_{|m|+2n-1}^{m} (\theta, \varphi),
\]

where \( s = 0 \) or \( 1 \).
Expansions of \( b \) and \( c \) are similar to that of \( a \). Note that we have assumed that the series may be truncated.

It is now convenient to introduce a vectorial notation for the unknown coefficients \( e.g. \ a = (a_1, a_2, \ldots) \). Then our equations take the following form

\[
\frac{r}{dr} + (2 - V_g) a + \varepsilon V_g u - \hat{A} \cdot b + \varepsilon V_g \hat{AC} \cdot c = 0, \tag{2.23}
\]

\[
\frac{d}{dr} (1 - \alpha) u + \left[ \left( \frac{N}{\Omega_0} \right)^2 - \hat{UA} \cdot \right] a + \hat{UB} \cdot b + \hat{UC} \cdot c = 0, \tag{2.24}
\]

\[
\hat{BB} \cdot b + \hat{BC} \cdot c = \hat{BA} \cdot a + \hat{A} \cdot u, \tag{2.25}
\]

\[
\hat{CB} \cdot b + \hat{CC} \cdot c = \hat{CA} \cdot a. \tag{2.26}
\]

To get Eq. (2.26) from Eq. (2.14) we eliminated \( u \) with the help of Eq. (2.25) neglecting all terms of the order of except the ones occurring in \( \hat{CC} \).

In order to obtain explicit forms of the matrix elements we have to evaluate the following angular integrals

\[
M_{knj}^a = \int Y_{(a,n)} \mu^{2k} Y_{(a,j)} \, d\mu \, d\varphi,
\]

\[
K_{knj}^{ac} = \int Y_{(a,n)} \mu^{2k+1} Y_{(c,j)} \, d\mu \, d\varphi,
\]

\[
L_{knj}^{ac} = \int Y_{(a,n)} \mu^{2k} \sin \vartheta \frac{\partial Y_{(c,j)}}{\partial \vartheta} \, d\mu \, d\varphi,
\]

\[
N_{knj}^a = \int Y_{(a,n)} \mu^{2k+1} \sin \vartheta \frac{\partial Y_{(a,j)}}{\partial \vartheta} \, d\mu \, d\varphi,
\]

where \( Y_{(a,n)} = Y_{lm} \) with \( l = l_n = |m| + 2n - 2 + s \), and \( Y_{(c,n)} = Y_{lm} \) with \( l = l_n = |m| + 2n - 1 - s \).

To this aim we use the following well-known formulae

\[
\mu Y_{lm}^m = J(l + 1) Y_{l+1}^m + J(l) Y_{l-1}^m,
\]

\[
\sin \vartheta \frac{\partial Y_{lm}^m}{\partial \vartheta} = l J(l + 1) Y_{l+1}^m - (l + 1) J(l) Y_{l-1}^m,
\]

where \( J(l) = \sqrt{\frac{L^2 - m^2}{4l^2 - 1}} \).

With their help we get a recurrent relation

\[
M_{k+1,nj}^a = J(l_n^a) J(l_j^a - 1) M_{kn,j-1}^a
\]

\[
+ [J^2(l_j^a) + J^2(l_j^a + 1)] M_{knj}^a + J(l_j^a + 2) J(l_j^a + 1) M_{kn,j+1}^a. \tag{2.28}
\]

The remaining integrals may be expressed in terms of \( M \) as follows

\[
K_{knj}^{ac} = J(l_j^a + 1) M_{kn,j+1-\pi}^a + J(l_j^a) M_{kn,j-\pi}^a. \tag{2.29}
\]
\[ L_{knj}^{ac} = l_j^a J(l_j^a) M_{kn,j+1}^{a} + (l_j^a + 1) J(l_j^a) M_{kn,j}^{a} \]
\[ N_{knj}^{a} = -(l_j^a + 1) J(l_j^a - 1) M_{kn,j-1}^{a} \]
\[ + [l_j^a J^2(l_j^a) - (l_j^a + 1) J(l_j^a)] M_{kn,j}^{a} + l_j^a J(l_j^a + 2) J(l_j^a + 1) M_{kn,j+1}^{a}. \]

For the integrals with the reversed order of \(a\) and \(c\) superscripts we have
\[ K_{knj}^{ca} = K_{knj}^{ac}, \]
\[ I_{knj}^{ca} = -I_{knj}^{ac} + 2kK_{knj}^{ac} - (2k + 2) K_{knj}^{ac}. \]

Note that we have \(M_{knj}^{a} = 0\) if \(|n - j| > k\).

It is now easy to obtain the following form of the matrix elements \(A_{nj} = j^a \delta_{nj}\), where \(j_j = (l_j^a + 1) l_j^a\).

\[ AC_{nj} = mK_{knj}^{ac} h_{6,k}, \]
\[ UA_{nj} = -\omega^2 \delta_{nj} + M_{knj}^{a} \left( \frac{d \Omega_{k}^{(2)}}{d \ln r} - h_{1,k} \right) - M_{k+1,nj}^{a} \frac{d \Omega_{k}^{(2)}}{d \ln r}, \]
\[ UB_{nj} = N_{knj}^{a} h_{8,k} + m M_{knj}^{a} h_{2,k}, \]
\[ UC_{nj} = I_{knj}^{ac} h_{2,k} + mK_{knj}^{ac} h_{8,k}, \]
\[ BA_{nj} = [(2k + 1) M_{knj}^{a} - (2k + 3) M_{k+1,nj}^{a} - N_{knj}^{a} h_{8,k} + m M_{knj}^{a} h_{2,k}, \]
\[ BB_{nj} = \omega^2 A_{nj} + j_j^a \left[ M_{knj}^{a} (h_{1,k} + 2k \Omega_{k}^{(2)}) - 2k M_{k+1,nj}^{a} \Omega_{k}^{(2)} \right] \]
\[ - m M_{knj}^{a} (h_{5,k} + 4m \Omega_{k}^{(2)}) + N_{knj}^{a} h_{4,k}, \]
\[ BC_{nj} = K_{knj}^{ac} (mh_{4,k} - j_j h_{2,k}) - I_{knj}^{ac} (h_{5,k} + 4m \Omega_{k}^{(2)}) \]
\[ CA_{nj} = K_{knj}^{ac} [mh_{8,k} + h_{3,k} - 2h_{2,k} - 2m \Omega_{k}^{(2)} - m (2k + 2) \Omega_{k+1}^{(2)}] \]
\[ - K_{k+1,nj}^{ac} h_{3,k} - I_{knj}^{ca} h_{2,k}, \]
\[ CB_{nj} = K_{knj}^{ac} (m^2 h_{3,k} - j_j h_{2,k}) - I_{knj}^{ca} h_{5,k}, \]
\[ CC_{nj} = \omega^2 j_j^a \delta_{nj} + M_{knj}^{a} (j_j^a h_{1,k} - mh_{5,k}) + mN_{knj}^{c} h_{3,k} \]
\[ + \varepsilon m^2 \sum_{q = j - 2k - s}^{j + 2k + 1} \sum_{q = 0}^{k} \sum_{k = 0}^{k} \sum_{s = 0}^{k} M_{k+1,nj}^{c} h_{6,q} h_{7,q-k}. \]

Implicit summation over \(k\) between 0 and \(2k_T\) is assumed, unless indicated otherwise.

All the matrices have \(n_T \times n_T\) dimension, but nonzero elements occur only in the following ranges
\[ |j - n| < 2k_T + 2 \text{ in } \hat{UA}, \hat{UB}, \hat{BA}, \hat{BB} \]
\[ n + s - 2 - 2k_T < j < n + s + 2k_T + 1 \text{ in } \hat{AC}, \hat{UC}, \hat{BC} \]
\[ n - s - 1 - 2k_T < j < n - s + 2k_T + 2 \text{ in } \hat{CA}, \hat{CB} \]
\[ |j - n| < 4k_T + 1 \text{ in } \hat{CC}. \]
An assumption of spherical rotation leads to a significant simplification. In particular, we have then \( h_3 \equiv h_4 \equiv 0, \ h_5 \equiv h_2, \ h_6 \equiv h_7 \). The only nonzero elements are now

\[
AC_{n,n+s-1} = mJ(l_n^a) h_6, \quad AC_{n,n+s} = mJ(l_n^a + 1) h_6,
\]

\[
UA_{n,n-1} = -J(l_n^a) J(l_n^a - 1) \frac{d\Omega^2}{d\ln r}, \quad UA_{n,n+1} = -J(l_n^a + 2) J(l_n^a + 1) \frac{d\Omega^2}{d\ln r},
\]

\[
UA_{n,n} = -\omega^2 - h_1 + [1 - J^2(l_n^a) - J^2(l_n^a + 1)] \frac{d\Omega^2}{d\ln r},
\]

\[
UB_{n,n-1} = (l_n^a - 2) J(l_n^a) J(l_n^a - 1) h_7, \quad UB_{n,n+1} = -(l_n^a + 3) J(l_n^a + 2) J(l_n^a + 1) h_7,
\]

\[
UB_{n,n} = mh_2 + [l_n^a J^2(l_n^a + 1) - (l_n^a + 1) J^2(l_n^a)] h_7,
\]

\[
UC_{n,n+s-1} = J(l_n^a) [m h_7 - (l_n^a - 1) h_2], \quad UC_{n,n+s} = J(l_n^a + 1) [m h_7 - (l_n^a + 2) h_2],
\]

\[
BA_{n,n-1} = -(l_n^a + 1) J(l_n^a) J(l_n^a - 1) h_7, \quad BA_{n,n+1} = l_n^a J(l_n^a + 2) J(l_n^a + 1) h_7,
\]

\[
BA_{nn} = [1 - (l_n^a + 3) J^2(l_n^a + 1) + (l_n^a - 2) J^2(l_n^a)] h_7 + mh_2, \quad (2.35)
\]

\[
BB_{nn} = (\omega^2 + h_1) l_n^a - mh_2,
\]

\[
BC_{n,n+s+1} = -(l_n^a + 1)(l_n^a - 1) J(l_n^a) h_2, \quad BC_{n,n+s} = (l_n^a + 2) l_n^a J(l_n^a + 1) h_2,
\]

\[
CA_{n,n-s} = J(l_n^c) [m h_7 - (l_n^c + 1) h_2], \quad CA_{n,n-s+1} = J(l_n^c + 1)(m h_7 + l_n^c h_2),
\]

\[
CB_{n,n-s} = -(l_n^c + 1)(l_n^c - 1) J(l_n^c) h_2, \quad CB_{n,n-s+1} = -(l_n^c + 2) l_n^c J(l_n^c + 1) h_2,
\]

\[
CC_{n,n-1} = \varepsilon m J(l_n^c) J(l_n^c - 1) h_7 \left(m h_6 - \frac{l_n^c - 2}{l_n^c - 1} h_2\right),
\]

\[
CC_{n,n+1} = \varepsilon m J(l_n^c + 2) J(l_n^c + 1) \left(m h_6 - \frac{l_n^c + 3}{l_n^c + 2} h_2\right),
\]

\[
CC_{n,n} = \lambda_n^c (\omega^2 + h_1) - mh_2 + \varepsilon m h_7 \left[J^2(l_n^c) \left(m h_6 - \frac{l_n^c + 1}{l_n^c} h_2\right) + J^2(l_n^c + 1) \left(m h_6 - \frac{l_n^c}{l_n^c + 1} h_2\right)\right].
\]

Since Eqs. (2.25) and (2.26) are algebraic they may be used to eliminate \( b \) and \( c \). We thus obtain from Eqs. (2.23) and (2.24)

\[
r \frac{da}{dr} + (2 - V_g) a + \widehat{AA} \cdot a + \widehat{H} \cdot u = 0, \quad (2.36)
\]

\[
r \frac{du}{dr} + (1 - \alpha) u + \frac{N^2}{\Omega_0^2} a + \widehat{UU} \cdot u = 0, \quad (2.37)
\]
where

\[
\dot{H} = \dot{A} \cdot \dot{B} \cdot \dot{A} - \epsilon V_g (\dot{E} + \dot{A} C \cdot \dot{C} \Gamma),
\]

\[
\dot{A} A = - \dot{A} \cdot \dot{B} + \epsilon V_g \dot{A} C \cdot \dot{C} 
\]

\[
\dot{G} = - \dot{U} A + \dot{U} B \cdot \dot{B} + \dot{U} C \cdot \dot{C},
\]

\[
\dot{U} U = \dot{U} B \cdot \dot{B} \cdot \dot{A} + \dot{U} C \cdot \dot{C} 
\]

\[
\dot{B} = (\ddot{B} B - \dot{B} C \cdot \ddot{C}^{-1} \cdot \dot{C} B)^{-1},
\]

\[
\dot{B} = \dot{B} L \cdot (\dot{B} A - \dot{B} C \cdot \ddot{C}^{-1} \cdot \dot{C} A),
\]

\[
\dot{C} \Gamma = -(\ddot{C}^{-1} \cdot \dot{C} B) \cdot (\ddot{B} \cdot \dot{A}),
\]

\[
\dot{C} C = \ddot{C}^{-1} \cdot \dot{C} A - \ddot{C}^{-1} \cdot \dot{C} B \cdot \dot{B} 
\]

and \( \dot{E} \) is the unit matrix.

Eqs. (2.36)-(2.37) represent system of differential equations which together with appropriate boundary conditions define an eigenvalue problem for the frequency \( \omega \). In the case of uniform rotation our equations reduce to those obtained by Berthomieu et al. (1978) (their Eqs. A5-A8). However, in the subsequent analysis they made an unjustified simplification of their Eq. A8 corresponding to our Eq. (2.31) by neglecting terms resulting from the Coriolis force which may be done only in the limit \( |N| \gg \Omega \).

### 3. An Asymptotic Decomposition

We now consider a region where \( \Omega / |N| \ll 1 \). It is clear that the center must always be excluded from such region and, in practice, convection zones except of their outer most parts.

We introduce a small parameter \( \delta = \Omega_0 / |N_0| \), where \( N_0 \) is certain characteristic value of Brunt-Vaisala frequency, and new variables defined as follows

\[
x = \frac{\ln r}{\delta}, \quad a_H = \frac{\dot{\tilde{Q}}^{-1} \cdot a}{\delta}, \quad u_H = \dot{\tilde{Q}}^{-1} \cdot u, \quad (3.1)
\]

where \( \dot{\tilde{Q}} \) is the matrix diagonalizing \( \tilde{H} \), i.e.

\[
\dot{\tilde{Q}}^{-1} \cdot \tilde{H} \cdot \dot{\tilde{Q}} = \tilde{D} \quad (3.2)
\]

and \( \tilde{D} \) contains eigenvalues of \( \tilde{H} \) on its diagonal.

Then we get

\[
\frac{da_H}{dx} - \tilde{D} \cdot u_H + \delta (2 - V_g + \dot{\tilde{Q}}^{-1} \cdot \dot{\tilde{A}} \cdot \dot{\tilde{Q}}) a_H = 0, \quad (3.3)
\]

\[
\frac{du_H}{dx} - \left( \frac{N}{|N_0|} \right)^2 a_H + \delta (1 - \alpha \dot{\tilde{Q}}^{-1} \cdot \dot{\tilde{U}} \cdot \dot{\tilde{Q}}) u_H + \delta^2 (\dot{\tilde{Q}}^{-1} \cdot \dot{\tilde{G}} \cdot \dot{\tilde{Q}}) a_H = 0. \quad (3.4)
\]
These equations have now quasidiagonal form, and thus they can be asymptotically decomposed (see e.g. Feshchenko et al. 1967). In the lowest order approximation in terms of $\delta$ the system splits into independent second order equations. Each of them may be written in the form

$$\frac{da_{H,n}}{dx} - \lambda_n u_{H,n} = 0,$$  \hspace{2cm} (3.5)

$$\frac{du_{H,n}}{dx} + \left( \frac{N}{|N_0|} \right)^2 a_{H,n} = 0,$$  \hspace{2cm} (3.6)

where $\lambda_n = \lambda_{nn}$ are eigenvalues of matrix $\hat{H}$, and should not be confused with $\lambda^a$ and $\lambda^e$ used in section 2. It may be seen from Eqs. (2.38) and (2.34) that matrix $\hat{H}$ and therefore its eigenvalues depend on local values of $\Omega(r)$ but not on their radial derivatives, $\chi$, and parameters of the unperturbed model.

If $\lambda_n$ is sufficiently large in the sense that will be explained below an asymptotic solution of this equations can be obtained in the form

$$a_{H,n} = \frac{1}{\sqrt{\chi_n}} \left[ C_1 \exp(iS_n) + C_2 \exp(-iS_n) \right],$$  \hspace{2cm} (3.7)

$$u_{H,n} = i \frac{\sqrt{\chi_n}}{\lambda_n} \delta \left[ C_1 \exp(iS_n) - C_2 \exp(-iS_n) \right],$$  \hspace{2cm} (3.8)

where $\chi_n = \sqrt{\lambda_n (N/\Omega_0)^2}$, $S_n = \int \chi_n \frac{dr}{r}$, $c_1$ and $c_2$ are arbitrary constants.

The condition for validity of this solution can now be expressed as follows

$$|\chi_n| \gg \left| \frac{r}{\chi_n} \frac{d\chi_n}{dr} \right|.$$

Berthomieu et al. (1978) calculated eigenvalues of $\hat{H}$ as functions of $\Omega/\psi$. Let us recall that $\psi = \omega + m\Omega$. As it follows from the foregoing discussion their results remain valid in the case of $\Omega$ dependent on $r$. It is interesting to notice that $\lambda$'s are eigenvalues of the operator in Laplace's Tidal Equation (see e.g. Chapmen and Lindzen 1970) and in this context some of them were calculated long time ago.

We recalculated some branches of the eigenvalues for the purpose of present discussion. In Figs. 1a and 1b we show the dependence of $\lambda$ on $\Omega/\psi$ for $m = 0$ and 2, respectively. Qualitatively, the behavior for others $m \neq 0$ values is the same as for $m = 2$. In Figs. 2a and 2b we use $\tilde{\lambda} = \lambda \psi^2/\Omega^2$ as the ordinate which allows us see better behavior of $\lambda$ for small values of $\Omega/\psi$. Values of $\tilde{\lambda} \geq 1$ correspond to gravity waves. Berthomieu et al. (1978) thoroughly discussed this case.

For $|\Omega/\psi| > 1/2$ there are negative eigenvalues. Such eigenvalues describe inertial waves. If $m \neq 0$ there are branches of $\lambda$ crossing axis of $\Omega/\psi$ at
Fig. 1a. The eigenvalues of $H$ matrix as functions of the inverse of the oscillation frequency in the corotating system, $\psi = \omega + m\Omega$ for $m = 0$. Continuous and dashed lines correspond to $l+m$ even ($s = 0$) and to $l+m$ odd ($s = 1$), respectively.

Fig. 1b. The same as Fig. 1a, but for $m = 2$.

$l(l+1)/2m$ where $l = m, m+1, \ldots$. Such eigenvalues describe toroidal waves. When $\lambda \gg O(\varepsilon)$ the branch corresponds again to the gravity wave. It is clear that the asymptotic solution given by eq. (3.11) is not applicable in some vicinity of $\lambda = 0$. In the following section we discuss this case in some details.

These properties of the eigenvalues are relevant only to the case of $\Omega$ independent of $\vartheta$. In general, $\lambda$'s may be complex which, however, by itself does not invalidate the asymptotic solutions. A problem arises only if there are multiple eigenvalues. Feshchenko et al. (1967) describe a method of a partial decomposition suitable in such cases.

Since the condition $\Omega/|N| \ll 1$ is not fulfilled in the whole stellar interior we
cannot expect, even in the case of uniform rotation, that individual \( \tilde{\lambda} \)-values fully describe a global oscillation mode. To calculate eigenfrequencies and eigenfunctions we have to solve numerically Eqs. (3.6) and (3.7) and combine it with the asymptotic solution in the region of its validity. We describe the method used by us in Section 5.

Here we propose the following definition of various types of modes

1. gravity-modes dominated in the sense of contribution to total energy by the solution corresponding to \( \lambda \gg O(\varepsilon) \)

2. inertial-modes dominated by solution corresponding to \( \lambda \leqslant 0 \) and the modes corresponding to \( \lambda = O(\varepsilon) \) which have been previously called quasi-toroidal or \( r \) modes. Thus, in this class we have all modes which owe their existence or become nontrivial due to the inertial forces.
4. Quasi-toroidal Modes

Assuming $\Omega \ll |N|$ it is possible to derive equations describing quasi-toroidal oscillations which are characterized by the following order-of-magnitude relations

$$a = O(\varepsilon) c, \quad b = O(\varepsilon) c.$$

Neglecting in Eq. (2.15) all terms of the order of $\varepsilon$ we get the following equation for $c$

$$\psi^2 \mathcal{V}_H^2 C - 2m\psi \sin \vartheta \frac{\partial \Omega}{\partial \mu} \frac{\partial c}{\partial \vartheta} + 2m\psi \left( \mu \frac{\partial \Omega}{\partial \mu} + \Omega \right) c = 0,$$

(4.1)

where $\psi^2 = \omega^2 + h_1 = (\omega + m\Omega)^2$.

There may exist unstable modes described by this equation if $\Omega$ depends on colatitude. We will discuss them in Paper III of this series. Here we will limit ourselves to case of $\Omega = \Omega(r)$. The case of uniform rotation was considered in the papers quoted in the Introduction. Our generalization follows the method developed by Provost et al. (1981). From Eq. (4.1) we get

$$c = c_n(r) Y_l^m, \quad \text{where } l = |m| + 2n - s + 1$$

and

$$\psi_n = \omega_n + m\Omega(r) = 2m\Omega(r)/\Lambda, \quad \text{where } \Lambda = l(l+1).$$

Just as in the case of uniform rotation the radial structure of the mode is undetermined at this stage. In order to find it we have to consider higher order terms in Eq. (2.5), where we use

$$\Omega(r) = \Omega(r_0) + \varepsilon_0 D(r),$$

$$\omega_r = -m\Omega(r_0) \left( 1 - \frac{2}{\Lambda} \right) + \varepsilon_0 \sigma_n,$$

with $\varepsilon_0 = \Omega_0^2 r_0/g(r_0)$ and $D(r_0) = 0$.

It is clear that in nonuniformly rotating stars such approach may be used only in some small vicinity of $r_0$, with the exception of the case $l = m = 1$.

Choosing $\Omega(r_0) = \Omega_0$ and using the last of Eqs. (2.26) we obtain

$$(l^2 - 1)J(l) b_{n-s} + (l+2)lJ(l+1) b_{n-s+1} + J(l) \left( m \frac{h_2}{h_2^2} - l - 1 \right) a_{n-s}$$

$$+ J(l+1) \left( m \frac{h_1}{h_2} + 1 \right) a_{n-s+1} - \frac{\varepsilon_0}{2} \Lambda \left( \sigma + mD \left( 1 - \frac{2}{\Lambda} \right) \right)$$

$$+ \frac{f h_1}{2} \left[ J^2(l) \left( mh_6 - \frac{l+1}{l} \right) + J^2(l+1) \left( mh_6 - \frac{l}{l+1} h_2 \right) \right] c = 0,$$

(4.2)

where $f = \varepsilon/\varepsilon_0 = g_0 r/gr_0$. 

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From Eqs. (2.23)-(2.25) with use of Eqs. (2.35) we derive the following expressions

\[ u_{n-s+k} = \frac{l+1-k}{l+k} J(l+k) y, \tag{4.3} \]

\[ a_{n-s+k} = \frac{\varepsilon_0 f}{\alpha} \left\{ \left( r \frac{d}{dr} + 1 - \alpha \right) u_{n-s+k} + J(l+k) \left[ \frac{h_7}{h_2} + (2k-1)(l+1-k) \right] y \right\}, \tag{4.4} \]

\[ b_{n-s+k} = \frac{1}{(l+2k)(l+2k-1)} \left\{ \left( r \frac{d}{dr} + 2 - V_g \right) a_{n-s+k} + \varepsilon_0 f V_g \left[ u_{n-s+k} + mJ(l+k) \frac{h_6}{h_2} y \right] \right\}, \tag{4.5} \]

where \( y = h_2 c, \) and \( k = 0, 1. \) Using Eqs. (4.4) and (4.5) we get after some algebra

\[ r^2 \frac{d^2 y}{dr^2} + r \frac{dy}{dr} \frac{d \ln (fr^4/\varepsilon_0)}{d \ln r} - \frac{\alpha}{\alpha f} \frac{d \ln (fr^4/\varepsilon_0)}{d \ln r} + \frac{A}{2} \gamma_1 \left( 1 - \frac{C}{A} \right) + \frac{m^2}{4A} (2\gamma_1 - \alpha h_6) + \frac{\alpha}{8mAf} \left[ \sigma + mD \left( 1 - \frac{2}{A} \right) \right] \]

\[ + \frac{BA}{4A} \left[ \frac{\alpha d(h_6 f)}{d \ln r} + \frac{A+2}{2} \gamma(\alpha h_6 - 2\gamma) - \gamma \gamma_1 + 2 \frac{d \ln \gamma}{d \ln r} \right] = 0 \tag{4.6} \]

where

\[ \gamma = \frac{d \ln \Omega}{d \ln r}, \quad \gamma_1 = 2V_g \frac{d \ln (fr^4/\varepsilon_0)}{d \ln r}, \]

\[ A = \left( \frac{l+1}{l} \right)^2 J^2(l) + \left( \frac{l}{l+1} \right)^2 J^2(l+1), \quad B = \frac{l+1}{l} J^2(l) + \frac{l}{l+1} J^2(l+1), \]

\[ C = \frac{l^2}{l+1} J^2(l+1) - \frac{(l+1)^2}{l} J^2(l). \]

This equation reduces to Eq. (11) of Provost et al. (1981) if we set \( \gamma = D = 0. \) In that paper as well as in the subsequent ones devoted to the same problem the equivalent equations were assumed to be valid within the whole stellar interior. However, Provost et al. (1981) found that in the polytropic model with \( \alpha \equiv 0 \) the equation admits only trivial solution \( \gamma \equiv 0. \) Saio (1982) noted that the solution for the radial components of the displacement becomes divergent at the points where \( \alpha = 0 \) for \( r \neq 0, \) unless \( l = m. \)
In fact a problem arises at \( r = 0 \) too because, as one may see from Eq. (4.4), \( a_{n-l} \rightarrow \infty \) except if \( J(l) = 0 \). This means that Eq. (4.6) alone cannot be used to describe quasi-toroidal modes corresponding to \( l > m \) in the whole model.

The outer boundary condition may be derived from Eq. (4.6) for any \( l > m \). Assuming that in an isothermal atmosphere \( V_g \gg 1 \) we get for the solution that is not exponentially increasing

\[
\frac{dy}{dr} = 0.
\]

The same boundary condition applies to the case \( \Omega r/v_s \gg 1 \) provided that \( r \) is replaced by \( \tilde{r} \) defined in section 2.

It may be seen from Eq. (4.6) that in the case of \( \Omega = \Omega(r) \) and \( l > 1 \) moving away from \( r_0 \) in the direction of decreasing \( \Omega/D < 0 \) implies rapid oscillations in \( c(r) \) which sooner or later invalidate the scaling. On the other hand in the region where \( D \) is large and positive the solution becomes asymptotically exponential which means that the modes do not propagate there. Furthermore, this implies that if there is a minimum in \( \Omega(r) \) curve then there are quasi-toroidal modes trapped in vicinity of such a minimum and they are fully described by Eq. (4.6).

Let us note that weak dependence of frequency on the radial structure implies that the group velocity for toroidal waves is very low. Therefore for a finite amount of time this waves can be always regarded as localized at \( r_0 \).

5. Numerical Method

In the situation when one expects rapid variability of the solution of differential equations occurring in certain interval it is useful to combine analytical solutions valid in the region of rapid variability with direct numerical solution outside it. This is the situation that we encounter in our case where it caused by variations in \( \Omega/N \) ratio within stellar models. The method we are going to describe is directly applicable only to the case of spherical rotation. We believe, however, that its essence remain generally valid.

It is convenient to apply a numerical integration scheme to the system written in the form of Eqs. (3.3) and (3.4) with \( \delta = \Omega_0/|N_0| \) set to 1, because this allows for an easy fitting the numerical solution to the asymptotic ones. At the center, however, the original system becomes decomposed, and one easily finds that for the convergent solution we have \( a_n \sim u_n \sim r^{l-1} \) for \( r \to 0 \). This together with Eq. (3.1) permits to derive the boundary condition for variables \( a_{l,n} \) and \( u_{l,n} \) in any desirable form.

A numerical integration is neccessary in some vicinity of the center, in the regions where \( x \approx 0 \) due to convection and in the outermost layers where on account of rapid variability in the parameters of the equilibrium model the
inequality given by Eq. (3.9) may not be satisfied. Solution for \( a_H \) and \( u_H \) components corresponding to \( \lambda \approx 0 \) may always be obtained only numerically. The fitting to the asymptotic solution corresponding to an imaginary \( \kappa_n \) consists in choosing an appropriate branch of the exponent and plays a role of a boundary condition. Thus, for stars with the radiative core the initial outward integration is needed only for the components corresponding to \( \lambda_n > 0 \) to determine phases in the trigonometric solutions.

To get the inward solutions we have to specify outer boundary conditions. This is the place where the adiabaticity assumption causes the greatest concern. The situation is better if the rotation is fast enough so that we may have \( V_g \varepsilon \gg 1 \) and \( V_g \gg \sqrt{\lambda_n \Omega^2 / N^2} \). In such cases all solution are evanescent and the eigenfrequencies are quite insensitive to the exact form of the boundary condition. Here, however, we are interested primarily in the opposite case and then we have a good boundary condition for the component corresponding to \( \lambda \approx 0 \) as discussed in the preceding section. In our calculation concerning the sun, which we will report in the next paper of this series, we specified the remaining conditions by choosing increasing inward branch of the exponential solutions in the superadiabatic zone if \( \lambda_n > 0 \) and in the atmosphere if \( \lambda_n < 0 \).

Each of inward and outward solutions involves \( n_r \) constants to be determined together with the eigenfrequency by \( 2n_r \) continuity conditions. It is convenient to impose them in the radiative zone close to the interface. An exact location of this place may be different for different components. The solution coming from the convective region is given numerically, while that from the radiative region is given by Eqs. (3.7) and (3.8) with specified \( C_1/C_2 \) ratio if \( \lambda_n > 0 \) and one of \( C \)'s set to zero if \( \lambda_n < 0 \).

6. Conclusions

Use of a truncated spherical-harmonic expansion appears to be the best method of solving the equations for low frequency oscillations in slowly rotating stars. We have derived the system of ordinary differential equations for the coefficients in this expansion valid in the case of nonuniform rotation. In the regions of stellar interior where \( |N| \gg \Omega \) there exist simple asymptotic solutions of this system. If there is no angular dependence in \( \Omega \) the solutions fall into the following three categories

(i) trigonometric if \( N^2 > 0 \) and exponential if \( N^2 < 0 \), describing internal gravity waves;

(ii) exponential if \( N^2 > 0 \) and trigonometric if \( N^2 < 0 \), describing inertial waves;

(iii) relatively slowly varying and nearly toroidal, describing Rossby waves; such solutions occur only very close to certain discrete frequencies and only for \( m \neq 0 \). Since the above strong inequality cannot be satisfied in the whole stellar
model a single eigenmode is, in general, described by a superposition of various solutions.

By inclusion of some higher order terms in the distortion we made the system applicable to calculation of the radial structure for quasi-toroidal modes. Such modes were previously studied only in uniformly rotating stars and were shown to be described by a single $Y_l^m$-dependence in the toroidal component of the displacement and $Y_l^{m_1}$-dependence in its poloidal component. We have argued that, in fact, for a uniform rotation a mode may be quasi-toroidal everywhere only if $l = m$. If $\Omega = \Omega(r)$ a mode may be quasi-toroidal only in some narrow zone, unless $l = m = 1$.

In the subsequent papers of this series we will use the formalism developed here to study inertial modes in the solar model and stability of the rotation law.

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