THE BOUNDARY VALUE PROBLEM OF THE SOLAR
FORCE-FREE MAGNETIC FIELD WITH CONSTANT $\alpha$ AND ITS
ANALYTICAL SOLUTION

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Abstract. In this paper we present a physical model which uses boundary conditions which seem to
 correspond more appropriately to actual situations. A boundary value problem of solar force-free magnetic
 field with constant $\alpha$ has been specified to represent the discretely concentrated characteristics of the
 longitudinal magnetic field on the photosphere. A unique analytical solution for the problem is obtained by
 a more strict method in mathematical physics. The most distinctive feature of our method is to make the
 solution be the superposition of the fields of single sources which are described by the physical parameters
 of corresponding sunspots on the photosphere, such as their position, strength, decay rates and the extent
 of the same polarity. The solution enables us to make an analytical description of the configuration of the
 magnetic field in the chromosphere and corona, and to investigate more conveniently its development as
 the foot points on the photosphere evolve.

1. Introduction

In order to study solar flares, radio bursts, and other phenomena of solar activities, one
 needs to investigate the magnetic field configurations of active regions on and above the
 photosphere. In recent years some theoretical methods which are based on the
 assumption of a force-free field with constant $\alpha$ and on observations of the longitudinal
 magnetic field on the photosphere, have been used to solve this problem. Several
 solutions were obtained by various authors, e.g. Nakagawa and Raadu (1972), Seehafer
 of the solutions have been used to study the three-dimensional structure of active regions
 productive of flares. To some extent, the calculated magnetic field configuration matches
 the actual structure of chromospheric fibrils and the free energy derived from the
 calculated field accounts reasonably for the energy release in flares.

However, it is well known that the boundary conditions of the problem are insufficient
 in that the solutions obtained are not unique (Chiu and Hilton, 1977; etc.). A unique
 solution for this problem needs to present a physical model described by reasonable
 boundary conditions. As Seehafer (1982) has shown, specifying the boundary conditions
 may have a great influence on the structure of the force-free magnetic field, and
 on the excess energy stored. Therefore, further study on this aspect is needed.

In this paper, we suggest an alternative way to make the solution of the boundary
 value problem unique. The consideration and motivation to do so is as follows.

(1) The supposition of zero net magnetic flux of an active region (Nakagawa and
 Raadu, 1972; Chiu and Hilton, 1977) may not be reasonable. In order to satisfy this
 supposition in a practical calculation one would often need to enlarge or reduce the size
of the active region and even increase or decrease the field strength in certain areas more or less arbitrarily. This kind of treatment may distort the observed magnetic field on the photosphere, and may have considerable influence upon the solution. As a general rule, the net magnetic flux of an active region may, in fact, not be zero (Seehafer, 1982). We speculate that the nonzero net magnetic flux may have some important implication for the occurrence of flares, radio bursts, and other solar activities because it simply relates to the interactions between different active regions. For a more reasonable representation of magnetic fields of active regions we would consider this problem without the supposition of zero net magnetic flux.

(2) It is well known that all magnetic flux through the photosphere is located in discrete elements with a field strength ranging from 3 kG (in sunspots) to not less than about 1 kG (in facular and network elements) (Zwaan, 1978). This concentrated characteristic is shown by magnetograms of high resolution. It is difficult to represent the discrete concentrations of magnetic flux by a trigonometric series. So the two-dimensional Fourier expansion may not be a good choice for an analytical description of observed photospheric magnetic fields. Figure 1 is the magnetogram of 16 May, 1981, basically constructed from the sunspot drawing made by Yunnan Observatory and supplemented with some details from the Kitt Peak full-disc magnetogram published in SGD and from the structure of an H\(\alpha\) filtergram of Beijing Astronomical Observatory.

![Magnetogram of longitudinal field](image)

*Fig. 1. The magnetogram of longitudinal field on 16 May, 1981, constructed basically from the sunspot drawing made by Yunnan Observatory.*

The spatial resolution is hard to estimate, perhaps no less than 10 arcsec; and the measured magnetic field strength suffered quite big errors, e.g., several hundred gauss for sunspot umbra. For such a spatial resolution, even one thousand Fourier terms could
not adequately represent the detail distribution of all sunspots, such as their position and strength. For a resolution of some arcsec ones (typical of magnetographic measurements) and a linear extent of the region considered of around a hundred arcseconds, one has to use over a thousand Fourier terms. However, the assumption of force-free field with constant $a$ itself is an approximation of real field configurations. In considering such situations, we think that a preparatory analytical description would be desirable to represent the main characteristics of active regions, such as the position, extent and strength of sunspots, and the neutral line direction and location. It is because of this consideration that we prefer to use the analytical source functions given by formula (2.1) of Section 2 to describe approximately the characteristics of the high concentration of magnetic flux in an active region.

(3) Numerical solutions are useful in constructing the three-dimensional structure of the force-free magnetic field. However, in this way we could not easily find the explicit association between the characteristics of the photospheric field and the configuration of the chromospheric and coronal fields. An analytical solution has advantages not only in making the calculation simple, but also in explaining clearly the intrinsic physical situation. This is why we would prefer to obtain an analytical solution. In this case, as the boundary condition on the photosphere, all the sunspots could be described by source functions with the corresponding parameters; the solution would then include these parameters directly. These parameters can be changed to represent various boundary conditions. Therefore, instead of repeatedly obtaining the numerical solution of the Helmholtz equation for each new boundary value, finding a solution now becomes a simple routine of changing the parameters of the single sources. The evolution of the chromospheric and coronal field configuration with that of the photospheric foot points can then be simply obtained by the substitution of new group parameters for old ones.

(4) The solution presented previously by some authors (Nakagawa and Raadu, 1972; Chiu and Hilton, 1977) is not unique, because the boundary conditions given are insufficient. Therefore, we would need to add some more reasonable boundary conditions and make a better model to fit the actual physical picture. A unique analytical solution may be useful to carry out investigations of the magnetic field of active regions.

Considering the above situation we present here a physical model which seems to correspond more suitably to real physical conditions. By applying a new method in mathematical physics presented by Chen (1981, 1983), we obtain a unique analytical solution for this problem. It enables us to make an analytical description of the magnetic field in the chromosphere and corona, and to study its developments as the magnetic field foot points on the photosphere evolve.

2. Formulation of the Physical Model

The Cartesian coordinate system $x$, $y$, $z$ is used as follows: the $(x, y, 0)$ plane coincides with the photosphere and the $z$ axis denotes height and directs toward the high atmosphere (see Figure 1).

Observations indicate that the basic features of the longitudinal magnetic field on the
photosphere are manifested by the varying distributions of single strong sunspots with their penumbras. We may consider them as the sources by which the whole magnetic field on and above the photosphere is produced.

Observations show that, in an active region, the magnetic flux with north polarity may not be equal to that with south polarity. We suspect that some of the unbalanced flux penetrates into outside regions. On the other hand, some part of the flux of the region may come from outside. It is obvious that the magnetic field of an active region does not close fully, and the magnetic field outside the active region may be non-zero.

In accordance with the observed facts as mentioned above, we propose the following physical model:

1. We suppose that the net magnetic flux of an active region is not zero.

2. We assume that, where there is no other source outside the active region, the magnetic field strength outside the active region $B$ tends to decrease outwards mainly from the border of active region, when $|x| \to \infty$, $B \to 0$; and when $|y| \to \infty$, $B \to 0$.

3. Generally speaking, in the direction $z$ the magnetic field decreases as the distance to the photosphere increases. Suppose that at a large enough height, say, $z > 2h$, the magnetic field strength vanishes. This means that at $z > 2h$, $B \equiv 0$; but at $0 \leq z \leq 2h$, $B \equiv B$.

4. According to observations, we may consider the sunspots as the sources of the field and use a set of analytical functions which represent sources on the photosphere to describe this magnetic field. It may be written as follows:

$$B_x(x, y, 0) = \sum_{m=1}^{k} B_{zm}(x, y, 0)$$

$$= \sum_{m=1}^{k} \Omega_m \exp(-\mu_m x_m - \nu_m y_m) \cos(\omega_m x_m + \sigma_m y_m), \quad (2.1)$$

where $k$ is the total number of the sunspots (or other discrete magnetic features) in the active region; $X_m = |x - a_m|$, $Y_m = |y - b_m|$; $a_m$, $b_m$ express respectively, the centre coordinates of the $m$th source in the $(x, y, 0)$ plane; $\Omega_m$ denotes the strength of magnetic field at $x = a_m$, $y = b_m$, $z = 0$ where the strength of longitudinal component takes the maximum or minimum value, i.e., the field strength of the sunspot centre on the photosphere; $\mu_m$ and $\nu_m$ express the decreasing rates of the field and $\omega_m$ and $\sigma_m$ denote the parameters describing the extent of the magnetic region with the same polarity for the $m$th source. We can obtain these data directly from observations. Table I shows the parameters of each source which are obtained from the observed magnetogram of 16 May, 1981 shown in Figure 1. The total number of sources is 50. The first column in Table I shows the number of sources and columns 2–8 show the parameters as mentioned above. Figure 2 is the magnetogram calculated using formula (2.1) with the parameters listed in Table I.

5. Assuming that the magnetic field for each single sunspot is symmetrically distributed with $z (X_m = 0, Y_m = 0)$ as its symmetry axis, we then have a transverse magnetic
### TABLE I

The parameters of each source which are obtained from observed magnetogram of 16 May, 1981, shown in Figure 1 ($\Omega$, in $10^2$ G; $a$, $b$, in $10^4$ km)

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<th>$b$</th>
<th>$\Omega$</th>
<th>$\mu$</th>
<th>$\nu$</th>
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The field $B_{xm} = 0$, $B_{ym} = 0$ at $X_m = 0, Y_m = 0, z = z$. However, here the total transverse field

$$B_x(a_m, b_m, z) = \sum_{m=1}^{k} B_{xm}(a_m, b_m, z) \neq 0,$$

$$B_y(a_m, b_m, z) = \sum_{m=1}^{k} B_{ym}(a_m, b_m, z) \neq 0.$$ 

On the basis of the physical model as mentioned above for the force-free magnetic field determined by the $m$th sunspot only, the equations may be written in the following form:

$$\mathbf{J} \times \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \alpha \mathbf{B}. \quad (2.2)$$

The boundary conditions may be written as follows:

$$B_z(x, y, 0) = \Omega \exp(-\mu X - \nu Y) \cos(\omega X + \sigma Y)$$

when $z = 0$, $|X| = |x - a|$, $|Y| = |y - b|$; \quad (2.3)

$$\mathbf{B} = 0 \quad \text{when} \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad z > 2h; \quad (2.4)$$

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Fig. 2. The calculated magnetogram by formula (2.1) with the parameters listed in Table I. It represents the basic characteristics of observed magnetic field shown in Figure 1. (The zero-gauss contour \(-\cdots\cdots\cdots\) has been truncated where the field strength is too weak.)

\[
\begin{align*}
\mathbf{B} & \to 0 \quad \text{when} \quad |x| \to \infty , \quad -\infty < y < \infty , \quad 0 \leq z \leq 2h ; \\
\mathbf{B} & \to 0 \quad \text{when} \quad -\infty < x < \infty , \quad |y| \to \infty , \quad 0 \leq z \leq 2h ; \\
B_x(a, b, z) &= B_y(a, b, z) = 0, \quad \text{at} \quad X = 0, \quad Y = 0, \quad z = z .
\end{align*}
\]

The parameters \(J, B, \Omega, \ldots, a, b,\) etc., in the above expressions should all have the suffix \(m\) which for simplicity and convenience has been omitted.

Up to now, our physical model has been proposed and its mathematical formulation has been completed. In the next section, we shall present the solution of this problem which satisfies the above partial differential equations and all boundary conditions by means of the integral transform.

3. The Solution for the Problem of Force-Free Magnetic Field with Constant \(\alpha\)

As illustrated by Nakagawa and Raadu (1972), in the case of a force-free magnetic field with a constant \(\alpha\), the magnetic induction \(\mathbf{B}\) can be expressed by a scalar function \(\varphi\) as

\[
\mathbf{B} = \nabla \times \nabla \times (\varphi \mathbf{e}_z) + \alpha \nabla \times (\varphi \mathbf{e}_z) = B_x \mathbf{e}_x + B_y \mathbf{e}_y + B_z \mathbf{e}_z ,
\]
where $\varphi$ is satisfied by the Helmholtz equation

$$\left( \nabla^2 + \alpha^2 \right) \varphi = 0 \quad .$$

(3.2)

Thus, we may reduce the solution of the problem to determine the function $\varphi(x, y, z)$ that satisfies the equations and boundary conditions mentioned above.

We introduce the parameter $h$ and define the function $\varphi(x, y, z)$ as follows: in the spatial domain $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \leq z \leq 2h$, $\varphi(x, y, z) \equiv \varphi(x, y, z)$; but in the domain $-\infty < x < \infty$, $-\infty < y < \infty$, $2h < z < \infty$, $\varphi(x, y, z) \equiv 0$. That is to say, we suppose that the magnetic field vanishes when $z > 2h$.

According to the formulae (3.1) the components of $\mathbf{B}$ may be written in scalar form as follows:

$$B_x(x, y, z) = \frac{\partial^2 \varphi}{\partial x \partial z} + \alpha \frac{\partial \varphi}{\partial y} \quad , \quad B_y(x, y, z) = \frac{\partial^2 \varphi}{\partial y \partial z} - \alpha \frac{\partial \varphi}{\partial x} \quad ,$$

$$B_z(x, y, z) = -\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} \quad .$$

(3.3)

Making use of the integral transform under the condition of satisfying Equation (3.2) and the boundary condition (2.3), and carrying out the transformations and inversions, we finally obtain:

$$\varphi(x, y, z) = -\frac{\Omega}{2} \left\{ \frac{1}{\gamma_1^2} \exp(-\mu_1 X - v_1 Y + i\eta_1 z) + \frac{1}{\gamma_2^2} \exp(-\mu_2 X - v_2 Y + i\eta_2 z) \right\} +$$

$$+ \sum_{n=0}^{\infty} \sin C_n z \left\{ A_n \left[ \exp(-\mu_1 X - C_{n\mu_1} Y) + \exp(-\mu_2 X - C_{n\mu_2} Y) \right] +$$

$$+ \Gamma_n \left[ \exp(-v_1 Y - C_{n\nu_1} X) + \exp(-v_2 Y - C_{n\nu_2} X) \right] \right\} ,$$

(3.4)

where

$$\mu_1 = \mu - i\omega \quad , \quad \nu_1 = v - i\sigma \quad , \quad \mu_2 = \mu + i\omega \quad , \quad \nu_2 = v + i\sigma \quad ,$$

$$\eta_1 = \sqrt{\gamma_1^2 + \alpha^2} \quad , \quad \gamma_1^2 = \mu_1^2 + \nu_1^2 \quad , \quad \eta_2 = \sqrt{\gamma_2^2 + \alpha^2} \quad , \quad \gamma_2^2 = \mu_2^2 + \nu_2^2 \quad ,$$

$$C_n = (2n + 1) \pi/2h \quad , \quad C_{nx} = C_n - \alpha^2 \quad , \quad C_{n\mu_1} = \sqrt{C_{nx} - \mu_1^2} \quad ,$$

$$C_{n\mu_2} = \sqrt{C_{nx} - \mu_2^2} \quad , \quad C_{n\nu_1} = \sqrt{C_{nx} - \nu_1^2} \quad , \quad C_{n\nu_2} = \sqrt{C_{nx} - \nu_2^2} \quad ,$$

$$n = 0, 1, 2, \ldots .$$

Deriving $B_x(x, y, z)$ and $B_y(x, y, z)$ from formulae (3.3) and (3.4), and satisfying the condition (2.7), we have

$$A_n = (2v\delta - \varepsilon G_{\nu})/\mathcal{H}_n \quad , \quad \Gamma_n = (2\mu\varepsilon - \delta H_{\mu})/\mathcal{H}_n \quad ,$$

(3.5)

where

$$\delta = \Omega(\mu_1 D_1 + \mu_2 D_2)/C_n^2 h \quad , \quad \varepsilon = \Omega(\nu_1 D_1 + \nu_2 D_2)/C_n^2 h \quad ,$$

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\[ D_1 = (C_n - i\eta_1 e^{-i\eta_1} \sin C_n h) \eta_1^2/(C_n^2 - \eta_1^2) \gamma_1^2, \]
\[ D_2 = (C_n + i\eta_2 e^{i\eta_2} \sin C_n h) \eta_2^2/(C_n^2 - \eta_2^2) \gamma_2^2, \]
\[ \mathcal{M}_n = 4\mu v - G_v H_\mu, \quad G_v = C_{n\nu_1} + C_{n\nu_2}, \quad H_\mu = C_{n\mu_1} + C_{n\mu_2}; \]

\( a \) and \( b \) denote the coordinates of a certain sunspot; \( \Omega, \mu, v, \omega, \sigma \) are given by observation.

In practical evaluations, the second term in formula (3.4) converges quickly. In general, it is accurate enough to take the first two terms and the third term is already negligible. Therefore, it enables us to simplify the calculations greatly.

Putting \( X = 0, Y = 0, z = 2h \) and supposing that there the longitudinal magnetic field decreases to \( B_z = 0.001 \) G then we obtain

\[ h = [5 + \log \Omega + \log (\cos 2R_c h)]/2R, \log e, \]

where the unit of \( \Omega \) is 100 G, and

\[ R_c = R \cos \theta, \quad R = R \sin \theta, \quad R = 4\sqrt{p^2 + q^2}, \quad \theta = \frac{1}{2} \arctg |p/q|, \]
\[ p = 2(\mu \omega + \nu \sigma), \quad q = t + \alpha^2, \quad t = \mu^2 + \nu^2 - \omega^2 - \sigma^2. \]

Using a method of successive approximation, the value of \( h \) can be obtained with sufficient accuracy. For any known source its corresponding \( h \) can be derived from formula (3.6). In the case of many sources emerging at the same time, we take, among \( h_m, h \) to be the maximum value as the boundary of definition domain for the function \( \varphi(x, y, z) \) in direction \( z \). In general, the source which has maximum \( \Omega \) or adequate group of parameters \( \Omega, \mu, v, \omega, \sigma \) may acquire maximum \( h \). The definition domain for \( \varphi(x, y, z) \) as mentioned above indicates that when \( z > 2h \), \( \varphi(x, y, z) \equiv 0 \), then we have \( B = 0 \). It suggests that the condition (2.4) has been satisfied. From the formulæ (3.3) and (3.4) one can see that the magnetic field vanishes quickly as \( |x| \to \infty \) (or \( |y| \to \infty \), or \( |x| \to \infty \) and \( |y| \to \infty \)). Therefore the conditions (2.5) and (2.6) are satisfied.

Till now, we have sought the function \( \varphi(x, y, z) \) which satisfies all of the equations and boundary conditions (2.2)–(2.7). So the function \( \varphi(x, y, z) \) and the corresponding physical quantities \( B_x, B_y, B_z \), etc., have been determined.

Assuming that there is another arbitrary solution for this problem, \( \bar{\varphi}(x, y, z) = \varphi(x, y, z) + \varphi^*(x, y, z) \), we can prove that \( \varphi^*(x, y, z) \equiv 0 \), if the corresponding equations and boundary conditions are satisfied. Hence, \( \bar{\varphi}(x, y, z) \equiv \varphi(x, y, z) \), that is to say, \( \varphi(x, y, z) \) obtained in this paper is the unique analytical solution of the problem.

At first sight the curve denoting the function \( B_x \) or \( B_y \) does not look smooth at \( x = a, y = b, z = z \); but, in fact, it is smooth; although the function \( \varphi(x, y, z) \) includes the factors \( \exp(-\mu_1 X - \nu_1 Y), \exp(-\mu_1 X - C_{n\mu_1} Y), \exp(-\nu_1 Y - C_{n\nu_1} X) \), etc. Because, in fact, at \( x = a, y = b, z = z \), we have the derivatives:

\[ \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi(x, y, z)}{\partial x} = \varphi'_x(a - 0, b - 0, z) = \varphi'_x(a + 0, b + 0, z) = 0, \]
\[
\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi(x, y, z)}{\partial y} = \varphi_y'(a - 0, b - 0, z) = \varphi_y'(a + 0, b + 0, z) = 0, \\
\frac{\partial^2 \varphi}{\partial x \partial z} = \frac{\partial^2 \varphi(x, y, z)}{\partial x \partial z} = \varphi_{xz}''(a - 0, b - 0, z) = \varphi_{xz}''(a + 0, b + 0, z) = 0, \\
\frac{\partial^2 \varphi}{\partial y \partial z} = \frac{\partial^2 \varphi(x, y, z)}{\partial y \partial z} = \varphi_{yz}''(a - 0, b - 0, z) = \varphi_{yz}''(a + 0, b + 0, z) = 0, \\
\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi(x, y, z)}{\partial x^2} = \varphi_{xx}''(a - 0, b - 0, z) = \varphi_{xx}''(a + 0, b + 0, z) = f_1(z), \\
\frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial^2 \varphi(x, y, z)}{\partial y^2} = \varphi_{yy}''(a - 0, b - 0, z) = \varphi_{yy}''(a + 0, b + 0, z) = f_2(z).
\]

Therefore, from formulae (3.3) we have \( B_x(a, b, z) = 0, B_y(a, b, z) = 0, B_z(a, b, z) = f(z) \), where \( f(z) = -f_1(z) - f_2(z) \). That is to say, although the individual factors \( \exp(-\mu X - \nu Y), \cos(\omega X + \sigma Y), \exp(-\mu_1 X - C_{n\mu_1} Y), \exp(-\nu_1 Y - C_{n\nu_1} X), \) etc., have no derivatives at \( x = a, y = b, z = z \), the function \( \varphi(x, y, z) \) has derivatives when we relate these factors in combinations as shown in formula (3.4). The current components are:

\[
J_x(a, b, z) = \alpha(\nabla \times \mathbf{B})_x = \alpha \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) = 0, \\
J_y(a, b, z) = \alpha(\nabla \times \mathbf{B})_y = \alpha \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) = 0, \\
J_z(a, b, z) = \alpha(\nabla \times \mathbf{B})_z = \alpha \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = \alpha^2 B_z(a, b, z).
\]

It is obvious, from the above explanation, that the functions expressing \( B_x, B_y, B_z \) are continuous. Moreover, the partial derivatives of \( B_x, B_y, B_z \) for \( x, y \), and \( z \) are continuous too at \( x = a, y = b, z = z \).

This means that the functions denoting \( B_x, B_y, B_z \) are smooth and the functions expressing \( J_x, J_y, J_z \) are continuous everywhere in the whole spatial domain considered, and that the magnetic field runs parallel to the current. Consequently, no singular point exists in the solution presented here.

The total field, as the superposition of the fields of the single sources, is a constant \( \alpha \) force-free magnetic field in a space: \(-\infty < x < \infty, -\infty < y < \infty, 0 \leq z \leq 2h\). Since each of the terms in the functions expressing \( B_x(x, y, z), B_y(x, y, z) \) and \( B_z(x, y, z) \) derived from function \( \varphi(x, y, z) \) includes the multiplier factors \( \exp(-\mu X - \nu Y), \)
exp\((-\mu X - \lambda_c Y\)), \exp\((-\nu Y - \rho_c X\)), \text{ etc.}, \) where \(\mu, \nu, \lambda_c, \rho_c\) are the non-zero positive reals, the field \(B\) must vanish very fast as \(|x| \to \infty\), or \(|y| \to \infty\), or \(|x| \to \infty\) and \(|y| \to \infty\). Accordingly, the magnetic energy content of the spatial domain considered is finite.

4. Discussion

Our solution would give an analytical description of the magnetic field in the spatial domain considered. The physical model, the method for solving the problem and the results derived from this solution may have the following meanings:

(1) We use a set of source functions to describe the observed longitudinal field on the photosphere which consists of the contributions due to the single strong sunspots or other magnetic features, cf. formula (2.1). The general similarity of Figure 2 to Figure 1 shows that the representation of the observed magnetogram by source functions (2.1) with parameters listed in Table I is quite successful. For any active region we can use such a set of source functions to describe the observed field mathematically. The only differences between different active regions are in the number of sources and corresponding parameters describing each of them.

(2) It should be pointed out that we use the function of source to describe the observed longitudinal magnetic field on the photosphere, and the summation of force-free magnetic fields determined by each single source as the whole solution of the problem. The physical picture obtained shows that for the force-free magnetic field determined by the \(m\)th source at \(x = a_m, y = b_m, z = z\) its transverse field is always equal to zero whereas the total transverse magnetic field does not equal zero. That is

\[
B_{xm}(a_m, b_m, z) = B_{ym}(a_m, b_m, z) \equiv 0, \quad \text{but} \quad B_{xm(m-1)}(a_m, b_m, z) \neq 0,
\]

\[
B_{ym(m-1)}(a_m, b_m, z) \neq 0; \quad B_{xm(m+1)}(a_m, b_m, z) \neq 0,
\]

\[
B_{ym(m+1)}(a_m, b_m, z) \neq 0; \quad \ldots; \quad B_{xk}(a_m, b_m, z) \neq 0,
\]

\[
B_{yk}(a_m, b_m, z) \neq 0, \quad m \neq k.
\]

Hence, here the total transverse field

\[
B_x(a_m, b_m, z) = \sum_{m=1}^{k} B_{xm}(a_m, b_m, z) \neq 0,
\]

\[
B_y(a_m, b_m, z) = \sum_{m=1}^{k} B_{ym}(a_m, b_m, z) \neq 0.
\]

Thus, we have established a new suitable boundary condition (2.7), which is a necessary condition for finding a unique solution and so enables us to overcome an important difficulty in some other studies, which give solutions which are not unique as the boundary conditions are not sufficient.

(3) The solution presented in this paper includes the position, field strength, extent and other parameters of the sources (sunspots or other magnetic features) observed on the photosphere. That is, it summarises the features of the photospheric foot points from
which the chromospheric and coronal magnetic fields expand. Therefore, if by observation we find the evolution of the foot points or emergence of new foot points, then the solution enables us to describe the evolution of the corresponding force-free magnetic field in the chromosphere and corona merely by changing the corresponding parameters in the solution or adding the contribution of the new emerging sources. Thus, our method would provide a convenient way to discuss the evolution of the chromospheric and coronal field when the photospheric field evolves.

(4) As long as the position of each source, its field strength and decay rate and the extent of the same polarity, etc., are found by observations, the unique analytical solution for the force-free field problem of the active region can immediately be written as $B = \sum_{m=1}^{k} B_m$. This result is much simpler than those of previous studies. It may be convenient for frequent analysis and discussion of solar activities.

(5) It is known from observations of photospheric magnetic field of active regions that the field strength may reach the order of $10^3$ G. Outside the active region the field strength decreases, but it does not equal zero everywhere. Moreover, two or many active regions may emerge at the same time. In accordance with these facts, we may infer that the magnetic force lines from the active region do not necessarily all come back to the same region; some lines may get into the outside. In the case of simultaneous emergence of two or more active regions, the magnetic force lines may connect with each other. Therefore, influence and interference between the active regions may take place. In this paper we have assumed that the net magnetic flux may not be zero, instead of assuming zero net magnetic flux. Therefore, the results given here may be directly used for the case when two or more active regions appear at the same time and the interactions between several active regions can be investigated straightforwardly.

From the above discussion and analyses it appears that the solution given in this paper can be directly applied to calculations of the magnetic field structure of actual active regions.

Continued study will be published at a later date.

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