POSITION PAPER:

Phase Mixing of Alfvén Waves

Takashi Sakurai
Department of Astronomy, University of Tokyo

1. Introduction

In the theory of coronal heating by means of Alfvén waves, a process called 'the phase mixing' gives a very natural and efficient way of dissipating the wave energy. In the following I will describe briefly the basic scheme of phase mixing and its consequences (Rae and Roberts, 1981; Heyvaerts and Priest, 1983; Sakurai and Granik, 1984). The phase mixing is a process in which the Alfvén oscillations of adjacent field lines become more and more out of phase when the Alfvén speed is non-uniform, thereby creating smaller and smaller scales which accelerate the dissipative processes. This process works both in the case of standing waves in coronal loops and in the case of propagating waves in the open field regions. Here I will concentrate on the case of standing oscillations in magnetic loops.

2. Basic Equations

As a simple model I will assume a uniform magnetic field $B=(0,0,B)$ between the two endplates at $z=0$ and at $z=l$. I will also assume that the effects of gas pressure and gravity are negligible, and the density $\rho$ is a function of $x$. Therefore the Alfvén speed $V_A(x)$ is non-uniform, being a function of $x$. The $z$-axis is identified as along the loop axis, $x$ is the radial distance from the loop axis, and the $y$-direction corresponds to the azimuthal direction around the loop axis. The resonance frequency of each field line is $\omega_A(x)=\pi V_A(x)/l$ (fundamental resonance).

Suppose the loop is shaken at its footpoints by the driver velocity field

$$v_0 = v_0 \exp(i\omega t + ik_\perp \cdot r_\perp), \quad k_\perp=(k_x,k_y) . \quad (1)$$

Then the loop resonates to this driver when $\omega=\omega_A$. To give a rough idea, the resonance period is $2\pi/\omega=100s$ if $V_A=1000km/s$ and $l=5\times10^4km$. The amplitude of the oscillation at $z=l/2$ is

$$v_x(z=l/2) = \exp(i\omega t + ik_y y) V(x) \quad (2)$$

and $V(x)$ satisfies the following equation:

$$\begin{align*}
(cV')' - ck_y^2 V - \frac{i(\eta+\nu)}{\omega_A} V &= \frac{2}{\pi} k_y (k_\perp \times v_0)_z \\
\text{(wave terms)} &\quad \text{(dissipation)} \quad \text{(driver)}
\end{align*} \quad (3)$$

where $\eta$ = magnetic diffusivity, $\nu$ = viscosity, prime (′) denotes $d/dr$, and $c=\omega_0/\omega_0(x)-1$ vanishes at the resonance point $\omega_A(x)=\omega$. By comparing the wave terms with the dissipative term, the latter becomes important when the length scale is smaller than $l_0$:

$$l_0 = \left(\frac{\eta+\nu}{\omega_A^2} \right)^{1/2} \quad (4)$$
3. Evolution of Wave Profile: Phase Mixing

In the following I will use a particular profile of $\omega_A(x)$ as shown in Fig.1. First let us neglect dissipation. Then (3) is singular at the resonance point $x=x_A(\omega)$ and $V$ diverges as $V \sim \ln |x-x_A|$. This fact leads to the idea of resonant absorption (Ionson, 1978). However this divergence, or a very large amplitude of waves localized at the resonance point, is a consequence of the assumed driver which has a monochromatic (fixed) frequency $\omega$. In a realistic case the driver has a broad power spectrum, and the time evolution of the wave amplitude is given by the superposition of such singular eigenfunctions $\nu_\omega(x)$. That is,

$$V(x,t) \sim \int A(\omega) \nu_\omega(x) \exp(i\omega t) d\omega.$$  \hspace{1cm} (5)

Following the same treatment as in the Landau damping of plasma oscillations, the asymptotic behavior of (5) is found as

$$V(x,\text{large } t) \sim \alpha \exp(i\omega_s t - \gamma_s t - |k_y x|) + \beta \exp(i\omega_\omega(x) t)/t$$ \hspace{1cm} (6)

where $\alpha$ and $\beta$ are slowly-varying function of $x$ which depend on the initial conditions. Correspondingly $\nu_y$ which is proportional to $dV/dx$ is given as

$$\nu_y \sim V \sim -\alpha k_y \exp(i\omega_s t - \gamma_s t - |k_y x|) + i\omega_\omega(x) \beta \exp(i\omega_\omega(x) t).$$ \hspace{1cm} (7)

The first terms in (6), (7) represent the surface waves with frequency $\omega_s$,

$$\omega_s = \omega_M + \omega_m - \frac{1}{2} |k_y|$$ \hspace{1cm} (8)

which decay at the rate $\gamma_s$,

$$\gamma_s = \frac{\pi}{2} k_y |x_A| \omega_M - \omega_m - 1.$$ \hspace{1cm} (9)

This "dissipationless damping" means that the wave energy is transferred from the surface waves to the body waves, which are the second terms in (6), (7). The body waves oscillate mainly in the $y$ direction ($|\nu_x| \ll |\nu_y|$), and they undergo a phase mixing. Namely, because of the $x$-dependent
frequency of body waves, the effective \( x \)-wavenumber

\[
k_{x,\text{eff}} = \frac{\partial}{\partial x} (\omega_\lambda (x,t) = \omega_\lambda t)
\]

becomes larger and larger as time proceeds. This process continues until the effect of dissipation becomes important when the length scales of waves are as small as \( l_0 \) given by (4). I will therefore define the phase mixing time \( \tau_{\text{mix}} \) by \( k_{x,\text{eff}} = 1/l_0 \), that is,

\[
\tau_{\text{mix}} = \frac{1}{l_0 \omega_\lambda} = (\eta + \nu)^{-1/3} \omega_\lambda^{-2/3}
\]

4. Energy Dissipation Rate and Wave Amplitude

Based on the argument given above, I can estimate the heating rate expected for the phase mixing of Alfven waves (see Fig. 2). If the loop is initially at rest and a driver \( u_b \) with frequency \( \omega \) and duration \( \tau_1 \) is applied on the footpoints at \( t = 0 \), the velocity field in the loop grows linearly in time (because of resonance) like \( u_b \sim u_b \omega t \), and after \( t = \tau_1 \) the oscillation keeps its amplitude \( u_b \sim u_b \omega \tau_1 \) until \( t = \tau_{\text{mix}} \). After \( t = \tau_{\text{mix}} \) the wave decays so that the life time of the wave excited by a single driving pulse is of the order of \( \tau_{\text{mix}} \). If the driving pulses apply randomly with a mean interval \( \tau_2 \), the expected mean square amplitude is

\[
\langle u_b^2 \rangle = \langle u_b \rangle^2 \frac{\tau_{\text{mix}}}{\tau_2}
\]

which is proportional to \( (\eta + \nu)^{-1/3} \). Similarly, the average dissipation rate of wave energy is

\[
\epsilon_H \sim \rho \langle \eta + \nu \rangle \langle \frac{\partial u_x}{\partial x} \rangle^2 \sim \rho \langle \eta + \nu \rangle \frac{\langle u_b^2 \rangle}{l_0^2} \sim \frac{B^2}{8\pi} \frac{\langle u_b^2 \rangle}{l_0^2} \frac{\tau_{\text{mix}}}{\tau_2}
\]

which does not depend on \( \eta \) and \( \nu \) (Ionson, 1982).

For example by equating (13) with the scaling law of Rosner et al. (1978) which requires

\[
\epsilon_H = 4 \times 10^{-3} \left( \frac{T_{\text{max}}}{10^6 \text{K}} \right)^{3/2} \left( \frac{L}{10^3 \text{cm}} \right)^{-2} \text{ erg cm}^{-3} \text{s}^{-1}
\]

the temperature \( T_{\text{max}} \) at the loop apex is found as

\[
\left( \frac{T_{\text{max}}}{10^6 \text{K}} \right)^{3/2} = 0.5 \times \left( \frac{B}{1 \text{G}} \right)^2 \left( \frac{u_b}{1 \text{km s}^{-1}} \right)^2 \frac{\tau_{\text{mix}}}{\tau_2}
\]

which is in a million-degree range. Further if \( a = 100 \text{km} \), \( 2\pi / a = 100 \text{ s} \), \( \Delta \omega / a \sim 1 \) and classical values of \( \eta \) and \( \nu \) are used, it is found that \( \tau_{\text{mix}} = 2 \times 10^4 \text{s} \sim 5 \text{ hours} \), \( l_0 = 100 \text{ m} \), and the average velocity amplitude \( \sqrt{\langle u_b^2 \rangle} \sim 60 \text{ km s}^{-1} \). Apparently \( T_{\text{max}} \) is rather long and \( \sqrt{\langle u_b^2 \rangle} \) is too large. It is unlikely that the wave oscillates stably with such a large velocity shear (60 km s\(^{-1}\) in a 100 m scale), and Heyvaerts and Priest (1983) suggested that shear instabilities will set in much earlier and promote the dissipation. A more detailed model is needed to make a quantitative comparison with the observable effects. One of the fundamental (qualitative) characteristics of the phase mixing is that the wave velocity field \( v \) is perpendicular to both \( B \) and \( \nabla \omega_\lambda \). This can be an detectable effect in the high resolution observations by SST.
References


Fig. 2
R. Stein: 1) What makes energy flow into resonance region? What is the physical process? 2) Does phase mixing occur only for the Alfvén mode or does it also occur for other modes?

T. Sakurai: 1) Each field line picks up energy from the corresponding resonant frequency in the driver power spectrum, so that no localized (thin) resonance region is expected to arise. 2) If the beta value \((8\pi\rho/B^2)\) is finite, another singular point appears in the equation, but I don't know whether it leads to phase mixing or not. I think the phase mixing takes place in Alfvén waves because each field line can oscillate independently as an Alfvén wave. For other modes I am not sure.

H.C. Spruit: You gave an estimate of "hours" for the time needed for the wave structure to be scrambled by phase mixing. This means that the wave has to bounce a large number of times between the endplates. As Joe Hollweg has discussed, the reflection at each bounce is far from perfect (90% may be), so that, the wave has only 10 or so bounces available. Doesn't this reduce the importance of phase mixing for coronal heating?

T. Sakurai: The phase mixing time of the order of hours is based on the classical resistivity/viscosity. If, for example, shear instabilities or tearing instabilities set in during the phase mixing process (as was suggested by Heyvaerts & Priest), the phase mixing time could be shortened substantially. As to the argument on the escape of waves from a resonant cavity, Hollweg's model for example assumes that the wave modes are Alfvén modes in the coronal part as well as in the lower atmosphere. On the other hand if the driving motion is powerful enough not to be disturbed by the coronal oscillations, the concept of transmission coefficients might be irrelevant and the phase mixing will proceed to the end no matter how long it takes.

J.H. Thomas: HenK's (Dr. Spruit's) question assumes that phase mixing only occurs for resonant modes in loops. But even when there is no reflection from the opposite footpoint, if you excite the loop continually you will produce phase mixing as the waves in adjacent regions propagate with different speeds (i.e., the case where \(\omega = \alpha k\) and \(\omega\) is fixed so \(k\) is different where \(a\) is different).