THE RESTRICTED 3-BODY PROBLEM WITH RADIATION PRESSURE

J. F. L. SIMMONS, A. J. C. MCDONALD, and J. C. BROWN
Department of Astronomy, The University, Glasgow, G12 8QQ

(Received November, 1982; accepted December, 1984)

ABSTRACT. The restricted 3-body problem is generalised to include the effects of an inverse square distance radiation pressure force on the infinitesimal mass due to the large masses, which are both arbitrarily luminous. A complete solution of the problems of existence and linear stability of the equilibrium points is given for all values of radiation pressures of both luminous bodies, and all values of mass ratios. It is shown that the inner Lagrange point, $L_1$, can be stable, but only when both large masses are luminous. Four equilibrium points, $L_6$, $L_7$, $L_8$, and $L_9$ can exist out of the orbital plane when the radiation pressure of the smaller mass is very high. Although $L_8$ and $L_9$ are always linearly unstable, $L_6$ and $L_7$ are stable for a small range of radiation pressures provided that both large masses are luminous.

1. INTRODUCTION

In this paper we give a complete treatment of the equilibrium points in the classical circular restricted three body problem modified by radiation pressure from the two finite masses. It is assumed that the only effect of radiation is to modify the magnitude of the inverse square law forces of the finite masses acting on the infinitesimal third mass. (We thus neglect for example the relativistic Poynting Robertson effect which may be treated as a perturbation - cf. Chernikov (1970), Schuerman (1980).) The treatment may also be applicable to some stellar wind problems where the wind forces have a similar modifying effect on the inverse square gravitational forces experienced by infinitesimal particles (cf. Budding, 1980, for example). The considerable complexity and interest of the problem over the classical treatment of Lagrange points (without radiation pressures) arises physically from the fact that although the gravitational terms in the potential are scaled by radiation pressure, the rotational term is unmodified since the orbit of the finite masses is unaffected by radiation pressure. Thus rich new possibilities arise for the existence and stability of equilibrium points since the radiation pressure factors decouple the values of the inverse square law and rotational terms in the potential.

The problem was first discussed in two largely overlooked papers by Radzievskii firstly in respect of changes in the geometry of the Roche surfaces of the positions of the collinear equilibrium points (Radzievskii,
1950), and secondly a more complete treatment of the behaviour of the equilibrium points, (Radzievskii, 1953). In both papers, however, Radzievskii, who is primarily interested in the solar problem, only treats a limited range of radiation pressure, in particular when only one massive body is luminous and does not consider the question of the linear stability of the equilibrium points. However Radzievskii (1953) did note the existence of two out of the plane equilibrium points (L_6 and L_7) though because the less massive body was taken to be non-luminous, as is reasonable for the solar system, he failed to show the existence of two further equilibrium points (L_8, L_9) out of the orbital plane. The stability of L_1, L_2, and L_3, the collinear equilibrium points, and L_4 and L_5 was investigated by Chernikov (1970), who also discussed the modifications of the results brought about through the Poynting Robertson effect, but again only for the Sun-planet problem. Perezhogin (1976) demonstrated that for this latter problem the out of plane solutions were always unstable.

Although Schuerman (1980) considered the general problem where both massive bodies exert a radiation pressure, his analysis was in part incomplete and incorrect. In particular his conclusions concerning the existence and stability of the collinear Lagrange points are wrong. Also he neglects the domain of very high radiation pressures and the appearance of equilibrium points out of the orbital plane. In this paper we therefore reconsider the whole problem for arbitrary values of the two radiation pressures and the mass ratio. Seen as a whole the problem displays a number of interesting features that are not apparent in previous treatments.

After introducing the basic formulation, we consider in Section 2 the existence and location of the various equilibrium points as a function of three parameters. In Section 3 we analyse their stability against small perturbations. In Section 4 we discuss our results and their possible applications.

Many basic texts present the analysis of the classical Lagrange point problem. Here we start from a presentation and notation closely similar to that of Danby (1962, p. 187ff.). In particular we choose the unit of mass equal to the sum of the primary masses m_1 + m_2, the unit of length equal to their separation and the unit of time such that G = 1. Then the angular frequency ω of the circular orbit, and hence of the rotating coordinate system Oxyz, is also unity. For definiteness we also take μ = m_2 / (m_1 + m_2) so (0 ≤ μ ≤ 1). The forces experienced by an infinitesimal test particle at rest in the rotating coordinate system are then derivable from the potential

\[ V(x, y, z) = -\left(\frac{1}{2}(x^2 + y^2) + \frac{\alpha(1 - \mu)}{r_1} + \frac{\beta \mu}{r_2}\right), \tag{1} \]

where

\[ r_1 = \{(x + \mu)^2 + y^2 + z^2\}^{1/2} \]
\[ r_2 = \{(1 - x - \mu)^2 + y^2 + z^2\}^{1/2}. \]  \tag{2}
THE RESTRICTED 3-BODY PROBLEM WITH RADIATION PRESSURE

(The centre of mass is at (0, 0, 0), m_1 at (-\(\mu\), 0, 0) and m_2 at 
(1 - \(\mu\), 0, 0); Oxy is the orbital plane) and the constant factors \(\alpha\), \(\beta\) represent the effects of radiation pressure from the two finite masses. These are related to the notation of Schuerman (1980) by

\[
\begin{align*}
\alpha &= 1 - \beta_1, \\
\beta &= 1 - \beta_2,
\end{align*}
\]

where \(\beta_1, 2\) are the ratios of the magnitudes of radiation to gravitational forces from \(m_1, m_2\). Thus \(\beta_1, 2 = 0\) (\(\alpha = \beta = 1\)) represents the classical problem; \(0 < \beta_1, 2 < 1\) \((1 > \alpha, \beta > 0)\) represents reduction of the gravitational forces by radiation, and \(\beta_1, 2 \geq 1\) \((\alpha, \beta \leq 0)\) implies that radiation has overwhelmed gravity. In general therefore radiation pressure turns all equations for the three body problem into a 3-parameter \((\alpha, \beta, \mu)\) set instead of the classical single parameter \((\mu)\) set.

We note that the complete range of physically possible \(\alpha, \beta\) is

\(-\infty < \alpha, \beta \leq 1\) (i.e. \(0 \leq \beta_1, 2 < \infty\)). Schuerman (1980) mentions the possibility of negative \(\alpha, \beta\) (i.e. \(\beta_1, 2 > 1\)) only briefly but notes possible application to spacecraft parking (O'Neill, 1974) and propulsion (Friedman, 1976) in the solar system (with \(\beta = 1\) i.e. \(\beta_2 = 0\) for the planetary mass) using 'sails' of large cross section mass ratio. In fact the possibilities here are more wide ranging than mentioned by Schuerman (1980) as we show in Sections 2 and 3. In addition, there may be applications to accumulation of matter around binaries, particularly where the stars are of luminous late type so that large enough grains may exist in their neighbourhood for \(\alpha\) and/or \(\beta\) to be negative.

2. LOCATION AND EXISTENCE OF THE EQUILIBRIUM POINTS

We shall discuss the existence and location of the triangular \((y \neq 0)\), collinear \((y = 0, z = 0)\) and out of orbital plane \((z \neq 0, y = 0)\) points separately in Sections 2.1, 2.2, and 2.3. The location of the equilibrium points discussed in Section 2 are given in general by solution of \(\nabla V = 0\) or, explicitly

\[
\begin{align*}
-V_x &= (1 - A)x - (1 - \mu)\mu(a/r_1^3 - \beta/r_2^3) = 0, \\
-V_y &= (1 - A)y = 0, \\
-V_z &= -Az = 0,
\end{align*}
\]

where

\[
A = \frac{\alpha(1 - \mu)}{r_1^3} + \frac{\beta\mu}{r_2^3}.
\]

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System
As we shall see below, the existence and stability of the equilibrium solutions is critically governed by the numerical value of $A$. All solutions, however, derive from the same general equation (Equations (4a), (4b), (4c)) and thus it is understandable that at critical values of the parameters one type of equilibrium point transforms into another, and thus it will be necessary to make an overall synthesis of the behaviour of all equilibrium points after each has been dealt with separately. Our aim in Sections 2.1, 2.2 and 2.3 is to establish the regions in parameter space for which equilibrium points of each type exist, and also the critical value of the parameters at which points coalesce, separate, appear and disappear. The question of the precise location of the equilibrium points will be dealt with where it is important, but otherwise left to the next section where it is essential for establishing the stability or instability of the points.

For later convenience we introduce now the parameters $\delta_1 = \alpha^{1/3}$ and $\delta_2 = \beta^{1/3}$ as employed by Schuerman (1980).

2.1. Triangular Points ($L_4$, $L_5$)

If $y \neq 0$ it follows from Equation (4b) that $A = 1$, which is consistent with Equation (4c) only if $z = 0$. Substitution of $A = 1$ into (4a) and (5) then gives

$$\begin{align*}
\lambda_1 &= \alpha^{1/3} = \delta_1 \\
\lambda_2 &= \beta^{1/3} = \delta_2
\end{align*}$$

which shows that the triangular points exist for $\delta_1 \geq 0$ and $\delta_2 \geq 0$. In fact as shown by Schuerman (1980) the points lie at the intersections of circles defined by (6) and so exist only provided further that $\delta_1 + \delta_2 \geq 1$. When the equality holds, $L_4$ and $L_5$ become degenerate with the inner Lagrangian point $L_4$. The inequality is always satisfied in the particular case considered by Radzievskii (1953) where one of the stars is non-luminous, say $\delta_2 = 1$, in which case $L_4$ and $L_5$ move from their classical equilateral positions onto the luminous mass and there coalesce with $L_1$ as the radiation pressure increases (i.e. $\delta_1$ decreases to zero).

2.2. Collinear Points $L_1$, $L_2$, $L_3$

If $y = 0$ two possibilities arise. Firstly, on the x-axis, $z = 0$ and so by Equation (4a)

$$\begin{align*}
x - \frac{\delta_1^3 (1 - \mu)(x + \mu)}{|x + \mu|^3} + \frac{\delta_2^3 \mu(1 - x - \mu)}{|1 - x - \mu|^3} = 0
\end{align*}$$

solution of which leads to a set of collinear points which we discuss in this section. The classical case corresponds to $\delta_1 = \delta_2 = 1$, and as is well known gives two outer ($L_2$, $L_3$) and one inner ($L_1$) equilibrium point.

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System
The second case arises from the fact that it is still possible to satisfy Equations (4a) and (4c) even for \( z \neq 0 \) (i.e. out of plane equilibrium) provided \( A = 0 \), in which case

\[
x - (1 - \mu)\mu \left( \frac{\delta_1^3}{r_1} - \frac{\delta_2^3}{r_2} \right) = 0, \tag{8a}
\]

\[
A = \frac{\delta_1^3(1 - \mu)}{r_1} + \frac{\delta_2^3\mu}{r_2} = 0. \tag{8b}
\]

Clearly Equation (8b) can be satisfied if and only if \( \delta_1 \) and \( \delta_2 \) have different signs, which is not true for the classical case. Any solution of Equation (7) that also satisfies Equation (8b) (and we will see that such exist) must of course satisfy Equation (8a). Thus we can expect a tri-furcation point at which the out of plane solutions appear on the \( x \)-axis. Discussion of these out of plane solutions is deferred to the next section.

The following physical arguments can be used to describe the behaviour of the collinear points. (A rigorous mathematical treatment will be given later.) Suppose that \( \delta_2 > 0 \). As the radiation pressure from star 1 increases (\( \delta_1 \) decreases from its classical value of 1) the inner Lagrangian point \( L_1 \) must move towards star 1 as the net force from star 1 decreases. In the special case where \( \delta_2 = 1 \) the \( L_1 \) point ultimately moves onto the luminous star when \( \delta_1 + 0 \), for when the net force from star 1 is zero, the gravitational force from star 2 and the centrifugal force are exactly balanced at the position of star 1.

Consider now \( L_3 \). This will move in the positive \( x \) direction towards star 1 as \( \delta_1 + 0 \). However small \( \alpha \) is, provided it is positive, the attractive force from star 1 can always be made arbitrarily large by moving the test particle near enough to star 1 and thus be made to counter-balance the net effect of the centrifugal and attractive force from star 2. However, when \( \delta_1 = 0 \) this is no longer possible and \( L_3 \) must disappear (except when \( \delta_2 = 1 \) in which case the \( L_3 \) and \( L_1 \) points are at \( x = -\mu \)). Directly \( \delta_1 \) becomes negative and the force from star 1 repulsive, however, it is possible to balance the centrifugal force and the attractive force from star 2, and a new equilibrium point appears to the right of star 1 and continues towards star 2 as \( \delta_1 \) decreases until it coalesces with \( L_1 \). Because of the discontinuity at \( \delta_1 = 0 \) we shall call this new point \( L^*_3 \). Any further increase in the radiation pressure from star 1 will cause the degenerate equilibrium point to disappear. As \( \delta_1 \) decreases from 1 to negative values the point \( L_2 \) will get arbitrarily close to star 2. The case where \( \delta_2 < 0 \) can be dealt with similarly. Finally, if \( \delta_2 = 0 \), then \( L_2 \) cannot exist.

Let us now establish rigorously the location and behaviour of the equilibrium points \( L_1 \), \( L_2 \), and \( L_3 \).

Equation (7) yields the following equations valid in the three different domains \( D_3 \), \( D_1 \), and \( D_2 \) given respectively by \( x < -\mu \), \( -\mu < x < 1 - \mu \), and \( x > 1 - \mu \).
\[ f_{3\delta_1}(x) = x + \frac{\delta_1^3(1 - \mu)}{(x + \mu)^2} + \frac{\delta_2^3\mu}{(1 - x - \mu)^2} = 0, \quad x < -\mu, \quad (9a) \]
\[ f_{1\delta_1}(x) = x - \frac{\delta_1^3(1 - \mu)}{(x + \mu)^2} + \frac{\delta_2^3\mu}{(1 - x - \mu)^2} = 0, \quad -\mu < x < 1 - \mu, \quad (9b) \]
\[ f_{2\delta_1}(x) = x - \frac{\delta_1^3(1 - \mu)}{(x + \mu)^2} - \frac{\delta_2^3\mu}{(1 - x - \mu)^2} = 0, \quad x > 1 - \mu. \quad (9c) \]

For later convenience when we shall consider the functions as families of curves we have introduced the subscripts \( \delta_1 \) and \( j = 1, 2, \) or 3 corresponding to these different domains.

We now establish by using Descartes' Rule of Signs (cf. Buckingham, 1957) that for any values of \( \mu, \delta_1, \) and \( \delta_2 \) there is only one solution in both the outer domains \( D_3 \) and \( D_2. \)

Consider the domain \( D_2 \) \( (x > 1 - \mu). \) Since we wish to establish that there exists only one zero of \( f_{2\delta_1}(x) \) that is greater than \( 1 - \mu \) it is useful to shift the origin to the secondary through the change in variable \( y = x - (1 - \mu). \) Equation (9c) yields the polynomial equation

\[ y^5 + (3 - \mu)y^4 + (3 - 2\mu)y^3 + (1 - \mu)(1 - \delta_1^3 - \delta_2^3\mu)y^2 + \]
\[ + \left( -2\delta_2^3\mu \right)y + (\delta_2^2\mu) = 0. \quad (10) \]

We now investigate the number of changes in sign of the above polynomial.

Since \( \mu \leq 1 \) the coefficients of \( y^5, y^4, \) and \( y^3 \) are necessarily positive. The signs of the remaining coefficients depend on \( \delta_1 \) and \( \delta_2. \)

When \( \delta_2 \) is positive there will be only one change of sign no matter what the sign of the \( y^2 \) coefficient. If this coefficient happens to be zero there is still one change of sign. Hence there is only one positive root to Equation (10), and one collinear point exists in \( D_2. \)

When \( \delta_2 \) is negative all the polynomial coefficients are positive (N.B. \( \delta_1 \leq 1), \) so there are no changes of sign, and there can be no positive roots. Hence there is no collinear point in \( D_2. \)

On the grounds of symmetry we conclude that only one collinear point exists in \( D_3 \) if \( \delta_1 \) is positive, and none if \( \delta_1 \) is negative. Evidently, if both \( \delta_1 \) and \( \delta_2 \) are negative all collinear points must lie in domain \( D_1 \) in between the two masses.

If \( \delta_2 = 0, \) the domain in which \( f_{2\delta_1} \) is defined can be extended to include \( x = 1 - \mu \) since the singularity at this point then disappears (cf. Equation (9c)) as is the case with \( f_{1\delta_1} \) defined in \( D_1. \) No positive solutions of Equation (10) exist for \( \delta_2 = 0, \) and the degenerate solution at \( y = 0 \) is fictitious in the sense that it is not a solution of Equation (9c). (Only if \( \delta_1 = 1 \) is \( y = 0 \) a real solution of Equation (9c).)

Consider now the family of curves \( f_{3\delta_1}, f_{1\delta_1}, \) and \( f_{2\delta_1} \) restricted to their respective domains \( D_3, D_1, \) and \( D_2. \)
Strictly speaking these families of curves depend on 2 parameters \( \delta_1 \) and \( \delta_2 \) for any fixed \( \mu \). However, because of the symmetry of the problem we can restrict ourselves to the case where \( \delta_2 > 0 \). \( \delta_1 > 0 \) is then the only free parameter. The zeros of the functions \( f_{j\delta_1} \) \( (j = 1, 2, 3) \) confined to their respective domains will yield the equilibrium points.

We shall consider the cases of positive and negative \( \delta_2 \) separately.

Case A: \( \delta_2 > 0 \)

Figure 1a gives a schematic representation of the family of functions. The following properties can be easily established:

(i) \( f_{\delta_1}, f_{1\delta_1} \) are monotonic increasing functions of \( x \) when \( \delta_1 \geq 0 \).

\( f_{2\delta_1} \) is monotonic increasing except for extremely negative values of \( \delta_1 \).

(ii) \( f_{\delta_1}(x) \to \infty \) as \( x \to -\mu^- \) when \( \delta_1 > 0 \),

\( f_{\delta_1}(x) \to -\infty \) as \( x \to -\mu^- \) when \( \delta_1 < 0 \).

(iii) \( f_{1\delta_1}(x) \to +\infty \) as \( x \to -\mu^+ \) for \( \delta_1 > 0 \), \( f_{1\delta_1}(x) \to \infty \), as \( x \to 1 - \mu \) for \( \delta_1 > 0 \), \( x \to (1 - \mu)^- \) for all \( \delta_1 \).

(iv) \( f_{2\delta_1}(x) \to +\infty \) as \( x \to (1 - \mu)^+ \),

\( f_{2\delta_1}(x) \to +\infty \) as \( x \to \infty \).

(v) When \( \delta_1 \) becomes negative \( f_{1\delta_1} \) and \( f_{3\delta_1} \) immediately develop turning points. (For sufficiently negative \( \delta_1 \), \( f_{2\delta_1} \) also develops turning points, though no new roots appear, as shown immediately above.) Since \( f_{1\delta_1}(x) = 1 + 2A \) for \( i = 1, 2, 3 \) these turning points occur when

\[ A = \frac{1}{2}. \]

(vi) For fixed \( x, \mu, \) and \( \delta_2, f_{3\delta_1} \) is monotonic increasing in \( \delta_1 \) whilst \( f_{2\delta_1} \) and \( f_{1\delta_1} \) are monotonic decreasing, i.e.:

\[ \forall x, f_{3\delta_1}(x) > f_{3\delta_1}(x) \Leftrightarrow \delta_1 > \delta_1'. \]

\[ \forall x, f_{1\delta_1}(x) > f_{1\delta_1}(x) \Leftrightarrow \delta_1 < \delta_1'. \]

\[ \forall x, f_{2\delta_1}(x) > f_{2\delta_1}(x) \Leftrightarrow \delta_1 < \delta_1'. \]

The monotonic character of these functions does not depend on \( \delta_2 \) being positive.

(vii) \( f_{10} \) (i.e. \( f_{1\delta_1} \) for \( \delta_1 = 0 \)) can be extended to the interval \( -\infty < x < 1 - \mu \) and passes through the point \((-\mu, (\delta_2^3 - 1)\mu) \) and \((0, \delta_2^3/(1 - \mu)^2) \) in the \((x, f)\) plane.

The following results follow from these properties. We indicate the properties invoked in brackets.

As \( \delta_1 \) decreases to zero

One collinear point exists in \( \Delta_3 \) ((i) and (ii)). We call this point \( L_3 \), corresponding to the classical nomenclature. \( L_3 \) monotonically and arbitrarily closely approaches star 1 as \( x = -\mu \). ((i), (ii), and (vi)).

One collinear point exists in \( \Delta_1 \) ((i) and (iii)). We call this point \( L_1 \). \( L_1 \) moves monotonically towards star 1 at \( x = -\mu \) ((i), (iii), and (vi)).
Fig. 1a-b. Schematic representation of families of curves corresponding to \( f_j \delta_1 \), \( j = 1, 2, 3 \) for different \( \delta_1 \). \( \delta_2 \) is fixed and (a) positive, (b) negative. Collinear points corresponding to \( \delta_1 = 0.6 \) and \( \delta_1 = -0.2 \) are denoted by o and +, respectively. Curves for which \( \delta_1 \) is positive are denoted by \( \cdots \cdots \cdots \), and \( \delta_2 \) negative by thin continuous lines. When \( \delta_1 \) has critical value such that (a) \( L_1 \) and \( L_3 \) or (b) \( L_1 \) and \( L_2 \) disappear a dotted curve is used.
but does not reach it unless $\delta_2 = 1$:

One and only one collinear point exists in $D_2$. We call this point $L_2$. $L_2$ moves monotonically towards star 2 at $x = 1 - \mu$.

When $\delta_1$ takes the value zero

Comparison with $f_{30}$ shows that $L_3$ no longer exists except when $\delta_2 = 1$, in which case there is an equilibrium point at $x = -\mu$, where $L_3$ and $L_1$ coincide. $L_2$ uniquely exists.

When $\delta_1$ becomes negative

$f_{1\delta_1}$ immediately develops two zeros if $\delta_2 < 1$ (property (v)).

If $\delta_2 = 1$ the turning point must lie above the x-axis, as shown by comparison with $f_{10}$, and there will be no inner equilibrium points. The equilibrium point nearer star 1, which we call $L'_3$ can be made arbitrarily close to $x = -\mu$ by choosing $|\delta_1|$ small enough. As $\delta_1$ decreases $L'_3$ continues in the direction of star 1 ($x = \mu$) and $L_2$ in the direction of star 2 ($x = 1 - \mu$). With further decrease in $\delta_1$ the two inner equilibrium points ($L'_2$ and $L_1$) coalesce at some critical value, $\delta_1 \text{crit}$, of $\delta_1$ (when $A = -\frac{1}{2}$) and subsequently both points disappear.

Case B: $\delta_2 < 0$

Similar arguments can be made in this case but for brevity we simply show the plot of the families of curves Figure 1b. For $\delta_1$, $\delta_2 < 0$ all collinear points lie in the region $(-\mu, 1 - \mu)$. This result was also derived above from Descartes' Rule of Signs. For $\delta_1$, $\delta_2 < 0$ only $L'_2$ will exist. (We use $L_2$ instead of $L_2$ for the reasons explained in relation to the case where $\delta_2 > 0$.) As $\delta_1$ decreases to zero $L_3 \to x = -\mu$. A critical value of $\delta_1$, $\delta'_1$, say, is reached at which $L_1$ and $L'_2$ appear. ($\delta'_1 < 0$ if $\delta_2 < -(4^{1/3}/(1 - \mu)/3\mu^{1/3})$ and $\delta'_1 > 0$ if $\delta_2 < -(4^{1/3}/(1 - \mu)/3\mu^{1/3})$). $A = -\frac{1}{2}$ at this point. With further decrease in $\delta_1$, $L'_3$, and $L_1$ more towards each other and coalesce, after which only $L'_2$ exists.

The special case where $\delta_2 = 1$ has been adequately dealt with in the paragraphs above.

Figure 2 gives a schematic representation in $\delta_1$, $\delta_2$ space of the existence of the collinear points $L_1$, $L_2(L'_2)$, $L_3(L'_3)$.

The contour PQR represents the curve $A = -\frac{1}{2}$ evaluated at the $L_1$ point. If $\delta_1$ and $\delta_2$ are changed so that PQR is traversed, $L_1$ and $L'_2$, $L'_3$ disappear or appear, depending on the direction taken.

Case C: $\delta_2 = 0$

It can easily be seen that $L_3$ and $L_1$ behave as in case A, but $L_2$ ceases to exist for all $\delta_1$ since there is no singularity at $x = 1 - \mu$. 
Fig. 2. Schematic representation in $\delta_1$, $\delta_2$ space of the existence of collinear points $L_1$, $L_2(L_2')$, $L_3(L_3')$. The contour PQR represents $A = -0.5$ evaluated at the $L_1$ point. On this curve $L_2'$ or $L_3'$ is degenerate with $L_1$.

2.3. Equilibrium Points Out of the Orbital Plane

To satisfy Equation (4) with $z \neq 0$ we require, by (4c), $A = 0$ and so $y = 0$ by (4b). Thus Equations (8) represent the only possible cases of $z \neq 0$, any equilibrium points which may exist for $z \neq 0$ necessarily lying in the Oxz plane containing the two main masses and the rotation axis. Clearly (8b) cannot be satisfied if $\delta_1 = \delta_2 = 1$ and so there are no solutions with $z \neq 0$ in the classical case. However (8b) can be satisfied either if $\delta_1 = \delta_2 = 0$ or if $\delta_1$ and $\delta_2$ have opposite sign - i.e. if $\delta_1/\delta_2 < 0$.

If $\delta_1 = \delta_2 = 0$ then $x = 0$ by (8a) and $z$ can take any value; an infinite line of equilibrium solutions exists with a simple physical significance. If $\delta_1 = \delta_2 = 0$ a test particle experiences no inverse square law forces from the masses. If further the test particle lies anywhere on the $z$ (rotation) axis it experiences no centrifugal force either and so is in equilibrium. The configuration is self evidently unstable and of very restricted interest requiring the scarcely probable simultaneous cancellation of
gravity by radiation pressure for both main masses. This is impossible for any spherical test particle unless the masses have identical mass/luminosity ratios but could in principle provide a means of parking a sailed spacecraft (albeit unstably) at any height \( z \) above a binary for suitably sized and oriented radiation pressure sails!

If \( \delta_1/\delta_2 < 0 \), it is evident from the form of Equations (8) that solutions, if they exist, occur in pairs, corresponding to \( \pm z \), as is necessary from the symmetry of the system. Considerable simplification of Equation (8) can be obtained by writing them in terms of \( r_1 \) and \( r_2 \), although in doing this we have to realise that

\[
\begin{align*}
  r_1 &\geq 0 \quad \text{and} \quad r_2 \geq 0, \quad (11a) \\
  r_1 + r_2 &\geq 1, \quad (11b) \\
  |r_1 - r_2| &\leq 1, \quad (11c)
\end{align*}
\]

are just triangular inequalities.

The equality holds when the solutions of Equation (8) yield collinear solutions, i.e. \( z = 0 \). Although strictly speaking we are interested in the off-orbital plane points, we shall admit the equality, for this will show how the trifurcation takes place, i.e. where the out of plane points begin to move off or onto the \( x \)-axis.

Equation (8b) gives in terms of \( r_1 \) and \( r_2 \)

\[
\frac{r_1}{r_2} = k \quad \text{where} \quad k = \frac{-\delta_1}{\delta_2} \left( \frac{1 - \mu}{\mu} \right)^{1/3} \quad (12a)
\]

and

\[
x - (1 - \mu) \mu \left( \frac{\delta_1^3}{r_1^3} - \frac{\delta_2^3}{r_2^3} \right) = 0. \quad (12b)
\]

Equations (12a) and (12b), upon substituting

\[
x = \frac{r_1^2 - r_2^2 + 1 - 2\mu}{2},
\]

give

\[
(k^2 - 1)r_2^5 + (1 - 2\mu)r_2^3 + 2\mu\delta_2^3 = 0, \quad (13)
\]

where we have eliminated \( r_1 \).

Before dealing with the general case for arbitrary \( \delta_1, \delta_2, \mu \), consider the special case where \( k = 1 \).

\[
k = 1
\]

Equation (13) then has solution
\[ r_2 = -\left( \frac{2\mu}{1 - 2\mu} \right)^{1/3} \delta_2', \]

provided of course that \( \mu \neq \frac{1}{2} \). From (12a) \( r_1 = r_2 \). (When \( \mu = \frac{1}{2} \) and \( k = 1 \) it follows from Equation (13) that there is no solution.) The above solution will be physically acceptable only if all the conditions (11) are satisfied. (11c) is immediately satisfied since \( |r_1 - r_2| = 0 \). From (11a) and (11b) we have either

\[ \mu < \frac{1}{2} \quad \text{and} \quad \delta_2 \leq \left( \frac{1 - 2\mu}{16\mu} \right)^{1/3} \quad \text{or} \quad \mu > \frac{1}{2} \quad \text{and} \quad \delta_2 \geq \left( \frac{2\mu - 1}{16\mu} \right)^{1/3}. \]

Arbitrary \( \delta_1', \delta_2, \mu \) (\( k \neq 1 \))

Equation (13) can now be written

\[ f(r) = r^5 + \frac{1 - 2\mu}{k^2 - 1} r^3 + \frac{2\mu \delta_2^3}{k^2 - 1} = 0, \]  

(14)

where we have dropped the subscript on \( r \), and defined the quintic function \( f \).

The problem of determining the off-orbital plane equilibrium points is thus reduced to solving the quintic equation, with the subsidiary conditions that \( r_1, r_2 > 0 \), \( r_1 + r_2 \geq 1 \) and \( |r_1 - r_2| \leq 1 \). For brevity we shall restrict the analysis of this problem to the case where \( \delta_1 < 0 \) and \( \delta_2 > 0 \), but allow any value of \( \mu \) between 0 and 1. Because of the symmetry of the problem, solutions for negative \( \delta_2 \) and positive \( \delta_1 \) can be obtained by the interchange \( \mu \leftrightarrow 1 - \mu \).

The solution of the quintic Equation (14) can be found numerically using the standard Newton-Raphson technique, and indeed to establish the stability of these solutions (dealt with in Section 3) this calculation has to be carried out. That solutions do exist can be seen from the special case where \( \mu = 0.5 \), where we have immediately from Equation (14)

\[ r_2 = \left( \frac{\delta_2^3}{1 - k^2} \right)^{1/5}. \]

With \( \delta_2 > 0 \), necessarily \( k < 1 \) and \( |r_1 - r_2| < 1 \) is immediately satisfied for \( \delta_2 \leq 1 \). The condition \( r_1 + r_2 \geq 1 \) yields

\[ (\delta_1 - \delta_2)^4 \geq \delta_1 + \delta_2. \]

(15)

The region in the \( \delta_1, \delta_2 \) plane where solutions exist for \( \mu = 0.5 \) is thus given by Figure 3 where the bounding curves are \( \delta_1 = -\delta_2 \) and \( y = x^4 \) where \( y = \delta_1 + \delta_2 \) and \( x = \delta_1 - \delta_2 \). On the bounding curve \( y = x^4 \) the solution coincides with the inner collinear pt.

We wish to address ourselves to the problem of establishing the regions on the \( \delta_1, \delta_2 \) plane in which physical solutions exist for arbitrary values of \( \mu \). A priori, there is no reason why more than one solution for \( r_2 \) (two
Fig. 3. Existence of out of orbital plane equilibrium points when $\mu = 0.5$. In the shaded region there is one solution either side of the orbital plane, otherwise there is no solution, and $\delta_1 + \delta_2 = (\delta_1 - \delta_2)^4$ where the solution is on the line of centres (i.e. $y = 0, z = 0$).

equilibrium points) should not exist. However, by applying Descartes' Rule of Signs to Equation (14) we see that at most two positive roots exist provided

$$\frac{1 - 2\mu}{k^2 - 1} < 0 \quad \text{and} \quad \frac{2\mu}{k^2 - 1} > 0,$$

i.e. $\mu > \frac{1}{2}$ and $k > 1$. If $k < 1$ there is one positive root to Equation (14); if $k > 1$ and $\mu < \frac{1}{2}$ there is no positive root.

We show below that two solutions for $r_2$ are indeed possible for a limited range of $\delta_1$ and $\delta_2$, i.e. there are in all 4 out of plane equilibrium
points. In this analysis we make no attempt to solve Equation (14). Instead, we shall establish the results using general arguments about the form and zeros of the function $f$.

The 'topological' form of the function $f$ depends on the values of the three parameters $\mu$, $k$, $\delta_2$. In fact, provided $\delta_2 > 0$ only $\mu$ and $k$ are important. For all values of $\mu$, $k$, $\delta_2$ there is an inflection at $r = 0$, since

$$f'(r) = \left(5r^2 + 3\frac{(1 - 2\mu)}{k^2 - 1}\right)r^2.$$  

(16)

Other turning points occur at $r_t$ given by

$$r_t = \left(\frac{-3}{5}\frac{(1 - 2\mu)}{k^2 - 1}\right)^{1/2}$$

(17)

which is only possible for

$$k < 1 \text{ and } \mu < \frac{1}{2},$$

or

$$k > 1 \text{ and } \mu > \frac{1}{2}.$$  

Figures 4(i) to (iv) illustrate the four possible cases:

(i) $\mu < \frac{1}{2}$, $k < 1$,

(ii) $\mu < \frac{1}{2}$, $k > 1$,

(iii) $\mu > \frac{1}{2}$, $k < 1$,

and

(iv) $\mu > \frac{1}{2}$, $k > 1$.

In case (iv) three possibilities arise, depending on whether $f(r_t)$ is greater, lesser or equal to zero.

For any root $s$ to represent a physical solution, we require that

$$s > 0, \text{ and } r_* \leq s \leq r^*,$$

where

$$r_* = \frac{1}{1 + k} \text{ and } r^* = \frac{1}{|1 - k|},$$

where $r_*$ and $r^*$ are the lower and upper bounds on $r_2$ imposed by $r_1 + r_2 \geq 1$ and $|r_1 - r_2| \leq 1$. Whenever $s = r_*$ or $s = r^*$ the solution corresponds to the inner or outer collinear point at bifurcation; however, we see below that the second case is impossible.

We now determine the relative position of $r_*$ and $r^*$ in the four cases above by evaluating $f(r_*)$, $f(r^*)$, $f'(r_*)$, and $f'(r^*)$. Substitution of $r_*$ and $r^*$ into Equations (14) and (16) gives

$$f(r^*) = \frac{2}{1 - k^2} \left\{\frac{1}{(1 - k)^4} \left(k(1 - \mu) + \mu - \mu \delta^3_2\right)\right\}, \quad k < 1$$

(18a)
Fig. 4(i). \( \mu < \frac{1}{2}, k < 1. \)

Fig. 4(ii). \( \mu < \frac{1}{2}, k > 1. \)

Fig. 4(iii). \( \mu > \frac{1}{2}, k < 1. \)

Fig. 4. Schematic graphs of the function

\[ f(r) = r^5 + \left[ (1 - 2\mu)/(k^2 - 1) \right] r^3 + 2\mu \beta/(k^2 - 1). \]

Cases iv(a), iv(b), and iv(c) correspond to \( f(r_L) > 0, \)
\( f(r_L) < 0, \) and \( f(r_L) = 0, \) respectively. \( r^* = 1/|k - 1| \) is an upper bound on the physically acceptable solutions of \( f(r) = 0. \)
Fig. 4(iv)(a)

Fig. 4(iv)(b)

Fig. 4(iv)(c)

Fig. 4(iv). $\mu > \frac{1}{4}, k > 1.$
\[ f'(r^*) = \frac{k(8 - 6\mu) + 2 + 6\mu}{(k + 1)(k - 1)^4} \] (18b)

and

\[ f(r_\star) = \frac{2}{k^2 - 1} \left[ \frac{1}{(1 + k)^4} (k(1 - \mu) - \mu) + \mu \delta_2^3 \right], \quad \text{for } k > 1 \] (18c)

\[ f'(r_\star) = \frac{2}{(k + 1)^3(k^2 - 1)} (k(4 - 3\mu) - (1 + 3\mu)). \] (18d)

From Equations (18a) and (18b) it can be seen that \( f(r^*) > 0 \) (N.B.: \( 0 < \delta_2 < 1 \)) and \( f'(r^*) > 0 \) in all cases, and hence if a root exists, it must necessarily satisfy \( s < r^* \) (see Figure 4). Thus we need only to examine whether or not \( r^*_\star \leq s \).

(i) \( \mu < \frac{1}{2}, \ k < 1 \)

In this case for the one positive root \( s \) to be a solution

\[ r^*_\star \leq s \Leftrightarrow f(r_\star) \leq 0 \] (19)

or from Equation (18c)

\[ \frac{k(1 - \mu) - \mu}{(1 + k)^4} + \mu \delta_2^3 \geq 0. \] (20)

The equality will correspond to the inner collinear point \( L_1 \).

Defining

\[ g(\delta_1, \delta_2) = \delta_1(1 - \mu)^{4/3} + \delta_2\mu^{4/3} - (1 - \mu)^{1/3} + \delta_2\mu^{1/3} \]

Equation (20) yields the inequality

\[ g(\delta_1, \delta_2) \leq 0. \] (21)

(ii) \( \mu < \frac{1}{2}, \ k > 1 \)

Evidently, since the only root is negative, there is no physical solution, as seen above.

(iii) \( \mu > \frac{1}{2}, \ k < 1 \)

The only root is positive. To be physical \( s \geq r_\star \).

Once again

\[ r^*_\star \leq s \Leftrightarrow f(r_\star) \leq 0 \]

and from Equation (18c) it follows that since \( k < 1 \)
\[ r_s \leq s \leq \frac{k(1 - u) - u}{(1 + k)^4} + u\delta_2^3 \geq 0 \]  

or substituting for \( k \) in terms of \( \delta_1, \delta_2, \) and \( u \)

\[ g(\delta_1, \delta_2) \leq 0. \]  

(iv) \( u \geq \frac{1}{4}, \ k \geq 1 \)

(a) The possibility arises (iv)(a) of no positive solution and, in this case the necessary and sufficient condition is \( f(r_t) > 0 \). Substitution for

\[ r_t = \left( \frac{3}{5} \frac{2u - 1}{k^2 - 1} \right)^{1/2} \]

in Equation (14) gives

\[ f(r_t) = \frac{2u\delta_2^3}{k^2 - 1} - a \left( \frac{2u - 1}{k^2 - 1} \right)^{5/2} \]  

where

\[ a = \left( \frac{3}{5} \right)^{3/2} - \left( \frac{3}{5} \right)^{5/2}. \]

With the substitution for \( k \) in terms of \( \delta_1, \delta_2 \) and \( u \) in Equation (24) the condition for no solution, \( f(r_t) > 0 \), becomes

\[ h(\delta_1, \delta_2) > 0, \]

where we define

\[ h(\delta_1, \delta_2) = \delta_1^2(1 - u)^{2/3} - \delta_2^2u^{2/3} - (2u - 1)^{5/3} \left( \frac{a}{2} \right)^{2/3}. \]  

This gives a left-hand and a right-hand region circumscribed by the hyperbola \( h(\delta_1, \delta_2) = 0 \) shown in Figure 5. This hyperbola has asymptotes corresponding to \( |k| = 1. \)

(b) If \( f(r_t) < 0 \) there will be two positive roots \( s_1 > s_2 \) \((s_1 < s_2)\), and it follows that:

- there are two physical solutions \( \leq r_s \leq s_1 \);
- there is one physical solution \( s_1 < r_s \leq s_2 \);
- there is no physical solution \( r_s > s_1, s_2 \).

Reference to (iv)(b) allows us to rewrite:

Two physical solutions \( \leq f(r_s) \geq 0 \) and \( f'(r_s) < 0 \) and \( f(r_t) < 0 \).

One physical solution \( f(r_s) < 0 \) or \( f(r_s) = 0 \) and \( f'(r_s) > 0 \).

(N.B.: \( f(r_t) < 0 \) is satisfied already by \( f(r_t) < 0 \).)

No physical solution \( f(r_s) > 0 \) and \( f'(r_s) > 0 \) and \( f(r_t) < 0 \).
\[ \mu = 0.8 \]

\[ k = 2.125 \]

\[ (\frac{3}{64})^{\frac{3}{2}} \]

\[ \mu = 0.9 \]

\[ k = 2.85 \]

\[ k = 1 \]

Fig. 5a.

Fig. 5b.

Fig. 5a-b. Existence of out of the orbital plane equilibrium points for (a) \( \mu = 0.8 \) and (b) \( \mu = 0.9 \). Horizontal shading indicates two points, vertical shading four. Two equilibrium points exist on the bold lines and none on the fine lines.
From Equations (18c, d) we have:

Two physical solutions \( g(\delta_1, \delta_2) \leq 0 \) and \( k < \frac{1 + 3\mu}{4 - 3\mu} \) and \( h(\delta_1, \delta_2) < 0 \).

One physical solution \( g(\delta_1, \delta_2) > 0 \) or \( \left( g(\delta_1, \delta_2) = 0 \text{ and } k > \frac{1 + 3\mu}{4 - 3\mu} \right) \).

No physical solution \( g(\delta_1, \delta_2) < 0 \) and \( k > \frac{1 + 3\mu}{4 - 3\mu} \) and \( h(\delta_1, \delta_2) < 0 \).

(c) If \( f(r_t) = 0 \), we have a degenerate root, which will give one physical solution iff \( r_t \geq r_\ast \), and no solution iff \( r_\ast > r_t \).

These two conditions are equivalent to \( f'(r_\ast) \leq 0 \), and \( f'(r_\ast) > 0 \), respectively. Thus when \( h(\delta_1, \delta_2) = 0 \) there is one physical solution iff

\[
k < \frac{1 + 3\mu}{4 - 3\mu},
\]

and otherwise none (from Equation (18d)).

Combining (iv)(a), (b), and (c) with (i), (ii), and (iii) and the special cases when \( k = 1 \), or \( \mu = \frac{1}{2} \) we obtain the following picture for \( \delta_1 \leq 0 \) and \( \delta_2 > 0 \).

When \( \mu \leq \frac{1}{4} \)
There is one solution iff \( g(\delta_1, \delta_2) \leq 0 \) and \( k < 1 \). Otherwise there is no solution.

When \( \mu > \frac{1}{4} \)
There are two physical solutions iff

\[
1 < k < \frac{1 + 3\mu}{4 - 3\mu}
\]

and

\[
g(\delta_1, \delta_2) \leq 0 \text{ and } h(\delta_1, \delta_2) < 0.
\]

There is one solution iff either \( k \leq 1 \text{ and } g(\delta_1, \delta_2) = 0 \)

or \( k > 1 \) and \( g(\delta_1, \delta_2) > 0 \);

or \( 1 < k < \frac{1 + 3\mu}{4 - 3\mu} \) and \( g(\delta_1, \delta_2) = 0 \);

or \( 1 < k \leq \frac{1 + 3\mu}{4 - 3\mu} \) and \( h(\delta_1, \delta_2) = 0 \);

and otherwise there is no solution.

When \( \delta_1 = 0 \) and \( \delta_2 = 0 \), an infinite number of solutions exist along the \( z \)-axis, with \( z \) taking any value. When \( \delta_1 \neq 0 \) and \( \delta_2 = 0 \) there is no solution, as is evident from Equation (8b).
The complete picture for all values of $\delta_1$ and $\delta_2$ for any given $\mu$ can now be found by combining the results derived above for $\mu$ with those for $1-\mu$ for $\delta_2 > 0$, i.e., the system $(m_1, m_2, \delta_1, \delta_2)$ is the same as the system $(m_2, m_1, \delta_2, \delta_1)$.

The situation is represented in Figure 5, where the regions of two solutions, one solution and no solutions are plotted on the $\delta_1, \delta_2$ plane for several different values of $\mu$. From what has been said above it is clear that the plot for $m_1 = 1 - \mu$ can be obtained from $m_1 = \mu$ simply by interchanging $\delta_1$ and $\delta_2$.

We shall defer a discussion of the position in space of $L_6, L_7, L_8,$ and $L_9$ until Section 4.

3. STABILITY OF THE EQUILIBRIUM POINTS

The equation of motion for a test particle in the corotating frame is simply

\[ \ddot{r} + 2 \dot{\varphi} \times \dot{r} = -\nabla V, \]  

(26)

where $V$ is given by Equation (1) and $\varphi$ is the unit vector in the $z$-direction, and $\xi$ the projection of $\varphi$ on the $x, y$ plane. If we allow infinitesimal displacements $(\xi, \eta, \zeta)$ of the test particle from any one of the equilibrium points, the motion of the particle in the neighbourhood of the equilibrium point will be given by

\[ \dot{\xi} - 2\eta = -\xi V_{xx} - \eta V_{x\xi} - \zeta V_{x\zeta}, \]
\[ \dot{\eta} + 2\xi = -\eta V_{xy} - \xi V_{y\xi} - \zeta V_{yz}, \]
\[ \dot{\zeta} = -\xi V_{xz} - \eta V_{yz} - \zeta V_{zz}, \]  

(27)

where the second derivatives of $V$ are evaluated at the equilibrium points and can be obtained by differentiating Equations (4a), (4b), and (4c).

Explicitly, at any general point $(x, y, z)$ not necessarily an equilibrium point we have

\[ V_{xx} = -1 + A - 3\delta_1^3 (1 - \mu) \frac{(x + \mu)^2}{r_1^5} - 3\delta_2^3 \mu \frac{(x - 1 + \mu)^2}{r_2^5}, \]  

(28a)

\[ V_{yy} = -1 + A - 3 By^2, \]  

(28b)

\[ V_{zz} = A - 3 B z^2, \]  

(28c)

\[ V_{xy} = -3y \left\{ \delta_1^3 \frac{(1 - \mu)(x + \mu)}{r_1^5} + \delta_2^3 \mu \frac{(x - 1 + \mu)}{r_2^5} \right\}, \]  

(28d)

\[ V_{xz} = -3z \left\{ (1 - \mu) \delta_1^3 \frac{(x + \mu)}{r_1^5} + \mu \delta_2^3 \frac{(x - 1 + \mu)}{r_2^5} \right\}, \]

\[ = -3z \left\{ Bx + (1 - \mu) \mu \frac{\delta_1^3}{r_1^5} - \frac{\delta_2^3}{r_2^5} \right\}, \]  

(28e)

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System
\[ V_{YZ} = -3Byz, \]  
where

\[ B = \frac{\delta^3_1 (1 - \mu)}{r_1^5} + \frac{\delta^3_2 \mu}{r_2^5}. \]

We shall only be concerned with the values of the second derivative of \( V \) at the equilibrium points.

The trial solution

\begin{align*}
\xi & = \xi_0 \\
\eta & = \eta_0 \, e^{\lambda t} \\
\zeta & = \zeta_0 
\end{align*}

in Equation (27) yields the characteristic equation for the complex normal frequencies \( \lambda_j \)

\[
\begin{vmatrix}
\lambda_j^2 + V_{xx} & -2\lambda_j + V_{xy} & V_{xz} \\
2\lambda_j + V_{xy} & \lambda_j^2 + V_{yy} & V_{yz} \\
V_{xz} & V_{yz} & \lambda_j^2 + V_{zz}
\end{vmatrix} = 0
\]  

(29)

which is a sixth order polynomial in \( \lambda \). As we see below, its solution is considerably simplified when the values of \( V_{xx} \) etc. are substituted.

An equilibrium point will be stable if and only if the real parts of all the normal frequencies are non-positive. In Sections 3.1, 3.2, and 3.3, we examine this question for the triangular points \( L_4, L_5 \), the collinear points \( L_1, L_2, L_3 \) and the out of plane points \( L_6, L_7, L_8, L_9 \), respectively.

3.1. Triangular points \( L_4, L_5 \)

This case has been dealt with by Chernikov (1970) and Schuerman (1980), who derived the result that stability is obtained provided

\[ 36\mu(1 - \mu)b \leq 1, \]

(30)

where

\[ b = 1 - \left( \frac{\delta^2_1 + \delta^2_2 - 1}{2\delta_1\delta_2} \right)^2 \]

which of course includes the classical Routh result as a special case.

It is illuminating to find the region in the \( \delta_1, \delta_2 \) plane for which stability holds. To this end we rewrite Equation (30)

\[
\left( \frac{\delta^2_1 + \delta^2_2 - 1}{2\delta_1\delta_2} \right) \geq 1 - \frac{1}{36\mu(1 - \mu)}.
\]

(31)
Inequality (31) is immediately satisfied for all $\delta_1$, $\delta_2$ provided

$$36\mu(1 - \mu) \leq 1,$$

i.e. $\mu \geq \frac{1}{2} + \frac{\sqrt{2}}{3}$ or $\mu \leq \frac{1}{2} - \frac{\sqrt{2}}{3}$.

If this is not the case we can write Equation (31) as

$$\delta_1^2 + \delta_2^2 - 1 - 2\delta_1\delta_2a \geq 0 \quad (32a)$$

or

$$\delta_1^2 + \delta_2^2 - 1 + 2\delta_1\delta_2a \leq 0, \quad (32b)$$

where we have put

$$a = \sqrt{1 - \frac{1}{36\mu(1 - \mu)}}.$$

Reducing the above inequalities to canonical form gives

$$\left(\frac{\delta_1 - \delta_2}{\sqrt{2}}\right)^2 (1 + a) + (1 - a) \left(\frac{\delta_1 + \delta_2}{\sqrt{2}}\right)^2 \geq 1 \quad (33a)$$

or

$$\left(\frac{\delta_1 - \delta_2}{\sqrt{2}}\right)^2 (1 - a) + (1 + a) \left(\frac{\delta_1 + \delta_2}{\sqrt{2}}\right)^2 \leq 1 \quad (33b)$$

which has a neat geometrical interpretation, in terms of 'conjugate' ellipses in the $\delta_1$, $\delta_2$ plane, with semi major and minor axes of lengths $1/(\sqrt{1-a})$ and $1/(\sqrt{1+a})$, as shown in Figure 6.

Stability will hold for positive values of $\delta_1$, $\delta_2$ that lie outside the ellipse $E_+$ whose major axis lies along the line $\delta_1 = \delta_2$, or within the ellipse $E_-$ whose major axis lies along the line $\delta_1 = -\delta_2$. Of course, the triangular points must exist, which is the case for $\delta_1 + \delta_2 \geq 1$ and $\delta_1, \delta_2 \leq 1$. Thus in general we can expect two disjoint regions where stability holds (cf. Figure 6(a)). However, when $\mu < 0.9615$ (Routh's value) or $\mu > 0.0385$, the semi major axis $1/(\sqrt{1-a}) > \sqrt{2}$ and all points lying outside $E_+$ are excluded, so there is only one region where stability holds (cf. Figure 6(d)). When $a = 0$, $E_+$ and $E_-$ collapse onto each other and are circles. In this case the triangular points when they exist will be stable for all values of $\delta_1$, $\delta_2$ (cf. Figure 6(b)). The critical values of $\mu$ corresponding to $a = 0$ are given by $36\mu(1 - \mu) = 1$, and thus for all

$$\mu \geq \frac{1}{2} + \frac{\sqrt{2}}{3}, \quad \mu \leq \frac{1}{2} - \frac{\sqrt{2}}{3}$$

the region of stability is the triangle described by $\delta_1 + \delta_2 \geq 1$, $\delta_1 \leq 1$, $\delta_2 \leq 1$. On the other hand the most restricted region will be obtained when $1/(\sqrt{1+a})$ is a minimum, i.e. when $36\mu(1 - \mu)$ is maximum, implying $\mu = 0.5$. 
Fig. 6. Values of $\delta_1$ and $\delta_2$ for which triangular points are stable. When $\delta_1 + \delta_2 \leq 1$ the triangular points do not exist.

(a) $\mu = 0.97$ (or $\mu = 0.03$). There are two distinct regions.
(b) $\mu = 0.971$ (or $\mu = 0.029$). $L_4$ and $L_5$ are stable for all $\delta_1, \delta_2$.
(c) $\mu = 0.9615$ (or $\mu = 0.0385$). This is Routh's value. Stability just obtains at $\delta_1 = 1$, $\delta_2 = 1$, i.e. no radiation pressure from either star.
(d) $\mu = 0.95$ (or $\mu = 0.05$). Stability with no radiation pressure is now impossible. However for any value of $\mu$, $\delta_1$ and $\delta_2$ can always be chosen small enough so that $L_4$ and $L_5$ are stable.
3.2. Collinear Points $L_1, L_2, L_3$

Contrary to the findings of Schuerman (1980), there is a range of $\delta_1, \delta_2$ for which the inner Lagrange point, $L_1$, is stable, although $L_2$ and $L_3$ are always unstable.

This region was not discussed by Radzievskii or Chernikov (1970), who restricted themselves to $\delta_1$ or $\delta_2 = 1$, which necessarily yield unstable $L_1$.

Since $z = y = 0$ for all the collinear points, it follows from Equation (28) that all the mixed second derivations of the modified potential $V$ are zero at $L_1, L_2$, and $L_3$. From Equations (28) and (29) we obtain the characteristic equation

$$(\lambda^2 + A)(\lambda^4 + \lambda^2(2 - A) + (1 + 2A)(1 - A)) = 0. \quad (34)$$

The roots are thus given by

$$\lambda^2 = -A \quad (35a)$$

and

$$\lambda^2 = -\frac{(2 - A) \pm \sqrt{(2 - A)^2 - 4(1 + 2A)(1 - A)}}{2} \quad (35b)$$

and are completely determined by the value of $A$. The requirement for stability is that the real parts of the normal frequencies should all be non-positive. Hence for stability it follows from (35a) that

$$A \geq 0 \quad (36a)$$

and from (35b) that

$$(2 - A)^2 - 4(1 + 2A)(1 - A) \geq 0 \quad (36b)$$

and

$$-(2 - A) \pm \sqrt{(2 - A)^2 - 4(1 + 2A)(1 - A)} \leq 0 \quad (36c)$$

Equation (36b) reduces to

$$A \geq \frac{2}{3} \quad \text{or} \quad A \leq 0 \quad (37)$$

and Equation (36c) gives

$$\sqrt{9A^2 - 8A} \leq 2 - A. \quad (38)$$

The necessary and sufficient conditions for the inequality (38) to hold is

$$A \leq 2 \quad \text{and} \quad -\frac{1}{3} \leq A \leq 1. \quad (39)$$

Combining these conditions (39) with (37) yields the necessary and sufficient conditions for stability

$$\frac{2}{3} \leq A \leq 1 \quad \text{or} \quad A = 0. \quad (40)$$
A = 1 and A = 0 are critical cases in the sense that \( \lambda = 0 \) for both these cases.

We shall now establish the region of stability in the \( \delta_1, \delta_2 \) plane for \( L_1, L_2, \) and \( L_3 \).

Analysis of the upper limit \( A = 1 \) is considerably simplified by using the relation (which is valid for all points \( L_1, L_2, \) and \( L_3 \)) that follows from Equation (4a)

\[
1 - A = \left[ \frac{\mu(1 - \mu)}{x} \right] \left[ \frac{\delta_1^3}{r_1^3} - \frac{\delta_2^3}{r_2^3} \right] \quad \text{for} \quad x = x_0,
\]

(41)

where \( x_0 \) is the x-coordinate of \( L_1, L_2, \) or \( L_3 \).

(i) The inner Lagrange point \( L_1 \)

The possibility of stable \( L_1 \) points is shown by considering the special symmetric case \( \mu = \frac{1}{2} \) and \( \delta_1 = \delta_2 \). Then by symmetry \( L_1 \) occurs at \( x = 0 \), and Equation (5) gives \( A = 8\delta_1^3 \). Thus \( L_1 \) will be stable provided \( \frac{1}{2} \leq \delta_1 \leq \frac{1}{8} \) (or \( \frac{1}{2} \leq \delta_1 \leq \frac{1}{8} \)).

It is interesting to note that the upper limit of \( \delta_1 = \frac{1}{2} \) corresponds to the degenerate case where \( \delta_1 + \delta_2 = 1 \), and \( A = 1 \), that is, where the triangular and \( L_1 \) points coincide on the x-axis.

We now establish the full region of stable \( L_1 \) for arbitrary \( \mu \).

If we consider

\[
A = \frac{\delta_1^3(1 - \mu)}{r_1^3} + \frac{\delta_2^3\mu}{r_2^3}
\]

as a function of \( \delta_1 \) and \( \delta_2 \), the value of \( r_1 \) and \( r_2 \) corresponding to the equilibrium point \( L_1 \), then it is clear from Equation (7) and (8b) that \( A \) has a singularity at \( (\delta_1 = 0, \delta_2 = 1) \) and \( (\delta_1 = 1, \delta_2 = 0) \). We now investigate the region in \( \delta_1, \delta_2 \) the plane for which \( A \leq 1 \).

For \( \delta_1, \delta_2 \leq 0 \) it is evident that \( A \leq 0 \) and a fortiori \( A < 1 \). For \( \delta_2 > 0 \) and \( \delta_1 \leq 0 \) (\( \beta > 0, \alpha \leq 0 \)), \( x_0 < 0 \) (see Section 2.2) and Equation (41) gives \( 1 - A > 0 \). If \( \delta_1 = \delta_2 = 0 \) then \( x_0 = 0 \) and Equation (5) gives \( A = 0 \). By symmetry this also holds for \( \delta_1 > 0 \) and \( \delta_2 \leq 0 \). Thus we need only discuss the case where \( \delta_1 > 0, \delta_2 > 0 \), for otherwise \( A < 1 \) except at \( (\delta_1 = 0, \delta_2 = 1), (\delta_1 = 1, \delta_2 = 0) \) where \( A \) is singular.

\( \delta_1 > 0, \delta_2 > 0 \)

The \( L_1 \) point is determined by

\[
f_1(x_0) = 0,
\]

(42)

where

\[
f_1(x) = x - \frac{\delta_1^3(1 - \mu)}{(x + \mu)^2} + \frac{\delta_2^3\mu}{(1 - x - \mu)^2}
\]
(cf. Equation (9b), and note that we have suppressed the $\delta_1$ subscript). For our immediate purposes we need not evaluate $x_0$, which except for a few special cases is best done numerically. It is sufficient to establish that

\[
x_0 = 0 \Leftrightarrow \mu = \frac{\delta_1}{\delta_1 + \delta_2}, \tag{43a}
\]

\[
x_0 < 0 \Leftrightarrow \mu > \frac{\delta_1}{\delta_1 + \delta_2}, \tag{43b}
\]

\[
x_0 > 0 \Leftrightarrow \mu < \frac{\delta_1}{\delta_1 + \delta_2}. \tag{43c}
\]

We see this as follows: since $f_1$ is monotonic increasing in $(-\mu, 1-\mu)$ for $\delta_1 > 0$ and $\delta_2 > 0$

\[
x_0 = 0 \Leftrightarrow f_1(x_0) = f_1(0), \tag{44}
\]

\[
x_0 < 0 \Leftrightarrow f_1(x_0) < f_1(0),
\]

\[
x_0 > 0 \Leftrightarrow f_1(x_0) > f_1(0).
\]

From Equation (42)

\[
f_1(0) = -\frac{\delta_1^3(1-\mu)}{\mu^2} + \frac{\delta_2^3\mu}{(1-\mu)^2}. \tag{45}
\]

Noting that $f_1(x_0) = 0$ we obtain the desired result (43).

Consider then the three cases:

(a) $\mu = \frac{\delta_1}{\delta_1 + \delta_2},$

(b) $\mu > \frac{\delta_1}{\delta_1 + \delta_2}$

and

(c) $\mu < \frac{\delta_1}{\delta_1 + \delta_2}.$

Case (a): $\mu = \frac{\delta_1}{\delta_1 + \delta_2}$.

From Equation (43a) $x_0 = 0$ and direct substitution into expression (5) yields $A = (\delta_1 + \delta_2)^3$. Thus stability obtains iff $A < 1$, i.e. $\delta_1 + \delta_2 < 1$.

Case (b): $\mu > \frac{\delta_1}{\delta_1 + \delta_2}$

From Equation (43b) $x_0 < 0$. It follows from Equation (41) that
THE RESTRICTED 3-BODY PROBLEM WITH RADIATION PRESSURE

\[ A < 1 \Rightarrow E(x_0) < 0, \]  
\[ E(x) = \frac{\delta_1^3}{(x + \mu)^3} - \frac{\delta_2^3}{(1 - x - \mu)^3} \]  
\[ \text{in the domain } D_1 \text{ given by } -\mu < x < 1 - \mu. \]  
\[ \text{For all } \mu, \ E(x) \text{ is a monotonic decreasing function of } x \text{ in } D_1, \text{ provided of course that } \delta_1 > 0, \delta_2 > 0. \]  
\[ \text{Furthermore } E(x) \rightarrow -\infty \text{ as } x \rightarrow (1 - \mu)^-, \text{ and } E(x) \rightarrow +\infty \text{ as } x \rightarrow -\mu^+. \]  
\[ \text{The zero, } x_E, \text{ of } E(x) \text{ is given by} \]  
\[ x_E = -\mu + \frac{\delta_1}{\delta_1 + \delta_2} \]  
\[ \text{and thus, since } E(x) \text{ is monotonic decreasing in } D_1 \]  
\[ E(x_0) < 0 \Rightarrow x_0 > x_E. \]  

The numerical value of \( x_0 \) has not been determined and so cannot be directly compared to \( x_E \). However, from the monotonic character of \( f_1(x) \) in \( D_1 \), we have

\[ x_0 > x_E \Rightarrow f_1(x_0) > f_1(x_E). \]

Directly evaluating \( f_1(x_E) \) from Equation (42) and (48) we obtain

\[ f_1(x_E) = \frac{1}{\delta_1 + \delta_2} [\delta_1 - \mu(\delta_1 + \delta_2)][1 - (\delta_1 + \delta_2)^3]. \]

Noting that \( f_1(x_0) = 0 \), Equations (49), (50), and (51) yield

\[ E(x_0) < 0 \Rightarrow \frac{1}{\delta_1 + \delta_2} [\delta_1 - \mu(\delta_1 + \delta_2)][1 - (\delta_1 + \delta_2)^3] < 0. \]

Since

\[ \mu > \frac{\delta_1}{\delta_1 + \delta_2}, \]

the above double implication reduces to

\[ E(x_0) < 0 \Rightarrow \delta_1 + \delta_2 < 1 \]

and thus as in case (a), \( A < 1 \iff \delta_1 + \delta_2 < 1. \)

**Case (c):** \( \mu < \frac{\delta_1}{\delta_1 + \delta_2} \).

From (43c) \( x_0 > 0 \), and so
\[ A < 1 \iff E(x_0) > 0. \] (54)

Using precisely the same argument as in case (b) we have

\[ A < 1 \iff \mathcal{f}_1(x_E) > 0. \] (55)

Since

\[ \mu < \frac{\delta_1}{\delta_1 + \delta_2} \]

we have once again

\[ A < 1 \iff \delta_1 + \delta_2 < 1. \]

Thus the necessary and sufficient conditions for all \( \mu \) that \( A < 1 \) is that \( \delta_1 + \delta_2 < 1 \).

We now show that \( A = 1 \iff \delta_1 + \delta_2 = 1 \).

From (41) \( A = 1 \iff \) either \( x_0 \neq 0 \) and

\[ \frac{\delta_1^3}{r_1^3} = \frac{\delta_2^3}{r_2^3}, \text{ or } x_0 = 0 \text{ and } \mu = \frac{\delta_1}{\delta_1 + \delta_2} \] (56)

\[ \implies r_1 = \frac{\delta_1}{\delta_1 + \delta_2} \text{ and } r_2 = \frac{\delta_2}{\delta_1 + \delta_2}. \]

Substituting these values of \( r_1 \) and \( r_2 \) into Equation (5) we have

\[ A = (\delta_1 + \delta_2)^3 \text{ and hence } \delta_1 + \delta_2 = 1. \]

Conversely, if \( \delta_1 + \delta_2 = 1, r_1 = \delta_1 \) (or \( r_2 = \delta_2 \)) is a solution of Equation (9b).

For \( \delta_1, \delta_2 > 0 \) the solution is unique. Substituting \( r_1 = \delta_1, r_2 = \delta_2 \) into expression (5) for \( A \) we have \( A = 1 \).

The constraint \( \delta_1 \geq \frac{\delta_2}{3} \) imposes another restriction on the region in the \( \delta_1, \delta_2 \) plane for which stability holds, and this is done by calculating the \( x \) coordinate, \( x_0 \), of the \( L_1 \) point by the standard Newton-Raphson technique, and hence evaluating \( A \) from Equation (5). In fact, it is useful and instructive to look at the isocurves of \( A \) for different values of \( \mu \). We have seen already the importance of the \( A = -1 \) and \( A = 0 \) isocurves, which correspond with the disappearance of the \( L_1/L_2 \), \( L_3 \) points, and the appearance of the out of orbital plane points \( L_6 \) and \( L_7 \) respectively.

Singularity exist in \( A \) (considered as a function of \( \delta_1 \) and \( \delta_2 \) for fixed \( \mu \)) at \( (\delta_1, \delta_2) = (1, 0) \) and \( (\delta_1, \delta_2) = (0, 1) \). All isocurves converge on these singular points. It can be shown from Equation (9b) and Equation (5) that \( A = \mu \) and \( A = 1 - \mu \) give critical isocurves which are tangential to the \( \delta_2 \) axis and \( \delta_1 \) axis, respectively. Figure 7a, b, c, show isocurves of \( A \) for different values of \( \mu \) (\( \mu = 0.9, \mu = 0.8, \mu = 0.5 \)). In all the graphs, the isocurves increase in numerical value moving from right to left. Whenever \( \mu > \frac{\delta_2}{3} \), there will be a small region with negative \( \delta_1 \) for which stability of \( L_1 \)
Fig. 7a-c. Isocurves of \( A = \frac{\alpha(1 - \mu)}{r_1^3} + \frac{\beta \mu}{r_2^3} \) evaluated at the \( L_1 \) point.
holds, otherwise this is not the case. Outside the region delimited by PQRS in Figure 7, the \( L_1 \) point ceases to exist, as was discussed in Section 2.

It is interesting to note that precisely when the triangular points become degenerate and collapse into the x-axis, the collinear \( L_1 \) point, which hitherto was unstable, becomes stable. As the radiation pressure further increases from either star A will decrease below \( \frac{\delta_1}{\delta_2} \) at which point \( L_1 \) again becomes unstable. There is, of course, the special case where \( A = 0 \), which corresponds to the natural frequency \( \lambda = 0 \). This point will be critically stable. As the radiation pressure further increases, the out of plane point will emerge from the x-axis.

Clearly \( L_1 \) in the classical case, where \( \delta_1 = \delta_2 = 1 \), is unstable. Moreover, a necessary condition for the stability of \( L_1 \) is that both bodies should exert radiation pressure (i.e. \( \delta_1, \delta_2 \neq 1 \)), which means that for the Sun-planet systems \( L_1 \) is always unstable, as was found by Chernikov (1970).

(ii) \( L_2 \) and \( L_3 \) (\( L'_2 \) and \( L'_3 \))

\( L_2 \) and \( L_3 \), the outer Lagrange points are unstable for all radiation pressures and mass ratios. We prove this by considering different regions of radiation pressure and make use of the symmetry between star 1 and star 2.

\[ \delta_1, \delta_2 > 0 \]

Without loss of generality consider only \( L_2 \), which necessarily lies in the region \( D_2 \). We now prove by contradiction that \( A > 1 \).

The x-coordinate of \( L_2 \), \( x_0 \) say, satisfies Equation (9c) viz.

\[ x_0 - \frac{\delta_1^3 (1 - \mu)}{r_1} - \frac{\delta_2^3}{r_2} \mu = 0. \]

Evidently, since \( \delta_1, \delta_2 > 0 \), \( x_0 > 0 \) and from Equation (41) we have

\[ 1 - A \geq 0 \iff \left( \frac{\delta_1^3}{r_1} - \frac{\delta_2^3}{r_2} \right) \geq 0 \iff \left( \frac{r_1}{x_0} \right) \leq \left( \frac{\alpha}{\beta} \right)^{1/3} = \frac{\delta_1}{\delta_2}. \]  

(The r.h.s. is evidently false for \( \delta_1 \leq \delta_2 \) since \( r_1 > r_2 \), also \( x_0 = 0 \) is excluded.)

With

\[ f_2(x) = x - \frac{\alpha (1 - \mu)}{r_1^2} - \frac{\beta \mu}{r_2^2}, \]

it follows immediately that

\[ f_2'(x) > 0 \ \forall \ x > 1 - \mu. \]

Since \( f_2(x) \to -\infty \) as \( x \to (1 - \mu)^+ \) and \( f_2(x) \to \infty \) as \( x \to \infty \), \( f_2(x) \) must have one and only one zero in \( D_2 \) given by \( 1 - \mu < x < \infty \).
Let $x_p$ be defined in such a way that

$$\frac{r_1}{r_2} = \frac{x - x_p}{x_p} = \frac{\delta_1}{\delta_2}$$

from which it follows that

$$x_p = -\mu + \frac{\delta_1}{\delta_1 - \delta_2}.$$ 

Since $r_1/r_2$ is monotonic decreasing in $x$, it follows from (57) that

$$1 - A \geq 0 \iff x_0 \geq x_p.$$ 

Once again using the fact that $f_2$ is monotonic increasing we have

$$1 - A \geq 0 \iff f_2(x_0) \geq f_2(x_p).$$ 

By definition $f_2(x_0) = 0$ so (58) becomes, upon substitution for $x_p$,

$$1 - A \geq 0 \iff 0 \geq [\delta_1(1 - \mu) + \delta_2\mu] \frac{[1 - (\delta_1 - \delta_2)^2]}{\delta_1 - \delta_2}.$$ 

However, the expression for $f_2(x_p)$ is necessarily positive, and so $A > 1$. Thus $L_2$ is unstable for $\delta_1, \delta_2 > 0$, and by symmetry so also is $L_3$.

$\delta_1 < 0, \delta_2 > 0$

Consider $L_3$, which now appears within the interval $(-\mu, 1 - \mu)$ as $L'_3$, providing $\alpha > \alpha_{\text{crit}}$ (see Section 2, Figure 1a), where $\alpha_{\text{crit}}$ depends on $\mu$ and $\beta$. (N.B., when $\beta = \delta_2 = 1$, $\alpha_{\text{crit}}$ does not exist, and $L'_3$ does not appear within $(-\mu, 1 - \mu)$).

$$A(x) = \frac{\delta_1^3(1 - \mu)}{(x + \mu)^3} + \frac{\delta_2^3\mu}{(1 - x - \mu)^3}$$

is a monotonic increasing function of $x$ for given $\mu, \delta_2 > 0$, and $\delta_1 < 0$. But $A$ evaluated at the minimum of $f_1(x)$ takes the value $-\frac{1}{4}$, and so $A < -\frac{1}{4}$ when evaluated at $L'_3$. $L'_3$ is therefore unstable. It follows from Equation (41) that $L_2$ is also unstable, since $x_0 > 0$ and

$$\frac{\delta_1^3}{r_1} - \frac{\delta_2^3}{r_2} < 0.$$ 

$L_2$ will also be unstable when $\delta_1 = 0$.

$\delta_1 < 0, \delta_2 < 0$

All collinear points are now unstable since $A < 0$. 

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System
\[ \delta_1 = 0 \]

\( L_3 \) does not exist. \( L_2 \) is unstable for \( \delta_2 > 0 \) (see above), and necessarily \( A < 0 \) when \( \delta_2 < 0 \). If both \( \delta_1 = 0 \) and \( \delta_2 = 0 \) only \( L_1 \) exists.

### 3.3. The Out of Plane Points \( L_6, L_7, L_8, L_9 \)

In Section 2.3 we established the regions in which solutions off the orbital plane exist. For these equilibrium points Equation (29) reduces to

\[
\begin{vmatrix}
\lambda^2 + V_{xx} & -2\lambda & V_{xz} \\
2\lambda & \lambda^2 - 1 & 0 \\
V_{xz} & 0 & \lambda^2 + V_{zz}
\end{vmatrix} = 0, \quad (60)
\]

where we have used the fact that \( y = 0 \) and \( A = 0 \). This further simplifies to

\[
\lambda^6 + 2\lambda^4 + (V_{xx}V_{zz} + 1 + 4V_{zz} - V_{xz}^2)\lambda^2 + (V_{xz}^2 - V_{xx}V_{zz}) = 0, \quad (61)
\]

where the second order derivatives are given by Equation (28). The region in the \( \delta_1, \delta_2 \) plane for which stability holds can only be obtained by solving Equations (12a, b) numerically, and evaluating the roots of Equation (61), which is a cubic in \( \lambda^2 \).

Figure 8 shows the regions of stable solutions for different values of \( \mu, L_8 \) and \( L_9 \) when they exist are unstable for all \( \delta_1 \) and \( \delta_2 \) and for all values of \( \mu \). There is for every value of \( \mu \), however, a limited region of \( \delta_1, \delta_2 \) for which \( L_6 \) and \( L_7 \) are stable. This region is greatest in the special case when \( \mu = 0.5 \).

For \( \mu \neq 0.5 \) stability of \( L_6 \) and \( L_7 \) is only possible when the less massive body is the more luminous. Except in the case \( \mu = 0.5 \), stability can only arise when both bodies exert a radiation pressure on the test particle (i.e. \( \delta_1 \) and \( \delta_2 \neq 1 \)). Previous work by Pereszhogin (1976) established the instability of \( L_6 \) and \( L_7 \) for the Sun-planet systems, which of course is a special case of the results presented here.

### 4. OVERVIEW AND CONCLUSIONS

For any given \( \mu \), the existence, position and stability of the equilibrium points depends on \( \delta_1 \) and \( \delta_2 \). It is instructive to see how the variation of the radiation pressure from one star affects the motion and behaviour of all the equilibrium points as an integrated whole (cf. Figure 9). This behaviour is necessarily complicated, with the transformations, appearance and onset of stability of any one equilibrium point interlocking with the behaviour of the others.

By way of example, we shall consider the case \( \mu = 0.8 \) keeping the radiation pressure \( (\delta_2) \) associated with the secondary fixed, and \( \delta_1 \) varying
Fig. 8a-b. Existence and stability of equilibrium points for (a) $\mu = 0.5$ and (b) $\mu = 0.8$. As before horizontal shading indicates two out of the orbital plane equilibrium points, vertical shading four. Dotted region indicates stability. Heavy broken line shows $A = -0.5$ isocurve, determining the disappearance of the two inner Lagrange points ($L_1', L_2'$) or ($L_1$, $L_3'$).

from its classical value of 1 to negative values. We consider three representative values $\delta_2$:

(a) $\delta_2 = 1$; (b) $\left(\frac{2\mu - 1}{16\mu}\right)^{1/3} < \delta_2 < 1$; (c) $0 < \delta_2 < \left(\frac{2\mu - 1}{16\mu}\right)^{1/3}$;

(cf. Figure 8).
Fig. 9. Schematic diagram of the motion of the equilibrium point in 3 dimensional space with radiation pressure from the secondary ($\delta_2$) fixed at different values and radiation pressure of the primary ($\delta_1$) increasing. Points at which equilibria coalesce or appear are indicated by $\circ$. Three cases for $\mu = 0.8$ are shown corresponding to $\delta_2 = 1$, $0.6$, and $0.1$ and two cases for $\mu = 0.2$ corresponding to $\delta_2 = 1$ and $0.6$. Dotted line indicates stability of the equilibrium points.
\[ \mu = 0.8 \]
\[ \delta_2 = 0.1 \]

Fig. 9c.

\[ \mu = 0.2 \]
\[ \delta_2 = 1 \]

Fig. 9d.
The case where the primary is more massive, say $\mu = 0.2$, can also be obtained from the Figure 8.

We shall refer simultaneously to Figures 8 and 9.

\[ \mu = 0.8 \]

(a) $\delta_2 = -1$

At $\lambda_1$, $L_1$, $L_2$, $L_3$, $L_4$, $L_5$ exist at their classical values and are all unstable. As $\delta_1$ decreases to 0 (at $\lambda_2$) $L_1$, $L_3$ and $L_4$ and $L_5$ all converge onto the primary mass at $x = -\mu$, and subsequently disappear. As $\delta_1$ becomes negative $L_6$ and $L_7$ emerge from the x-axis off the orbital plane. $L_8$ and $L_9$ appear at $x = 0$, $y = 0$, $z = \pm$ when

\[ \delta_1 = -\left(\frac{\mu}{1 - \mu}\right)^{1/3} = -(4)^{1/3} \approx -1.6, \]

(at $\lambda_3$) and subsequently move in direction of decreasing $|z|$ until they merge with $L_6$ and $L_7$ respectively at $\delta_1 = -1.67$ given by Equation (25), and then disappear (at $\lambda_4$). At all times the equilibrium points are all unstable. Throughout $L_2$ exists and approaches the secondary mass at $x = 1 - \mu$, as in the cases below.
\( \delta_2 = 0.6 \)

L_1, L_2, L_3, L_4, L_5 all exist at \( \delta_1 = 1 \) (at B_1). As \( \delta_1 \) decreases L_1 moves in direction of primary, and L_4 and L_5 move along the arc of a circle radius \( \delta_2 \) and centred at \( m_2(x = 1 - \mu) \) towards the x-axis. L_4 and L_5 become stable at B_2 and remain so until they merge on the x-axis with L_1 at \( x = -0.4 \) (B_3). L_1 is now stable (L_4 and L_5 no longer exist), but at B_4 becomes unstable, moving all the time in the direction of the primary. At B_5 (\( \delta_1 = 0.41 \)) bifurcation takes place, and L_6 and L_7 move out from L_1 in the x-z plane. They are briefly stable until B_6 is reached, whereupon instability sets in. L_1 and L_3 merge and disappear (B_7). At

\[
\delta_1 = -\delta_2 \left( \frac{\mu}{1 - \mu} \right)^{1/3} = -0.6(4)^{1/3} = -0.95
\]

L_8 and L_9 appear (B_8) at \( x = 0, y = 0, z = \pm \infty \) and then move towards the orbital plane until they merge with L_6 and L_7 at \( \delta_1 = -1.1 \) (B_9) and subsequently disappear.

\( \delta_2 = 0.1 \)

Here the behaviour of L_1, L_2, L_3, L_4, and L_5 is similar, but now L_8 and L_9 appear at \( z = \pm \infty \) and coalesce with L_1. L_6 and L_7 do not appear.

\( \mu = 0.2 \)

This case can be described by Figure 8 as well.

We consider the radiation pressure \( \delta_2 \) from the more massive body as increasing, the radiation pressure \( \delta_1 \) from the less massive being fixed.

(a) Less massive body has no radiation pressure

In Figure 8 this corresponds to \( \delta_1 = 1 \). At D_1, L_1, L_2, L_3, L_4, and L_5 are in their classical positions. As the radiation pressure of the more massive body increases L_4, L_5, L_3, and L_1 all converge onto it and disappear (D_2). L_6 and L_7 appear at \( x = -0.2 \) (D_2) and move in direction of increasing \( |z| \) and towards \( x = 0 \). L_6 and L_7 disappear at \( x = 0, y = 0, z = \pm \infty \) when \( \delta_2 = -(4)^{1/3} \) (D_3).

No equilibrium point is stable.

(b) Radiation Pressure exerted by less massive body

When \( \delta_1 > 0 \), L_8 and L_9 fail to appear and L_6 and L_7 are never stable. Only for \( \delta_1 < 0 \) can L_8 and L_9 appear and L_6 and L_7 be stable (i.e. when the less massive body is extremely luminous).

For the Sun-planet system L_6 and L_7 exist though are unstable. For Sun-Jupiter \( \mu = 10^{-3} \). Furthermore since Jupiter will exert negligible radiation pressure on a small particle the collinear point L_1 cannot become
stable. The stability of $L_4$ and $L_5$ has been invoked to explain the presence of azimuthal asymmetry of the distribution of interplanetary dust in the solar system (Schuerman, 1980). Depending on the time scale of the instability of the $L_6$ and $L_7$ points, we might expect a similar distribution of particles along the trajectories of the $L_6$ and $L_7$ points.

For physically acceptable radiation pressure ($\delta_1$, $\delta_2$ values) the new and interesting results derived in this paper will only find natural application when the masses of the two bodies are of the same order of magnitude and both are luminous. The stability of the $L_1$ point in particular must have a bearing on any mass transfer within the binary system. A more complete understanding of this question could be obtained by studying the behaviour of the zero-velocity surfaces, with particular attention to the changes in the topological structure of these surfaces. This latter question is intimately related to the existence of the equilibrium points and in particular the values taken by the Jacobi constant at these points. We shall address ourselves to this question in a subsequent paper.

ACKNOWLEDGEMENTS

We wish to acknowledge the support of a Research Grant from the U.K. Science Research Council (J.F.L.S.) and a Carnegie Vacation Award (A.J.C.M.).

REFERENCES

Radzievskii, V. V.: 1950, Astron. Zh. 27, 250.
Radzievskii, V. V.: 1953, Astron. Zh. 30, 265.