CONTRIBUTION FUNCTIONS FOR ZEEMAN-SPLIT LINES, 
AND LINE FORMATION IN PHOTOSPHERIC FACULAE

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ABSTRACT

The transfer of polarized light in an inhomogeneous stellar atmosphere, and the formation of magnetically sensitive spectral lines, are discussed. A new method for the solution of the transfer equations is proposed. The method gives a natural definition of the contribution functions for Stokes' parameters, i.e. functions describing the contributions from different parts along the line-of-sight (LOS). The formalism includes all magneto-optical effects, and allows for an arbitrary variation of magnetic field, velocity field, temperature, density, etc. along the LOS.

To illustrate the method I discuss the formation of FeI λ 5250.2 in photospheric faculae. A potential-field model of a facular element is presented, and spectral profiles and contribution functions are computed for the Stokes parameters I, Q, and V.

1. INTRODUCTION

The use of spectral lines to determine the thermodynamic and magnetic structure of the solar atmosphere often requires knowledge of the position along the LOS from where the light, observed at a certain wavelength, originates. For non-magnetic lines, the mean height-of-formation can be computed from a model of the atmosphere, since the emergent intensity at each wavelength can be expressed as an integral over height h of the contribution function:

$$ C_I(h) = \frac{1}{\mu} \{ \kappa_c(h) B(h) + \kappa_\ell(h) S(h) \} \exp[-\tau(h)]. \tag{1} $$

Here B(h) is the Planck function, S(h) is the line source function, \( \kappa_c(h) \) and \( \kappa_\ell(h) \) are the line- and continuum opacity (per unit length), \( \tau(h) \) is the sum of line- and continuum optical depth as measured along the LOS, and \( \mu = \cos \theta \) defines the angle with the vertical direction.
It is not obvious how to extend this definition to magnetically sensitive lines. In this case the variations of the four Stokes parameters $I$, $Q$, $U$, and $V$ along the LOS are described by four coupled differential equations (e.g., Landi Degl'Innocenti and Landi Degl'Innocenti, 1972). In general these couplings seem to prevent us from writing the emergent intensities as simple integrals over height (for an iterative solution see Staude, 1969). Analytical solutions to the transfer equations can be obtained only for simple models of the line-formation process, in which the magnetic field is either weak (Landi Degl'Innocenti and Landi Degl'Innocenti, 1973), or nearly constant with height (Unno, 1956; Kjeldseth-Moe, 1968; Landi Degl'Innocenti and Landi Degl'Innocenti, 1977). These models do not give a good description of the strong, inhomogeneous magnetic field present on the sun. For more realistic models of magnetic- and atmospheric structure the radiative transfer equations must be solved by numerical integration (Beckers, 1969; Stenflo, 1971; Wittmann, 1974, 1977; Landi Degl'Innocenti, 1976). A disadvantage of the numerical method is, however, that all information about contribution functions and height-of-formation is lost.

In Sect. 2 I show how the four coupled transfer-equations may be formally integrated, leading to a general definition of the contribution function for each Stokes parameter. The solution is not restricted to certain magnetic- or atmospheric models, and includes magneto-optical effects.

In Sect. 3 I discuss the application of my method to a model of photospheric faculae.

2. FORMAL SOLUTION OF TRANSFER EQUATIONS

The tranport of polarized light in a magnetically sensitive spectral line is described by the matrix equation:

$$\frac{d}{d\tau_c} \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} (1+\eta_I) & \eta_Q & \eta_U & \eta_V \\ \eta_Q & (1+\eta_I) & \rho_V - \rho_U & \rho_Q \\ \eta_U & \rho_V & (1+\eta_I) & \rho_Q \\ \eta_V & \rho_U - \rho_Q & (1+\eta_I) & \rho_V \end{bmatrix} \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} - \begin{bmatrix} B+\eta_I s \\ \eta_Q s \\ \eta_U s \\ \eta_V s \end{bmatrix}, \quad (2)$$

where $I(\tau_c)$, $Q(\tau_c)$, $U(\tau_c)$, and $V(\tau_c)$ are the Stokes parameters at a certain wavelength in the line, and $\tau_c$ is the continuum optical depth as measured along the line of sight. The quantities $\eta_I$, $\eta_Q$, $\eta_U$, and $\eta_V$ are ratios of the line opacity in each Stokes parameter and the continuum opacity, while $\rho_Q$, $\rho_U$, and $\rho_V$ describe
magneto-optical effects. The $\eta$'s and $\rho$'s are functions of continuum optical depth $\tau_c$, and of wavelength in the line; they can be computed if the variations of magnetic field, temperature, density, etc. along the line of sight are known (e.g. Landi Degl'Innocenti, 1976). The emergent intensities $I(0), Q(0), U(0),$ and $V(0)$ can be computed by integrating Eq.(2) numerically. The integration starts at a large optical depth, and proceeds towards smaller depth.

The symmetries and anti-symmetries in the real-valued $4 \times 4$-matrix of Eq.(1) represent a certain redundancy in the formalism. This redundancy is removed by using Jones-calculus, in which the polarization state of the light, and the absorption- and emission properties of the medium, are described by complex $2 \times 2$ matrices. I define the matrices $D(\tau_c), A(\tau_c)$ and $F(\tau_c)$ by:

$$D = \frac{1}{2} \begin{bmatrix} I + Q & U + iV \\ U - iV & I - Q \end{bmatrix},$$

$$A = \frac{1}{2} \begin{bmatrix} 1 + \eta_I - \alpha_v & \alpha_u + i\alpha_v \\ \alpha_u - i\alpha_v & 1 + \eta_I - \alpha_Q \end{bmatrix},$$

and

$$F = \frac{1}{2} \begin{bmatrix} B + (\eta_I + \eta_Q)S & (\eta_u + i\eta_v)S \\ (\eta_u - i\eta_v)S & B + (\eta_I - \eta_Q)S \end{bmatrix},$$

where the complex quantities $\alpha$ are defined by:

$$\alpha_Q = \eta_Q - i\rho_Q,$$

$$\alpha_u = \eta_u - i\rho_u,$$

$$\alpha_v = \eta_v - i\rho_v.$$

Note, that the matrices $D$ and $F$ are hermitian, while $A$ is not. $D(\tau_c)$ describes the polarization state of the light. $A(\tau_c)$ is the Jones matrix (per unit optical depth in the continuum), and describes absorption-, polarization- and birefringent properties of the medium. $F(\tau_c)$ is the source-function matrix. The transport equations (2) can now be written in the form:

$$\frac{dD}{d\tau_c} = A \ D + D \ A^\dagger - F,$$

where $^\dagger$ denotes a hermitian conjugate (complex conjugation plus transposition).
To solve Eq. (7) I introduce the Jones matrix $T(\tau_c)$ for the entire atmosphere above the level $\tau_c$, which can be found from the differential equation:

$$\frac{dT}{d\tau_c} = A(\tau_c) \ T(\tau_c),$$

and the boundary condition that $T(0)$ at $\tau_c=0$ is the unit matrix. The hermitian conjugate of $T$ satisfies the equation:

$$\frac{dT^+}{d\tau_c} = T(\tau_c)^+ \ A(\tau_c)^+, \quad (9)$$

I also introduce a matrix $E(\tau_c)$, related to $D(\tau_c)$ by:

$$D = T \ E \ T^+. \quad (10)$$

Inserting this into Eq. (7), and using Eqs. (8) and (9), I obtain the following equation for $E(\tau_c)$:

$$\frac{dE}{d\tau_c} = - \ (T)^{-1} \ F \ (T^+)^{-1}. \quad (11)$$

Since $E(\tau_c)$ does not appear on the RHS of Eq. (11), and since the RHS vanishes for large optical depth (due to the exponential increase of $T$ and $T^+$), the matrix $E(0)$ at $\tau_c = 0$ can be found by direct integration of Eq. (11) over optical depth. However, $E(0) = D(0)$, since $T(0)$ is the unit matrix. The matrix $D(0)$, which describes the Stokes parameters emerging from the atmosphere, can therefore be written as an integral over depth:

$$D(0) = \int_0^\infty \ (T)^{-1} \ F \ (T^+)^{-1} \ d\tau_c. \quad (12)$$

The argument of this integral,

$$C(\tau_c) \equiv (T)^{-1} \ F \ (T^+)^{-1}, \quad (13)$$

is nothing but the contribution function of the matrix $D(0)$. The contribution functions for the emergent Stokes parameters $I(0)$, $Q(0)$, $U(0)$, and $V(0)$ can be found directly from the components of the $C$-matrix:

$$
\begin{align*}
C_I(\tau_c) & = C_{11}(\tau_c) + C_{22}(\tau_c), \\
C_Q(\tau_c) & = C_{11}(\tau_c) - C_{22}(\tau_c), \\
C_U(\tau_c) & = C_{12}(\tau_c) + C_{21}(\tau_c),
\end{align*}
$$

(14)
\[ C_V(\tau_c) = -i[C_{12}(\tau_c) - C_{21}(\tau_c)]. \]

These are contribution functions per unit continuum optical depth. To obtain the contribution functions per unit height, Eqs.(14) must be multiplied with \( \tau_c/\mu \).

To compute the \( C(\tau_c) \)-matrix we first have to solve differential equation (8) for \( T(\tau_c) \). In general this must be done numerically, e.g. using the Runge-Kutta method. The integration starts at \( \tau_c = 0 \), and proceeds to larger optical depth. Note, however, that the integration now involves eight coupled equations, since each component of the \( T \)-matrix has a real and an imaginary part. In contrast, the traditional method of Eq.(2) involves only four coupled equations. Thus in order to retain information about contribution functions it is necessary to solve a larger system of equations. After the contribution functions of Eq.(14) have been computed, the emergent intensity and polarization can be found by integration of these functions over depth.

It should be pointed out that it is not strictly necessary to use Jones-calculus in order to derive a formal solution of the form (12); a procedure similar to the one above can be applied directly to Eq.(2). However, because of the redundancy in the \( 4 \times 4 \) matrices, we then obtain a system of sixteen coupled equations. I conclude that the description in terms of complex \( 2 \times 2 \) matrices is more efficient.

3. PHOTOSPHERIC FACULAE

3.1 INTRODUCTION

In the following I present some preliminary results of a study that applies the method of Sect. 2 to a model of photospheric faculae. The small-scale magnetic field of the sun outside sunspots is concentrated into facular elements, or flux tubes, that have typical sizes on the order of a few hundred kilometers, and fieldstrengths of about 1500 gauss (Stenflo, 1973; Harvey, 1977; Zwaan, 1978). Faculae probably consist of vertical flux tubes that are pushed close together by flows in the convective zone below. To first approximation these flux tubes can be considered magnetostatic structures (Spruit, 1976a,b). The magnetic field is contained by a gas-pressure difference between the inside and the outside of the tubes. This pressure difference is a result of the so-called Wilson depression of the levels of constant optical depth inside the tubes, relative to those of the surrounding photosphere. The outward decrease of gas pressure causes the magnetic fieldlines of a flux tube to fan out with height, away from the vertical axis of the tube. Higher up, in the chromosphere and corona, the fields of different faculae merge, and fill the available volume. Waves travelling
upward along flux tubes are believed to play an important role in the heating of the higher atmosphere.

To determine the 3-dimensional structure of photospheric faculae, it is necessary to compare high-resolution observations of these features with predictions based on physically realistic models. The purpose of the present study is to construct a model of an isolated flux tube, and to predict what this structure would look like, when observed with a spectrograph or filtergraph at high spatial resolution.

3.2 FLUX TUBE MODEL

Instead of a more realistic flux-tube model, in which the magnetic field is cylinder-symmetric around a vertical axis, I use here the more simple, 2-dimensional model of a flux sheet, in which the magnetic vector lies in vertical x-z planes, and is independent of the horizontal y-coordinate (z = -h measures depth relative to the level \( \gamma_{0.5} = 1 \); \( \gamma_{0.5} \) is the continuum optical depth at \( \lambda = 0.5 \mu m \)). The flux sheet has boundaries at \( x = -R(z) \) and \( x = +R(z) \), outside of which the gas is field-free (\( R(z) \) is the depth-dependent "radius"). The field inside the boundaries is assumed to be a potential field, so that all electric currents (in the y-direction) are concentrated at the two boundaries. It follows from Ampere's law that the current density per unit length along the boundary is equal to \( (4\pi/c)B_0(z) \), where \( B_0(z) \) is the fieldstrength just inside the boundary. I assume magneto-static equilibrium, so that the gas pressures \( p_i(z) \) and \( p_e(z) \) inside and outside the sheet are both independent of \( x \). At the sheet boundary there is a jump in gas pressure, equal to \( p_e(z) - p_i(z) = [B_0(z)]^2 / 8\pi \).

To determine the location \( R(z) \) of the sheet boundary for given \( B_0(z) \), I use the following iterative method. All currents are located at the boundary, so the vector potential \( A(x,z) \) of the magnetic field inside the sheet can be expressed as an integral over the boundary surface. For an assumed shape \( R_1(z) \) the vector potential \( A_1(z) \) on the boundary \( x = +R_1(z) \) is computed. The assumed shape is correct only if \( A_1(z) = \) constant, so that fieldlines do not cross the boundary. If \( A_1(z) \) is not constant, the difference with the \( z=0 \) level, \( \Delta A_1(z) = A_1(z) - A_1(0) \), is used to correct the boundary shape: \( R_2(z) = R_1(z) - \Delta A_1(z) / B_0(z) \). Note, that the radius at \( z=0 \) does not change, so that \( R(0) \) is a free parameter of the model.
The temperature structure of the model is shown in Fig.1. Curve 1 gives the temperature far away from the sheet boundary, adopted from model C of Vernazza et al. (1981) and the convective-zone model of Spruit (1976a). Curve 3 gives the temperature on the mid-plane $x=0$, taken from one of the flux-tube models (model 2) of Spruit (1976b). Spruit's model assumed a tube radius $R(0)$ of 84 km, and a Wilson depression of 100 km, but actually his model was computed for a flux tube with cylindrical geometry. I use this model here only to illustrate the method, and I do not imply that it gives a realistic description of faculae. The gas pressures $p_e(z)$ and $p_t(z)$, necessary to compute the magnetic field, were found by solving the hydrostatic-equilibrium equation for the temperature models of curves 1 and 3. A vertical cross-section of the flux-sheet is shown in Fig.2; the level $\gamma_{0.5} = 1$ is indicated by horizontal lines.
Curves 2 of Fig. 1 gives the wall temperature of the gas just outside the sheet boundary (also taken from Spruit, 1976b). This model, and the hydrostatic pressures computed from it, were used for the radiative transport calculations described in the next section. Although the use of two different external models is not consistent with our assumption of hydrostatic balance, the rationale behind the approach is that the external pressure, $p_e(z)$, is determined mostly by the mean stratification of the atmosphere, and not by the immediate surroundings of the flux tube. The temperature reduction in the wall with respect to the mean atmosphere is important, however, for computing realistic intensities of the light emitted by the wall.

3.3 FORMATION OF FeI $\lambda$ 5250.2

In Fig. 2 three lines-of-sight (LOS), that lie in the x-z plane and make an angle of 25° with the vertical, are also shown. The full-drawn line intersects the $\zeta_{0.5} = 1$ level in the interior of the flux sheet, while the dotted- and dash-dotted lines cross the left boundary of the sheet at $z = +50$ km and $z = -200$ km. Spectral profiles
Fig. 3 Wavelength profiles of $I$, $Q$, $U$, and $V$ for FeI 5250.2, for three lines-of-sight of Fig. 2, and for LOS in quiet sun (dashed curve)
of the line FeI \( \lambda 5250.2 \) were computed in LTE for each LOS. The results are shown in Fig. 3, normalized according to the continuum intensity for each LOS. The dashed curve in Fig. 3a is for a LOS in the "quiet" sun away from the flux sheet, although it was still computed with the wall model (curve 2 of Fig. 1). The continuum intensities for the full-, dotted-, and dash-dotted LOS were respectively 1.37, 1.32, and 1.03 times the "quiet" value. Micro-turbulent velocities inside and outside the sheet were taken to be 1 km/s, and macro-turbulence was neglected. Systematic flows were also neglected, so that the wavelength-profiles of I, Q, and U are symmetric, while the profiles of V are anti-symmetric.

Figs. 4a, 4c, and 4e give the contribution functions for the intensity, as measured at line center, in the line wing (\( \Delta \lambda =30 \) mA), and in the continuum, respectively. The curves are normalized to their peak value. The dashed curves again refer to a LOS outside the flux sheet. Figs. 4b and 4d give contribution functions for linear- and circular polarization, measured respectively at line center and in the line wing, where these signals are strongest. In this case the normalization is according to the integral over each curve.

The dashed curve in Fig. 4a shows that the line-center intensity in the quiet sun is formed over a large range in depth, from about \( z=-400 \) to \( z=0 \) km (also see Lites, 1972). For the dash-dotted and dotted LOS I find a large jump in the contribution function at the level where the LOS crosses the wall: because of the lower opacity in the interior, most of the light comes from the external medium behind the wall. The full curves in Figs. 4a,c,e are nearly identical because the spectral line is so weak in the interior of the flux sheet (cf. Fig.3a).

Figs. 4b and 4d show that the contribution functions for Q and V change sign in the line-forming region. This is due to the fact that Q, U, and V measure intensity differences between two orthogonal modes of polarization (e.g. left- and right circular polarization for V). The opacity in these two modes is generally not equal, so that each mode is formed at a different depth in the atmosphere. Since the modes give opposite contributions to the difference signal, the contribution functions of Q, U, and V consist of a positive and a negative peak that are displaced in depth. The peak at larger depth is larger than the one at smaller depth because the Planck function increases with depth in the photosphere. Thus in LTE the observed polarization is due to the temperature gradient in the atmosphere.

4. DISCUSSION

It was shown that, for the case of polarized light, a formal solution of the transport equations can be obtained. This leads to a natural definition for the contribution functions of Stokes
Fig. 4 Contribution functions of I, Q, and V for FeI λ5250.2
parameters. The present formalism includes non-LTE effects due to scattering, but only in the simplest approximation, in which the scattering is isotropic, with complete redistribution in frequency and polarization. Differences in population between the sublevels with different quantum numbers in each atomic level (atomic level polarization) have also been neglected. The formalism seems adequate for the description of most non-resonance lines in the photosphere and low chromosphere, in particular the lines of neutral atoms frequently used in magnetic studies.

The facular model indicates that the effects of the walls are important not only for the continuum (Spruit, 1976b), but also for the formation of spectral lines. Thus one-dimensional models (e.g. Unno's equations) are not adequate to interpret the observation. A detailed comparison between models and observations is necessary to determine the thermodynamic structure of faculae.

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