CASCADE OF MAGNETIC ENERGY AS A MECHANISM OF CORONAL HEATING

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ABSTRACT

The generation of field-aligned electric currents in the solar corona is discussed in the context of coronal heating theory. A basic problem encountered in the current-heating theory is that large electric current densities are required to explain the observed energy losses of coronal loops. We propose that the required current densities are produced via a cascade process, in which "free" magnetic energy is transferred from large to small length scales in the corona as a result of the random motions imposed on the magnetic field lines by subphotospheric convective flows.

A model of the initial development of this cascade process is presented. The magnetic field is assumed to be initially uniform, extending between two parallel plates which represent the photosphere at the footpoints of the loop. The field is then perturbed by random motions at the two boundary plates. Neglecting magnetic diffusion, an expression for the power spectrum of the transverse magnetic field is derived. We find that the cascade of magnetic energy toward higher wavenumbers proceeds very rapidly in time and is accompanied by an exponential increase of the rms electric current density. In constrast, the stored magnetic energy increases only quadratically with time in our model.

We suggest that magnetic diffusion eventually leads to a statistically stationary state, in which there is a balance between the rate of energy input and the rate of dissipation. An estimate of the heating rate in this regime is obtained: \( E_H \approx 0.19B_0^2D/L^2 \), where \( B_0 \) is the coronal field strength, \( L \) is the length of the coronal loop, and \( D \) is the effective diffusion constant of the photospheric motions. Comparison with observations shows that \( E_H \) is too low by a factor of \( \sim 40 \). Several possibilities for explaining this discrepancy are discussed.

Subject headings: hydromagnetics — Sun: corona

1. INTRODUCTION

The nonradiative heating of the upper solar atmosphere has been a central problem in solar physics for many years. With the advent of high-resolution X-ray and UV observations in the 1970s, it has become clear that coronal and transition-region emissions are copatial with regions of strong magnetic field, and that magnetic fields play a crucial role in the heating process (see Vaiana and Rosner 1978; Chiuderi 1981; Kuperus, Ionson, and Spicer 1981). The observations have led to the development of a theory first put forward by Gold (1964), who suggested that coronal heating is due to the dissipation of field-aligned electric currents (Parker 1972, 1983a, b, 1986a; Tucker 1973; Rosner et al. 1978; Golub et al. 1980; Galeev et al. 1981; Sturrock and Uchida 1981; Heyvaerts and Priest 1984; Rabin and Moore 1984). The basic idea is that convective flows below the solar surface cause a random shuffling of the magnetic field lines that penetrate the solar surface, and thereby causes twisting and braiding of the coronal field. In "open" magnetic field structures the twists are able to propagate outward into interplanetary space with the Alfvén speed. However, in "closed" magnetic structures such as coronal loops, the magnetic stresses can be balanced by dynamical forces acting on the field lines at the photospheric footpoints, so that stresses are able to build up. The energy stored in the twisted magnetic field can be converted into heat by dissipation of the associated electric currents.

A problem with the current-dissipation theory is that very large current densities and very small spatial scales are required to explain the observed rate of coronal heating. This may be illustrated as follows. The observed radiative and conductive losses of active-region loops imply a nonradiative energy flux into the corona of the order (Withbroe and Noyes 1977):

\[ F_{\text{mech}} = 10^7 \text{ergs cm}^{-2} \text{s}^{-1}. \] (1)

Assuming this energy enters the loop at both footpoints and is distributed uniformly over the length \( L \) of the loop, the heating rate is

\[ E_H = \frac{2F_{\text{mech}}}{L} = 2 \times 10^{-3}L_{10}^{-1} \text{ergs cm}^{-3} \text{s}^{-1}, \] (2)

where \( L_{10} \) is the loop length in units of \( 10^{10} \) cm. Equating this rate with the expression for Joule dissipation, \( E_H = j^2/\sigma \), with \( j \) the electric current density and \( \sigma = 6 \times 10^{16} \) esu s\(^{-1}\) the classical conductivity of a 2 \( \times 10^6 \) K plasma, the required current density is

\[ j = 1.1 \times 10^7 L_{10}^{-2/3} \text{esu}. \] (3)

On the other hand, one can estimate the current density that will be generated if the loop is twisted over an angle of, say, \( \Delta \phi \) radians; if the average field strength in the loop is \( B_0 = 100 \) G, then

\[ j = \frac{c}{2\pi} \frac{B_0}{L} \Delta \phi = 4.8 \times 10^4 L_{10}^{-2} \Delta \phi \text{ esu}. \] (4)

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Comparison of equations (3) and (4) shows that the current generated by a twist $\Delta \phi = 1$ rad is by far insufficient to explain the observed coronal heating rate; to contribute significantly to coronal heating, the current density must somehow be increased by a factor $\sim 2 \times 10^5$. Since $j = (c/4\pi)V \times B$, this requires magnetic field variations on very small transverse length scales (see Tucker 1973).

The process by which small-scale structures are generated in the corona is not well understood. A key question is how the magnetic field responds to the slow, random motions of the photospheric footpoints, and in particular whether the field relaxes to an equilibrium state. Parker (1972, 1979, pp. 370–378, 1983a, b, 1986b; also see Yu 1973; Tsinganos, Distler, and Rosner 1984) argued that the existence of equilibrium requires a certain structural invariance of the magnetic field; for example, Parker (1972) considered the case of an initial uniform field, and found that perturbations of such a field, $B = B_0 z + b$, could be in equilibrium only if $\partial b/\partial z = 0$; i.e., the perturbed field must be invariant with respect to the coordinate along the initial field. The flow patterns in the solar photosphere are stochastic in nature, and the magnetic perturbations produced by these flows do in general not have an ignorable coordinate. Therefore, if the existence of an ignorable coordinate is a necessary condition for equilibrium, the magnetic field in closed coronal loops cannot reach equilibrium without changing its “topological” structure. Parker suggested that the relaxation toward equilibrium involves the formation of current sheets, i.e., regions of strong magnetic shear where reconnection of magnetic field lines takes place. The reconnection changes the “topological” structure of the field, thereby allowing it to approach an equilibrium state. However, since the photospheric footpoints are continually being shuffled around, the field never actually attains equilibrium, but remains in a state of continuous relaxation (“nonequilibrium”). According to this model, heating of the coronal plasma takes place in the current sheets that are embedded in the field.

In an earlier paper (van Ballegooijen 1985, hereafter Paper I) it was shown that the uniform-field problem considered by Parker (1972) allows equilibrium solutions that do not satisfy Parker’s invariance constraint. It follows that invariance of $B$ is not a necessary requirement for equilibrium in magnetic structures. Furthermore, the solutions were shown to be compatible with arbitrary patterns of motions of the photospheric footpoints: in response to the photospheric motions, the field can evolve through a series of equilibrium states without the necessity for reconnection. This suggests that the coronal magnetic field always remains close to an equilibrium state. Consequently, the magnetic field structure required for coronal heating must be a result of the time evolution of equilibrium states and cannot be due to the relaxation process by which these equilibria are reached.

In Paper I it was suggested that the quasi-static evolution of closed coronal structures involves a cascade process, in which “free” magnetic energy is transported to smaller length scales on the time scale of the photospheric motions. According to this model, photospheric flows can generate magnetic fine structure in the corona with a (transverse) length scale of order 10–100 m, even if the velocity field does not contain any power at such small spatial scales. The cascade is due to the stochastic nature of the motions: neighboring magnetic footpoints separate exponentially with time as a result of the randomly changing velocity gradients in the photosphere. This causes the coronal field to be “shredded” into finer and finer flux elements, with an exponential decrease of the typical size of these elements.

In the present paper we describe a statistical model of the cascade process. The goals are (1) to substantiate our hypothesis about the existence of such a process, and (2) to derive an estimate of the coronal heating rate. The model considers the initial development of the cascade process, neglecting magnetic diffusion. Although we are unable to give a fully self-consistent statistical treatment, the results suggest that the cascade proceeds very rapidly, and that the random motions of the photospheric footpoints cause an exponential increase of the rms electric current density.

The paper is organized as follows. In § II we describe the basic assumptions of the model, and we derive the equations governing the time evolution of the system. In § III we present our statistical model, and we derive an expression for the power spectrum of the transverse magnetic field fluctuations. In § IV we present numerical calculations of the power spectrum, which show that magnetic energy is transferred to larger wavenumbers via a cascade process. The effect of magnetic diffusion on the power spectrum is discussed, and an expression for the heating rate in the final, stationary regime is obtained. In § V we discuss the application of the present theory to the problem of coronal heating.

II. BASIC ASSUMPTIONS

Electric currents in the solar corona are generated as a result of the motions of magnetic footpoints in the photosphere. We consider the case in which the typical time scale $\tau$ of the photospheric motions is large compared with the time $L/v$, for an Alfvén wave to travel along a coronal loop from one photospheric footpoint to the other ($L$ is the length of the coronal loop). Since the magnetic footpoints are moved slowly, the coronal field can adjust to the changing boundary conditions, and therefore the motions do not lead to the generation of Alfvén waves. Hence the field evolves through a series of equilibrium configurations. Furthermore, since the gas pressure in the solar corona is small compared with the magnetic pressure, the equilibrium configurations are approximately “force-free,” with electric currents flowing parallel to the magnetic field lines:

$$V \times B = \alpha B .$$

Here $\alpha$ is a scalar which must be constant along field lines (since $V \cdot B = 0$).

Following Parker (1972) the curvature of the coronal loop is neglected: the initial magnetic field is assumed to be uniform, extending between two flat, parallel plates which represent the photosphere at the footpoints of the loop (see Fig. 1a). The system is then subjected to horizontal flows at the two boundary plates, which cause random motions of the magnetic footpoints. The velocity field is taken to be parallel to the boundary plates and incompressible. The flow is assumed to be characterized by a certain correlation length $l$ and a correlation time $\tau$, and we will assume that $l$ is small compared with the distance between the plates. For ordinary granulation the typical length and time scales are $l \approx 10^3$ km and $\tau \approx 300$ s, while for supergranulation $l \approx 3 \times 10^4$ km and $\tau \approx 10^5$ s. Hence our model will apply to loops with $L \sim 10^5$ km typical for large active regions.

As a result of the footpoint motions, the field lines become intertwined and braided (see Fig. 1b). At present, not much is known...
about the properties of such braided, force-free magnetic fields. In the following we briefly review some results of Paper I, where the basic equations governing such fields were derived. A Cartesian reference frame \((x, y, z)\) is adopted, with the boundary plates located at \(z = 0\) and \(z = L\). Then the initial field is given by \(B(x, y, z, 0) = B_0 \xi\). In the limit \(l \ll L\) the magnetic field \(B(x, y, z, t)\) can be expanded in powers of some small parameter \(\epsilon\), which can be defined as the ratio of transverse and longitudinal length scales of \(B(x, y, z, t)\) (\(\epsilon \sim l/L\)). In lowest order of \(\epsilon\), \(B\) makes only a small angle with the \(z\)-direction, and is given by

\[
B = B_0 (b_x, b_y, 1),
\]

where \(b_x(x, y, t)\) and \(b_y(x, y, t)\) are the dimensionless transverse field components (\(b_x \sim b_y \sim \epsilon\)), which satisfy

\[
\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = 0 .
\]

The scalar \(\alpha(x, y, z, t)\), which is a measure of the field-aligned electric current, is given by

\[
\alpha = \frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} .
\]

The condition of force-free magnetostatic balance requires that \(\alpha\) is constant along field lines:

\[
\frac{\partial \alpha}{\partial z} + b_x \frac{\partial \alpha}{\partial x} + b_y \frac{\partial \alpha}{\partial y} = 0 .
\]

In combination with equations (7) and (8), equation (9) determines the spatial structure of the magnetic field.

The force-free fields considered here have the special property that the magnetic energy in the transverse field is distributed uniformly in the \(z\)-direction. To demonstrate this, consider an area \(A\) in the \((x, y)\)-plane which is large compared to the correlation length \(l\), and define the average of the magnetic energy over \(A\):

\[
\begin{align*}
\langle B_z^2 \rangle &= \frac{B_0^2}{8\pi A} \int (b_x^2 + b_y^2) dxdy .
\end{align*}
\]

Since \(b_x\) and \(b_y\) satisfy condition (7), we introduce a potential function \(h(x, y, z, t)\) such that

\[
\begin{align*}
\frac{\partial h}{\partial x} &= b_x , \\
\frac{\partial h}{\partial y} &= -b_y .
\end{align*}
\]

It follows that the \(z\)-derivative of the energy vanishes:

\[
\begin{align*}
\frac{\partial}{\partial z} \left( \frac{\langle B_z^2 \rangle}{8\pi} \right) &= \frac{B_0^2}{4\pi A} \int \left( \frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial y} \frac{\partial^2 h}{\partial y^2} + \frac{\partial h}{\partial z} \frac{\partial^2 h}{\partial z^2} \right) dxdy = -\frac{B_0^2}{4\pi A} \int h \left( \frac{\partial^3 h}{\partial x^2 \partial z} + \frac{\partial^3 h}{\partial y^2 \partial z} + \frac{\partial^3 h}{\partial z^2} \right) dxdy \\
&= \frac{B_0^2}{4\pi A} \int h \frac{\partial h}{\partial z} dxdy + \frac{B_0^2}{4\pi A} \int h \left( \frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x \partial y} - \frac{\partial h}{\partial y} \frac{\partial^2 h}{\partial x \partial y} \right) dxdy + \frac{B_0^2}{8\pi A} \int \left( \frac{\partial (h^2)}{\partial x} \frac{\partial h}{\partial x} - \frac{\partial (h^2)}{\partial y} \frac{\partial h}{\partial y} \right) dxdy = 0 ,
\end{align*}
\]

where the first and last steps involve partial integration with respect to \(x\) and \(y\) (contributions from the boundary of \(A\) can be neglected since \(A \gg l^2\)), and where equation (9) has been used. The result that \(\langle B_z^2 \rangle /8\pi\) is independent of \(z\) is related to the fact that force-free fields represent a state of minimum magnetic energy (see Sakurai 1979); as one might expect, the lowest energy state is obtained when the energy is distributed evenly along \(z\).

The temporal evolution of the system is determined by the magnetic induction equation. We will assume that the conductivity of the coronal plasma is so large that magnetic diffusion and field-line reconnection can be neglected. This assumption is reasonable.
because the magnetic Reynolds number, \( R_m = \frac{l^2}{\eta} \), is of order \( 10^{10} \) for both granular and supergranular motions (\( \eta = \frac{c^2}{4\pi \sigma} \) is the magnetic diffusivity of the coronal plasma, assuming classical resistivity). Then the induction equation can be written as

\[
\frac{\partial B}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),
\]

where \( \mathbf{v} \) is the plasma velocity. Inserting equation (6) into the x- and y-components of equation (13), and assuming there are no motions along the z-direction (\( v_z = 0 \)), we obtain

\[
\frac{db_x}{dt} = \frac{dv_x}{dz},
\]

\[
\frac{db_y}{dt} = \frac{dv_y}{dz},
\]

where \( d/dz \) is the derivative along magnetic field lines and \( d/dt \) is the comoving time derivative. The z-component of equation (13) yields

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0,
\]

i.e., in the limit of small \( \epsilon \) the flow is incompressible. Together with condition (9), equations (14a), (14b), and (15) determine the temporal evolution of the system for given boundary conditions \( \mathbf{v}(x, y, 0, t) \) and \( \mathbf{v}(x, y, L, t) \).

It should be pointed out that, although we are working in first order of the parameter \( \epsilon \sim l/L \), the typical displacements \( \delta s \) in the boundary plates do not have to be small compared with the correlation length \( l \) (as long as \( \delta s \ll L \)). Therefore, equations (6)-(15) should correctly describe situations where the footpoints have been thoroughly intermixed, and the field has a complicated braiding pattern. Thus the intrinsically nonlinear character of the braiding process is preserved in the equations.

### III. Statistical Model

In the following we present a statistical model of the magnetic field in a coronal loop. We assume that the field is subjected to random motions of the photospheric footpoints, i.e., the velocity field \( \mathbf{v} = (v_x, v_y) \) and transverse magnetic field \( \mathbf{b} = (b_x, b_y) \) will be considered statistical variables. Since the evolution of the system is determined by the motions at the two boundary plates, the quantities \( \mathbf{v}(x, y, 0, t) \) and \( \mathbf{v}(x, y, L, t) \) should be considered as the independent statistical variables. Our goal is to compute the (spatial) power spectrum of the transverse magnetic field \( \mathbf{b} \) in the interior of the volume, given the power spectrum of the velocity \( \mathbf{v} \) at the two boundaries. For simplicity we assume (1) the average values of \( \mathbf{v} \) and \( \mathbf{b} \) are zero, (2) the statistical properties of \( \mathbf{v} \) and \( \mathbf{b} \) are independent of position in the \( (x, y) \)-plane (homogeneity), and (3) there is no preferred direction in the \( (x, y) \)-plane (isotropy). Furthermore, we assume that the velocities at the two boundary planes, \( \mathbf{v}(x, y, 0, t) \) and \( \mathbf{v}(x, y, L, t) \), are stationary statistical variables; i.e., the statistical properties of the footpoint motions are assumed to be independent of time.

Let us consider an arbitrary plane \( z = z_0 \) in the interior of the volume. Then the power spectrum of transverse magnetic fluctuations in the \( z = z_0 \) plane is defined as the Fourier transform of a correlation function:

\[
P(k_x, k_y, z_0, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} C_{\delta}(\Delta x, \Delta y, z_0, t)e^{ik_x \Delta x + ik_y \Delta y}d\Delta x d\Delta y,
\]

where \( k = (k_x, k_y) \) is the transverse wave vector, and \( \Delta r = (\Delta x, \Delta y) \) measures the difference in position between two points in the \( z = z_0 \) plane. The correlation function \( C_{\delta} \) is defined as

\[
C_{\delta}(\Delta x, \Delta y, z_0, t) = \langle b_x(x + \Delta x, y + \Delta y, z_0, t)b_x(x, y, z_0, t) \rangle,
\]

where the angle brackets denote a statistical average over an ensemble of all possible velocity fields \( \mathbf{v}(x, y, 0, t) \) and \( \mathbf{v}(x, y, L, t) \) at the boundary. Because of the assumption of isotropy, the power spectrum and correlation function of \( b_y \) are identical to those of \( b_x \), and the cross-correlation vanishes:

\[
\langle b_x(x + \Delta x, y + \Delta y, z_0, t)b_y(x, y, z_0, t) \rangle = 0.
\]

Using equation (8) and the assumption that average quantities do not depend on \( x \) and \( y \), it follows that the correlation function of \( z \) is given by

\[
C_{\delta}(\Delta x, \Delta y, z_0, t) \equiv \langle z(x + \Delta x, y + \Delta y, z_0, t)z(x, y, z_0, t) \rangle = -\left( \frac{\partial^2 C_{\delta}}{\partial (\Delta x)^2} + \frac{\partial^2 C_{\delta}}{\partial (\Delta y)^2} \right).
\]

Taking the Fourier transform, we obtain for the power spectrum of current-density fluctuations:

\[
P_{\delta}(k_x, k_y, z_0, t) = k^2 P_{\delta}(k_x, k_y, z_0, t),
\]

where \( k^2 = k_x^2 + k_y^2 \).

A straightforward computation of the correlation function \( C_{\delta} \) would require that we express \( b_x(x, y, z_0, t) \) in terms of the velocity fields \( \mathbf{v}(x, y, 0, t) \) and \( \mathbf{v}(x, y, L, t) \) at the two boundary plates. This would require a formal solution of the evolution equation (14a).
subject to the constraint of magnetostatic balance (eq. [9]). Unfortunately, this problem is a complicated, three-dimensional boundary-value problem (see Paper I), and a general solution is not known. Therefore we must resort to a more simplified and approximate method of deriving the correlation function. The basic idea is to consider the velocity at the $z = z_0$ plane, $\mathbf{v}(x, y, z_0, t)$, and the velocity gradient,

$$ \mathbf{a}(x, y, z_0, t) \equiv \left( \frac{\partial \mathbf{v}}{\partial z} \right)_{z=z_0}, $$

(21)

as the independent statistical variables. This approach essentially reduces the problem to a two-dimensional one, since only the level $z = z_0$ needs to be considered. Furthermore, we will assume that the velocity $\mathbf{v}(x, y, z_0, t)$ and the velocity gradient $\mathbf{a}(x, y, z_0, t)$ at $z = z_0$ are characterized by correlation lengths and correlation times equal to the correlation length $\ell$ and correlation time $\tau$ of the "photospheric" velocity fields, $\mathbf{v}(x, y, 0, t)$ and $\mathbf{v}(x, y, L, t)$. This seems reasonable, since the motions in the interior of the volume are dictated by the motions applied at the boundaries, so that the statistical properties of the motions must be similar.

The correlation function $C_a$ can now be determined as follows. Consider two points $\mathbf{R}_1$ and $\mathbf{R}_2$ in the $z = z_0$ plane, separated by

$$ \Delta \mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2. $$

(22)

The transverse field $b_a(\mathbf{R}_1, T)$ at time $T$ can be found by integrating equation (14a) with respect to time:

$$ b_a(\mathbf{R}_1, T) = \int_0^T a_a(\mathbf{r}_1(t), t) dt, $$

(23)

where $\mathbf{r}_1(t) = [x_1(t), y_1(t), z_0]$ is the path in the $z = z_0$ plane that ends at position $\mathbf{R}_1$ at time $T$. A similar expression holds for $b_a(\mathbf{R}_2, T)$, with the integration along a different path $\mathbf{r}_2(t)$. The correlation between the two $b_a$-values is

$$ C_a(\Delta \mathbf{R}, T) = \langle b_a(\mathbf{R}_1, T) b_a(\mathbf{R}_2, T) \rangle = \int_0^T \int_0^T \langle a_a(\mathbf{r}_1(t), t) a_a(\mathbf{r}_2(t'), t') \rangle dt dt'. $$

(24)

To evaluate the right-hand side of equation (24) we assume that the variable $a_a$ is characterized by the same correlation time $\tau$ as the velocity field at the boundaries. Furthermore, we will assume that the temporal correlation function of $a_a$ is Gaussian:

$$ \langle a_a(\mathbf{r}_1(t), t) a_a(\mathbf{r}_2(t'), t') \rangle = \langle a_a(\mathbf{r}_1(t), t) \rangle \langle a_a(\mathbf{r}_2(t'), t') \rangle \exp \left[ -\frac{(t' - t)^2}{2\tau^2} \right]. $$

Inserting this into equation (24), and assuming $T \gg \tau$, the integral over $t'$ can be explicitly evaluated:

$$ C_a(\Delta \mathbf{R}, T) = \tau \sqrt{2\pi} \int_0^T \langle a_a(\mathbf{r}_1(t), t) a_a(\mathbf{r}_2(t), t) \rangle dt. $$

(26)

The next step is to relate the correlation function for two comoving points, $\langle a_a(\mathbf{r}_1(t), t) a_a(\mathbf{r}_2(t), t) \rangle$, to the correlation function for two fixed points:

$$ C_a(\Delta \mathbf{s}, t) \equiv \langle a_a(s_1, t) a_a(s_2, t) \rangle, $$

(27)

where $s_1$ and $s_2$ are two fixed points in the $z = z_0$ plane, and $\Delta \mathbf{s} \equiv s_1 - s_2$. We introduce a distribution function,

$$ F(s_1, s_2, t; \mathbf{R}_1, \mathbf{R}_2, T) d^2 s_1 d^2 s_2, $$

which describes the probability that two fluid particles, which are located at positions $\mathbf{R}_1$ and $\mathbf{R}_2$ at time $T$ will be located near positions $s_1$ and $s_2$ at time $t$. We note that the average $\langle a_a(\mathbf{r}_1(t), t) a_a(\mathbf{r}_2(t), t) \rangle$ can be considered a double average: (1) over the ensemble of all possible functions $a_a(s, t)$ and (2) over the ensemble of all possible paths $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. Our assumption that $\mathbf{v}$ and $\mathbf{a}$ are statistically independent implies that these two ensembles are independent. Hence the comoving average entering equation (26) is given by the convolution of $C_a(\Delta \mathbf{s}, t)$ with the distribution function $F$:

$$ \langle a_a(\mathbf{r}_1(t), t) a_a(\mathbf{r}_2(t), t) \rangle = \int \int C_a(\Delta \mathbf{s}, t) \langle a_a(s_1, t) a_a(s_2, t) \rangle F(s_1, s_2, t; \mathbf{R}_1, \mathbf{R}_2, T) d^2 s_1 d^2 s_2. $$

(28)

This expression can be further simplified by introducing the integral of the function $F$ over all $s_2$:

$$ f(\Delta \mathbf{s}, t; \Delta \mathbf{R}, T) \equiv \int F(s_2 + \Delta \mathbf{s}, s_2, t; \mathbf{R}_1, \mathbf{R}_2, T) d^2 s_2. $$

(29)

Here we have again used the assumption that all statistical quantities are independent of position in the $z = z_0$ plane (homogeneity), so that the function $f$ depends only on $\Delta \mathbf{R}$, and not on $\mathbf{R}_1$ and $\mathbf{R}_2$ separately. Note that $f(\Delta \mathbf{s}, t; \Delta \mathbf{R}, T) d^2(\Delta \mathbf{s})$ can be interpreted as the probability that two fluid particles have a separation $\Delta \mathbf{s}$ at time $t$, given the fact that they are separated by $\Delta \mathbf{R}$ at time $T$. With equation (29), the comoving correlation of $a_a$ can be written as

$$ \langle a_a(\mathbf{r}_1(t), t) a_a(\mathbf{r}_2(t), t) \rangle = \int C_a(\Delta \mathbf{s}, t) f(\Delta \mathbf{s}, t; \Delta \mathbf{R}, T) d^2(\Delta \mathbf{s}). $$

(30)
By inserting this expression into equation (26), we obtain an expression for the correlation function of the magnetic field:

$$C_d(\Delta R, T) = \tau \sqrt{2\pi} \int_0^\infty C_d(\Delta s, t) f(\Delta s, t; \Delta R, T) d^2(\Delta s) dt.$$  \hspace{1cm} (31)

In the next two sections we discuss what to insert for the functions $C_d(\Delta s, t) and f(\Delta s, t; \Delta R, T). We return to the computation of the magnetic power spectrum in § IIIc.

a) The Correlation Function $C_d$

The present model is based on the assumption that the velocity, $v(x, y, z_0, t)$, and the velocity gradient, $a(x, y, z_0, t) \equiv dv/dz$, can be approximated as independent statistical variables; we now have to specify their statistical properties. Since our goal is to demonstrate the existence of a cascade process, we will assume that $v(x, y, z_0, t)$ and $a(x, y, z_0, t)$ are smooth functions of $x$ and $y$; i.e., we assume that the power spectra of $v_x$ and $a_x$ are characterized by a single peak at wavenumber $k \sim l^{-1}$, with negligible power at wavenumbers $k \gg l^{-1}$. Specifically, we assume that the correlation function of $a_x$ is a Gaussian:

$$C_d(\Delta s, t) = C_d(0, t) \exp \left[-\frac{(\Delta s)^2}{2\ell^2}\right],$$  \hspace{1cm} (32)

where $\Delta s = |\Delta s|$. It follows that the power spectrum of $a_x$ is also a Gaussian, with very little power at high wavenumbers. Of course, if the velocity field described by $v(x, y, z_0, t)$ and $a(x, y, z_0, t)$ would contain small-scale structures, this would directly generate a corresponding small-scale structure in the magnetic field $b(x, y, z_0, t)$; however, this would not be the result of a cascade process. Therefore, the present approach is a conservative one: additional, small-scale structures in the velocity field can only speed up the generation of magnetic fine structures.

We now turn to the determination the normalization factor $C_d(0, t)$ in equation (32). The function $C_d(0, t)$ describes the magnitude of the velocity gradient $a_x$, and therefore determines the rate at which magnetic energy is fed into the system. To demonstrate this fact more clearly, let us consider the special case when $r_i(t)$ and $r_f(t)$ in equation (30) refer to the same fluid particle ($\Delta R = 0$). Then the distribution function $f$ is a $\delta$-function:

$$f(\Delta s, t; 0, T) = \delta(\Delta s),$$  \hspace{1cm} (33)

so that equation (30) yields

$$\langle a_x^2[r(t), t] \rangle = C_d(0, t) \equiv \langle a_x^2[s, t] \rangle ;$$  \hspace{1cm} (34)

i.e., the comoving average of $a_x^2$ along a path $r(t)$ in the $z = z_0$ plane is equal to the average at a fixed position $s$. Inserting this into equation (26) with $\Delta R = 0$, we obtain

$$C_d(0, t) \equiv \langle b_x^2(T) \rangle = \tau \sqrt{2\pi} \int_0^T C_d(0, t) dt ,$$  \hspace{1cm} (35)

which shows that $C_d(0, t)$ is proportional to the rate at which magnetic energy is fed into the system.

Unfortunately, the magnetic energy $\langle b_x^2(t) \rangle$ depends in an essential way on the three-dimensional structure of the field, and therefore it is difficult to predict the correct time dependence of either $\langle b_x^2(t) \rangle$ or $C_d(0, t)$. Different scaling laws have been proposed in the literature: for example, Parker (1983b) proposes that the free magnetic energy increases quadratically with time, whereas Sturrock and Uchida (1981) propose a linear increase with time. Parker's scaling law derives from the intuitive idea that the field lines in a braided magnetic field configuration must be wrapped around each other such that the projection of each field line onto a plane $z = constant$ has a length of the order of the distance traveled by its footpoints (see Fig. 1 of Parker 1983b). Since this distance increases linearly with time, $b_x$ and $b_y$ also increase more or less linearly, so that $\langle b_x^2(t) \rangle$ must increase quadratically with $t$. On the other hand, Sturrock and Uchida (1981) argue that random twisting of a single flux tube leads to a net twist angle which increases as the square root of time (random walk of the twist angle). This argument assumes that the same flux tube is twisted for many correlation times; i.e., the tube retains its identity for a long time. The Sturrock-Uchida model implies that the transverse field at the photospheric boundaries changes randomly from one correlation time to the next. Hence flux tubes loose their identity after about one correlation time, and so the Sturrock-Uchida scaling does not apply in the context of our uniform-field model.

To define the functions $C_d(0, t)$ and $\langle b_x^2(t) \rangle$ more precisely, we now present a somewhat different derivation of Parker's scaling law. We first recall that, according to equation (12), the magnetic energy is distributed uniformly in $z$, so that the quantity $\langle b_x^2(T) \rangle$ in equation (35) is independent of $z$. This implies that the function $C_d(0, t)$ is also independent of $z$, a result we will use in the derivation which follows. Integration of the relation $dv_x/dz = a_x$ along the path $r(z)$ of a field line yields

$$v_x[R_{top}, t] - v_x[R_{bot}, t] = \int_0^L a_x[r(z), t] dz ,$$  \hspace{1cm} (36)

where $R_{top}$ and $R_{bot}$ are the footpoints of the field line in the boundary planes $z = L$ and $z = 0$. Taking the square of equation (36), and averaging over all possible field configurations, we obtain

$$2\mu^2 = \int_0^L \int_0^L \langle a_x[r(z), t] a_x[r(z'), t] \rangle dz dz' ,$$  \hspace{1cm} (37)
where we used the fact that motions in the top and bottom plane are uncorrelated, and where \( u \) is the rms velocity at the boundary:

\[
u^2 \equiv \langle v_z^2(R_{bot}, t) \rangle = \langle v_z^2(R_{top}, t) \rangle.
\]

In a strongly braided field configuration, the projection \([x(z), y(z)]\) of the path followed by a field line is probably far from a straight line, making several twists and turns on its way from \( z = 0 \) to \( z = L \). Let \( \Delta z \) be the typical distance over which the direction of the projected path is coherent (\( \Delta z \) can be defined, e.g., as the correlation length of the function \( b_x(z) = dx/dz \) along a field line). It seems reasonable to assume that the correlation length \( \lambda \) of \( a_x(z) \) is of the same order as \( \Delta z \). The correlation length \( \lambda \) can be defined as the width of the autocorrelation function of \( a_x(z) \); for example, if the correlation function is Gaussian:

\[
\langle a_x[r(z), t]a_x[r(z'), t] \rangle = \langle a_x^2[r(z), t] \rangle \exp \left[ -\frac{(z' - z)^2}{2\lambda^2(t)} \right],
\]

where we recognize the fact that \( \lambda \) (and \( \Delta z \)) may be a function of time \( t \).

Inserting expression (39) into equation (37) and assuming \( l(t) \ll L \), we obtain

\[
\frac{\lambda^2}{2\nu^2 t} = \frac{\lambda^2(t)\sqrt{2\pi}}{2\nu^2 t} \int_0^L \langle a_x^2[r(z), t] \rangle dz = \lambda(t) L C_d(0, t) \sqrt{2\pi},
\]

where we have used the fact that \( \langle a_x^2[r(z), t] \rangle = C_d(0, t) \) is independent of \( z \) (see above). The same expression is also valid for non-Gaussian correlation functions, provided the correlation length \( \lambda(t) \) is defined as \( \lambda(t) = \lambda(t) L C_d(0, t, t) \).

To derive \( C_d(0, t) \) from equation (40), we require an expression for the correlation length \( \lambda(t) \). As discussed above, we assume that \( \lambda(t) \) is of the order of the length \( \Delta z(t) \) of a typical elementary braid in the field lines. This length decreases with time, since the number of braids \( N(t) \) along a given field line increases while the length \( L \) of the field line remains approximately constant. Following Parker (1983b), we suggest that \( N(t) \propto t \) because the field lines are interwoven, so that braids which are introduced into the magnetic field structure at one time are unlikely to be "unwoven" at a later time. The analogy with weaving further suggests that the time \( t_b \) for a new layer of braids to be formed is approximately equal to the time it takes for a magnetic footpoint to be displaced over a distance of the order of the correlation length \( l \) of the displacement pattern. This time scale can be estimated as follows. Measured over one correlation time \( \tau \), the footpoint displacements are of the order \( u \tau \), where \( u \) is the rms velocity. Therefore, if \( u \tau \sim l \), the "braiding" time \( t_b \) is of the order of the correlation time \( \tau \). However, we want to allow for the possibility that the displacements over one correlation time are small compared with the velocity correlation length, \( u \tau \ll l \), in which case the displacement grows more slowly, via random walk. Therefore, in general the number of correlation times necessary to make a displacement of the order of the correlation length \( l \) is approximately \( (l/u\tau)^2 \). It follows that \( t_b \) is given by

\[
t_b = \frac{l^2}{u^2 t}.
\]

Combining the above results, we find that the correlation length of \( a_x(z) \) is given by

\[
\lambda(t) \sim \frac{L}{(t/t_b)} \sim \frac{L^2}{u^2 t}.
\]

Inserting this into equation (40), we obtain for the normalization factor:

\[
C_d(0, t) = \frac{2qu^4 t}{\nu^2 L^2},
\]

where we have introduced a dimensionless constant \( q \sim 1 \), to account for the uncertainty in our estimate of the correlation length \( \lambda(t) \). From equations (35) and (43) it follows that the rate at which magnetic energy is fed into the system increases linearly with time:

\[
\frac{d}{dt} \langle b_z^2(t) \rangle = \frac{2qu^4 t^2}{\nu^2 L^2},
\]

and that the stored magnetic energy increases quadratically:

\[
\langle b_z^2(t) \rangle = \frac{qu^4 t^2}{\nu^2 L^2}.
\]

This implies that the amplitude of the transverse field increases linearly with time:

\[
b_x \sim b_y \sim \frac{1}{L} \frac{t}{t_b},
\]

consistent with the result of Parker (1983b).

b) The Distribution Function \( f \)

We now determine the distribution function \( f(\Delta s, \Delta t; \Delta R, T) \), which describes the statistics of the relative separation of fluid particles in a stochastic, two-dimensional, incompressible flow. At time \( t = T \), consider two fluid particles separated by a distance
\( \Delta R = |R_1 - R_2| \), which is assumed to be small compared with the correlation length \( l \) of the flow. At an earlier time, \( t < T \), the distance between the particles is given by \( \Delta r(\theta) = |r_1(\theta) - r_2(\theta)| \), where \( \theta = T - t \) is the time difference. As long as \( \Delta r(\theta) \ll l \), the temporal variations of \( \Delta r(\theta) \) are caused by the gradients in the velocity field; therefore, each correlation time \( \tau \) the separation \( \Delta r(\theta) \) changes by a factor which is independent of the magnitude of \( \Delta r \). If we define a logarithmic measure of the separation,

\[
\mu(\theta) = \ln \left( \frac{\Delta r(\theta)}{\Delta R} \right),
\]

the change in \( \Delta r(\theta) \) over one correlation time corresponds to an additive change \( \delta \mu \) which is independent of the current value of \( \mu \). Measured over many correlation times, the changes \( \delta \mu \) add up randomly, causing a “random walk” in \( \mu \)-space; the resulting distribution function \( f(\mu, \theta) \) is a Gaussian (see Chandrasekhar 1943):

\[
f(\mu, \theta) d\mu = \frac{1}{\sqrt{2\pi} \gamma \theta} \exp \left[ -\frac{(\mu - \beta \theta)^2}{2\gamma \theta} \right] d\mu,
\]

where \( \beta \) and \( \gamma \) are two parameters which describe the linear variation of the average and variance of \( \mu \) as function of the time difference \( \theta \):

\[
\langle \mu \rangle = \beta \theta,
\]

\[
\langle \mu - \langle \mu \rangle \rangle^2 = \gamma \theta.
\]

The parameters \( \beta \) and \( \gamma \) depend on the statistical properties of the velocity field, but they are independent of \( \Delta R \). Approximate expressions for \( \beta \) and \( \gamma \) in terms of the rms velocity \( u \), correlation length \( l \), and correlation time \( \tau \) are derived in the Appendix. We find that in the limit of small velocity gradients (\( u/l < 1 \)), \( \beta \) and \( \gamma \) are approximately equal and are given by

\[
\beta \approx \gamma \approx \frac{u^2 \tau}{2\pi l^2}.
\]

Note that \( \beta \) and \( \gamma \) are of order \( t_0^{-1} \), where \( t_0 \) is the “braiding” time defined in equation (41).

For large time differences \( \theta \) the separation \( \Delta r(\theta) \) becomes larger than the correlation length \( l \). When the particles are separated by more than a correlation length, their velocities are uncorrelated, and \( \Delta r \) increases as the square root of the time difference (“random walk”). In this regime expression (47) is no longer valid, but this does not lead to gross inaccuracies since by inserting equation (47) into equation (31) the distribution function \( f(\mu, \theta) \) is convolved with the Gaussian function \( C_\mu \) (see eq. [32]), which provides a cutoff for separations larger than the correlation length \( l \). Therefore equation (47) will be adequate for the present purpose.

c) The Magnetic Power Spectrum \( P_b \)

Inserting equations (32), (43), and (47) into equation (32), we obtain an expression for the comoving correlation function of \( a_x \); inserting this expression into equation (26), we find for the correlation function of the transverse field:

\[
C_b(\Delta R, T) = \frac{2\eta u^2}{L^2} \int_0^T (T - \theta) \left\{ \int_{-\infty}^{+\infty} \exp \left[ -\frac{(\mu - \beta \theta)^2}{2\gamma \theta} - \frac{(\Delta \varphi)^2}{2l^2} - \frac{1}{2} k^2 \varphi^2 \right] d\mu \right\} d\theta,
\]

where we have written \( \Delta \varphi = \Delta R e^\alpha \). Taking the Fourier transform of equation (51) we obtain the final result of our calculation, namely, the power spectrum of magnetic fluctuations (see eq. [16]):

\[
P_b(k, T) = \frac{qu^2}{\pi l^2} \int_0^T (T - \theta) \left\{ \int_{-\infty}^{+\infty} \exp \left[ -\frac{(\mu - \beta \theta)^2}{2\gamma \theta} - \frac{1}{2} k^2 \varphi^2 \right] d\mu \right\} d\theta.
\]

The power spectrum of current-density fluctuations is simply given by \( P_s = k^2 P_b \) (see eq. [20]).

IV. CASCADE PROCESS

Expression (52) was evaluated numerically for the case \( \gamma = \beta \); the results are shown in Figures 2 and 3, which illustrate the temporal behavior of the power spectrum. In Figure 2a and 2b the power per unit \( \ln k \) in the magnetic spectrum, \( P_b(\ln k, T) \equiv k^2 P_b(k, T) \), is plotted as function of \( \log_\varphi(\kappa l) \); Figure 2a shows the spectrum plotted on a linear vertical scale, and Figure 2b shows the same spectrum plotted on a logarithmic scale. The different curves in each graph correspond to different values of \( \beta T \). Note that the peak power always remains at the wavenumber \( k_\varphi = l^{-1} \), where the magnetic energy is put into the system. However, some of the energy is transferred toward higher wavenumbers, i.e., there is a cascade toward smaller length scales. This cascade process takes place on the time scale \( \beta^{-1} \), which is of the order of the “braiding” time, \( t_\varphi = l^2/\eta u^2 \) (see eq. [50]).

Figure 3 shows the power spectrum of current-density fluctuations, \( P_s(\ln k, T) = k^2 P_s(\ln k, T) \). Note that the wavenumber of maximum power, \( k_{\max}(T) \), increases exponentially with time \( T \). This implies that the currents are distributed on a very small spatial scale, with positive and negative current channels separated by distances of order \( k_{\max}^{-1} \). Note that the value of the power, \( P_s(\ln k_{\max}, T) \), also increases exponentially. Since \( P_s(\ln k, T) \) is nearly symmetric around its maximum, \( k_{\max}(T) \) can be defined as

\[
\ln (k_{\max}) = \frac{1}{\langle x^2(T) \rangle} \int_{-\infty}^{+\infty} \ln (k) P_s(\ln k, T) d(\ln k),
\]
where

\[ \langle \chi^2(T) \rangle = \int_{-\infty}^{\infty} P_b(\ln k, T)d(\ln k) . \]  (54)

The evaluation of equations (53) and (54) using equation (52) is tedious but straightforward; we present here only the final results:

\[ \langle \chi^2(T) \rangle = \frac{q u^2 \tau^2}{\sigma^2 L^2 (\beta + \gamma)^2} \left[ e^{2(\beta + \gamma)T} - 1 - 2(\beta + \gamma)T \right] , \]  (55)

\[ \ln (k_{\text{max}, t}) \approx \frac{1}{2} \left( 1 - c_1 + \ln 2 \right) - \frac{2\gamma + \beta}{\beta + \gamma} + (2\gamma + \beta)T , \]  (56)

where \( c_1 = 0.5772 \ldots \) is Euler's constant. Equation (56) is an approximation which is valid for \( 2(\beta + \gamma)T > 1 \), and which is in good agreement with the numerical results of Figure 3.
Inserting approximation (50) for the parameters $\beta$ and $\gamma$ into equations (55) and (56), we obtain

$$\langle \pi^2(T) \rangle \approx \frac{q}{8\pi L^2} e^{4\sqrt{2\pi T(b)}},$$  \hspace{1cm} (57)

$$k_{\text{max}}(T) \approx 0.39 l^{-1} e^{3\sqrt{2\pi T(b)}},$$  \hspace{1cm} (58)

where $t_b$ is the "braiding" time defined in equation (41), and $q$ is a dimensionless factor of order unity. Equation (57) shows that the rms electric current density, $j_{\text{rms}}(T) = (c/4\pi)B_0\langle \pi^2(T) \rangle^{1/2}$, increases exponentially with time; in contrast, the stored magnetic energy increases only quadratically with time (see eq. [45]). The e-folding time of $j_{\text{rms}}(T)$ is approximately $t_b/[2(2\pi)^{1/2}]$.

With increasing time $T$, the power at high wavenumbers soon becomes so large that magnetic diffusion and energy dissipation can no longer be neglected. Diffusion becomes important when the diffusion time at the peak of the current-density spectrum, $t_d = [r\eta^2_{\text{max}}(T)]^{-1}$, becomes of the order of the time scale $t_b$ of the cascade process ($\eta$ is the magnetic diffusivity). This occurs at a time $T_\eta$ given by

$$T_\eta \approx t_b \frac{\ln R_m}{6\pi},$$  \hspace{1cm} (59)

where $R_m \equiv \eta^2/t_b$ is the magnetic Reynolds number. Since $T_\eta$ depends logarithmically on the magnetic Reynolds number, we do not have to wait very long for reconnection to become important, even though $R_m$ is very large: assuming $R_m \sim 10^{10}$, a value typical of the solar corona, we obtain $T_\eta \approx 1.5t_b(\beta T_1 \approx 4)$. This suggests that dissipation becomes important after only a few braids have been formed.

At some time shortly after $T_1$ the system will reach a statistically stationary state, in which the rate of energy input into the spectrum matches the rate of dissipation. The energy input rate, which increases linearly with time in the initial stages of the cascade (see eq. [44]), probably saturates to a constant value. The precise shape of the magnetic and current-density spectra in the statistically stationary state cannot be determined with the present model, since effects of magnetic diffusion were neglected. However, it is clear that the spectrum will have a sharp cut off at a wavenumber $k_{\text{cutoff}} \sim l^{-1}R_m^{1/2}$, since for wavenumbers $k > k_{\text{cutoff}}$ the diffusion time $t_d(k)$ is shorter than the time scale $t_b(\approx l^2/\eta^2)$ for energy transfer via the cascade process. For $k \ll l^{-1}R_m^{1/2}$ the diffusion time $t_d(k)$ is much larger than $t_b$.

Since there is only one time scale involved in the cascade process (namely the braiding time $t_b$), the time scale for energy transfer from wavenumber $ln(k)$ to wavenumber $ln(k) + 1$ is independent of $k$ (see Paper I). This implies that, for $l^{-1} \ll k \ll l^{-1}R_m^{1/2}$, the power spectrum $P_d(ln(k))$ will also be independent of $k$, since the energy flow through the spectrum must be constant in the stationary state. Since $P_d(ln(k)) = k^2P_{\beta}(ln(k))$, the current-density spectrum in the stationary regime will have a $k^2$ dependence, and therefore the maximum power will occur near the cutoff wavenumber. It follows that the rms current density in the stationary state is given by

$$j_{\text{rms}} \sim \frac{c}{4\pi} \frac{B_0}{L} R_m^{1/2},$$  \hspace{1cm} (60)

which is roughly a factor $R_m^{1/2}$ larger than the estimate of equation (4). Since $R_m \sim 10^{10}$, the enhancement is approximately the factor $2 \times 10^5$ that we needed (see § 1).

Fig. 3.—Power spectrum of electric current density fluctuations, $P_a(ln k, T)$, plotted as function of wavenumber for six different times ($\beta T = 1, 2, 3, 4, 5, 6$). Note the exponential increase with time of the peak power.
An estimate of the heating rate $E_H$ in the stationary regime can be obtained as follows. We assume that the saturation of the energy input rate occurs at a time $T \sim T_v$ so that $E_H$ is of order of the input rate (44) evaluated at time $T_v$. Using equations (59) and (41), we obtain from equation (44)

$$E_H \approx \frac{B_0^2}{8\pi} \frac{4\rho u^2 \tau^2 T_i}{L^2} \approx \frac{B_0^2}{8\pi} \frac{2\rho u^2 \tau \ln R_m}{3L^2 \sqrt{2\pi}},$$

(61)

where an extra factor of 2 has been included to account for the contribution of $\langle h^2 \rangle$ to the magnetic energy. It is interesting to note that the velocity field enters in expression (61) primarily through the product $u^2 \tau$, which is proportional to the effective diffusion constant:

$$D = \frac{1}{2} u^2 \tau \sqrt{2\pi}.$$  

(62)

Equation (62) is derived by comparing the rms displacement predicted by the diffusion model, $\langle x^2 \rangle = 2Dt$, with the rms displacement computed from

$$x(t) = \int_0^t v_x(t')dt';$$

as before, the correlation function of $v_x$ is assumed to be Gaussian:

$$\langle v_x(t + \Delta t)v_x(t) \rangle = u^2 \exp\left(-\frac{\Delta t^2}{2\tau}\right).$$

Assuming $R_m \sim 10^{10}$ and $q \sim 1$, the heating rate can be written as

$$E_H \approx 0.19 B_0^2 \frac{D}{L^2}. \quad (63)$$

The magnetic energy stored in the transverse field can be estimated in a similar way from equation (45):

$$\langle B_x^2 \rangle \approx \frac{B_0^2}{8\pi} \frac{q(\ln R_m)^2 L^2}{3\pi L^2} \approx 0.19 B_0^2 \frac{L^2}{L^2}. \quad (64)$$

Note that the "free" magnetic energy, $\langle B_x^2 \rangle/8\pi$, is small compared with the energy $B_0^2/8\pi$ of the background potential field, since by assumption $l \ll L$.

Equation (63) is based on the assumption that magnetic diffusion and reconnection limit the energy input rate to its value at time $T = T_v$. However, it should be noted that the magnetic energy at wavenumbers $kL > 10^5$, which is in principle available for dissipation, is only a small fraction of the total magnetic energy at $T = T_v$ (see the curve for $\beta T = 4$ in Fig. 2). Therefore, a balance between energy input and energy dissipation is not reached until some time $T_2 > T_v$. Since the energy input rate will continue to increase during the period $T_v < T < T_2$, the final dissipation rate $E_H$ will probably be larger than that given by equation (63), perhaps by as much as a factor of 10. Unfortunately, the precise value of $E_H$ cannot be computed with the present model, since diffusion effects were neglected.

V. DISCUSSION

We now discuss the application of the cascade model to the problem of coronal heating. At first sight, the model seems able to resolve the difficulty pointed out in the Introduction, namely, that the electric current densities needed to explain the observed energy losses of coronal loops are a factor $\sim 2 \times 10^5$ larger than the current densities produced by simple twisting. A more precise comparison with solar observations is complicated by the fact that the photospheric velocity field cannot be characterized by a single correlation length $l$: both granulation and supergranulation contribute to the effective diffusivity $D$ which enters in the expression for the heating rate (eq. [63]). Furthermore, observations show that at the photospheric level the vertical component of the magnetic field is highly inhomogeneous, whereas the present model neglects this inhomogeneity and assumes that $B_z(x, y)$ is nearly constant.

First, ignoring these complications, we obtain an estimate of the heating rate by using the diffusion constant derived from the observed rate of spreading of active regions. De Vore et al. (1985) measured $D$ by comparing the flux distributions of a number of active regions on successive solar rotations with predictions based on a flux-transport model. In addition to turbulent diffusion of magnetic flux (see Leighton 1964), the model includes effects of differential solar rotation and meridional flow. With a few exceptions, $D$ was found to be in the range $150-425$ km$^2$s$^{-1}$, in agreement with earlier measurements by Mosher (1977). Adopting the geometric mean value, $D = 250$ km$^2$s$^{-1}$, we obtain from equation (63):

$$E_H \sim 5 \times 10^{11} B_0^2 L^{-2}, \quad (65)$$

where $B_0$ is in gauss and $L$ is in centimeters. For $B = 100$ G and $L = 10^{10}$ cm we obtain

$$E_H \sim 5 \times 10^{-5} \text{ergs cm}^{-3} \text{s}^{-1},$$

which is about a factor 40 smaller than the observed coronal heating rate (eq. [2]). This suggests that the random motions which give rise to the spreading of active regions are not sufficient to explain the heating of active loops.

This problem is alleviated somewhat by the fact that magnetic flux is concentrated in a "network" at the boundaries between...
supergranular cells, so that the field strength in the photosphere is larger than in the corona (Sturrock and Uchida 1981). Figure 4 illustrates how this affects the diffusion constant. Since the magnetic field in the network flares out with height above the solar surface, field line motions induced by granulation at the base of the network appear amplified at larger height. For example, rotation at one of the footpoints of the loop indicated by the dashed lines in Figure 4 would lead to displacements in the lower corona which are larger than the corresponding displacements in the photosphere. As a result, the horizontal velocity is increased by a factor \( \sim (B_{\text{phot}}/B_0)^{1/2} \), where \( B_{\text{phot}} \) is the average field strength in the photospheric network and \( B_0 \) is the coronal field strength. Since \( D \) is quadratic in the velocity (see eq. [62]), the effective diffusion constant for coronal heating is given by

\[
D = B_{\text{phot}}^2 / B_0 \cdot (\frac{u}{c})^2 T \cdot (2\pi)^{1/2}
\]

where \( D_{\text{phot}} = \frac{1}{\pi} u^2 (2\pi)^{1/2} \) is the diffusion constant in the photosphere and \( u \) is the velocity of the magnetic features in the photosphere.

Since the magnetic field in the photosphere is concentrated in the regions between granules, \( u \) corresponds to the typical velocity with which the intergranular lanes move across the solar surface as a result of the growth and decay of granules (note that \( u \) may be significantly less than the flow velocities inside granules, which are several kilometers per second). Assuming \( u \approx 0.5 \) km s\(^{-1}\) and \( \tau \approx 5 \) minutes, we estimate \( D_{\text{phot}} \approx 100 \) km\(^2\) s\(^{-1}\), which yields

\[
E_H \sim 2 \times 10^{11} B_{\text{phot}} B_0 L^{-2}.
\]

The appropriate value of \( B_{\text{phot}} \) in active regions is difficult to estimate, since it depends on the surface area over which one is willing to average. An upper limit on \( B_{\text{phot}} \) is provided by field strength measurements based on the line-ratio method (see Stenflo 1984, and references therein), which suggest that the field strength in unresolved photospheric flux tubes is in the range 1000–1500 G. Assuming \( B_{\text{phot}} = 1000 \) G, \( B_0 = 100 \) G, and \( L = 10^{10} \) cm, we find \( E_H \sim 2 \times 10^{-4} \) ergs cm\(^{-3}\) s\(^{-1}\), which is still a factor 10 below the value of equation (2) (a more realistic value for \( B_{\text{phot}} \) would probably be in the range 200–500 G).

We conclude that the heating rate predicted by expression (63) is too low by at least a factor 10. There are several possible reasons for this discrepancy:

1. There are major uncertainties in our theoretical estimate of the heating rate. The numerical constant in expression (63) is based on our assumption that the rate of energy input in the statistically stationary regime is equal to the input rate at time \( T = T_1 \), when diffusion effects first become important; as pointed out in § IV this is probably an underestimate. Since our present analysis neglects magnetic diffusion, we are unable to predict at precisely which level the energy input rate will saturate. It is conceivable that the saturation level is significantly higher than the level at time \( T = T_1 \). Also the numerical constant \( q \) introduced in equation (43) may be significantly different from unity; the present two-dimensional model does not allow a more precise determination of its value.

2. It is possible that the random motions of photospheric magnetic fields are more vigorous than assumed above, in particular in growing active regions. We estimated that granulation causes random motions characterized by \( D_{\text{phot}} \approx 100 \) km\(^2\) s\(^{-1}\), but actual measurements of this quantity do not seem to be available. If observations of random motions within active regions should indicate a significantly larger diffusion constant, our estimate of the heating rate would increase accordingly.

3. Finally, it is possible that random motions of the photospheric footpoints are not the primary cause of coronal heating in active regions; periodic motions, in particular motions associated with Alfvén waves, may be a more important source of energy (see Hollweg 1981, 1984; Jonson 1982; however, see Spruit 1984). Unlike random motions, periodic footpoint motions do not lead to a buildup of field-aligned electric currents. The dissipation of Alfvén waves might involve processes like phase mixing, resulting from density inhomogeneities in the corona (see Jonson 1982; Heyvaerts and Priest 1983; Sakurai and Granik 1984). However, even if the random motions do not provide the bulk of the energy, they may still play an important role, since the propagation of Alfvén waves along the diverging field lines of a braided magnetic field will probably lead to a certain amount of phase mixing, even in the absence of density variations.
A direct comparison of wave-heating and current-heating models is difficult, since the relative amplitudes of random and periodic motions of magnetic fields are not well known. Time sequences of magnetograms with high spatial resolution, like those obtainable with the High Resolution Solar Observatory, are required to resolve some of these issues. Such observations would be especially useful if simultaneous observations of the X-ray corona were available.

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APPENDIX

In the following we derive approximate expressions for the parameters \( \beta \) and \( \gamma \), which describe the statistics of the separation of fluid particles in a stochastic, two-dimensional, incompressible flow (see eq. [47]). Let \([\Delta x(\theta), \Delta y(\theta)]\) be the relative separation of two fluid particles as function of the time difference \( \theta = T - t \), and let the separation at time \( T \) be given by \( \Delta x(0) = \Delta R \) and \( \Delta y(0) = 0 \). As long as the distance between the fluid particles, \( \Delta r = (\Delta x^2 + \Delta y^2)^{1/2} \), is small compared with the velocity correlation length \( \ell \), the variations of \([\Delta x(\theta), \Delta y(\theta)]\) are determined by the gradients in the velocity field:

\[
\frac{d(\Delta x)}{d\theta} = -\frac{\partial v'_x}{\partial x_j} \Delta x_j ,
\]

where we use tensor notation \((i = 1, 2)\). The solution of equation (A1) can be written as an infinite series:

\[
\Delta x(\theta) = \left( \delta_{ij} - \oint_0^\theta \frac{\partial v'_{x_i}}{\partial x_j} d\theta' + \oint_0^\theta \oint_0^{\theta'} \left( \frac{\partial v'_{x_i}}{\partial x_j} + \frac{\partial v'_{y_i}}{\partial x_j} \right) d\theta'' d\theta' + \cdots \right) \Delta x(0) ,
\]

(A2)

where \( v'_i \) denotes the velocity at time \( \theta = \theta' \).

We now consider the limit of small velocity gradients; i.e., we assume that \( |\partial v'_i/\partial x_j| \ll \tau^{-1} \), where \( \tau \) is the correlation time. Then only the first few terms in equation (A2) need to be considered. Retaining terms up to second order in the velocity gradients, we obtain for the quantity \( \mu \) defined in equation (44):

\[
\mu = \frac{1}{2} \ln \left[ \frac{\Delta x^2(\theta) + \Delta y^2(\theta)}{\Delta R^2} \right] = -\oint_0^\theta \frac{\partial v'_x}{\partial x} d\theta + \oint_0^\theta \oint_0^{\theta'} \left( \frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial x} \right) d\theta'' d\theta' + \frac{1}{2} \oint_0^\theta \oint_0^{\theta'} \left( -\frac{\partial v'_x}{\partial x} \frac{\partial v'_y}{\partial x} \right) d\theta'' d\theta' + \cdots ,
\]

(A3)

and for its square:

\[
\mu^2 = \oint_0^\theta \oint_0^{\theta'} \frac{\partial v'_x}{\partial x} \frac{\partial v'_y}{\partial x} d\theta'' d\theta' + \cdots ,
\]

(A4)

where the dots denote higher order terms.

To evaluate the averages of \( \mu \) and \( \mu^2 \), we assume that the temporal correlation function of \( \partial v'_x/\partial x \) is Gaussian:

\[
\langle \frac{\partial v'_x}{\partial x} \frac{\partial v'_x}{\partial x} \rangle = \left( \frac{\partial v'_x}{\partial x} \right)^2 \exp \left[ -\frac{(\theta - \theta')^2}{2\tau^2} \right] .
\]

(A5)

The spatial correlation function of \( v_x \) is also assumed to be Gaussian,

\[
\langle v_x(x + \Delta x)v_x(x) \rangle = u^2 \exp \left[ -\frac{(\Delta x)^2}{2\ell^2} \right] ,
\]

(A6)

so that the variance of \( \partial v'_x/\partial x \) is given by

\[
\langle \left( \frac{\partial v'_x}{\partial x} \right)^2 \rangle = -\langle v_x \left( \frac{\partial^2 v_x}{\partial x^2} \right) \rangle = -\left[ \frac{\partial^2}{\partial (\Delta x)^2} \langle v_x(x)v_x(x + \Delta x) \rangle \right]_{\Delta x = 0} = \frac{u^2}{\ell^2} .
\]

(A7)

Furthermore, since the statistical properties of the flow are assumed to be isotropic, the correlation function of \( \partial v'_x/\partial x \) is given by (see Tennekes and Lumley 1972):

\[
\langle \frac{\partial v'_x}{\partial x} \frac{\partial v'_y}{\partial x} \rangle = \frac{3}{\ell^2} \left( \frac{\partial v'_x}{\partial x} \frac{\partial v'_y}{\partial x} \right) .
\]

(A8)

Finally, because of the assumption of statistical homogeneity, the correlation between \( \partial v'_x/\partial y \) and \( \partial v'_y/\partial x \) is given by

\[
\langle \frac{\partial v'_x}{\partial y} \frac{\partial v'_y}{\partial x} \rangle = -\langle v_x \frac{\partial^2 v_x}{\partial x \partial y} \rangle = +\langle v_x \frac{\partial^2 v_x}{\partial x^2} \rangle = -\langle \frac{\partial v'_x}{\partial y} \frac{\partial v'_x}{\partial x} \rangle .
\]

(A9)
Inserting these expressions into equations (A3) and (A4), and assuming that $\theta \gg \tau$, we find that $\langle \mu \rangle$ and $\langle [\mu - \langle \mu \rangle]^2 \rangle$ are approximately equal:

$$ \langle \mu \rangle \approx \langle [\mu - \langle \mu \rangle]^2 \rangle \approx \int_0^\pi \int_0^{2\pi} \left( \frac{\partial v_x}{\partial x} \frac{\partial v_x}{\partial x} \right) d\theta d\phi \approx \frac{u^2 \theta r \sqrt{2\pi}}{l^2} , $$

which leads directly to equation (50).

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