THE EQUILIBRIUM STRUCTURE OF THIN MAGNETIC FLUX TUBES. I.

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ABSTRACT

We construct a model atmosphere inside a thin magnetic flux tube embedded in an arbitrarily stratified medium by solving the radiative transfer equation in the two-stream approximation for gray opacity. The atmosphere external to the flux tube is in hydrostatic equilibrium, and in radiative or convective equilibrium according to the Schwarzschild criterion. We assume that the magnetic field is sufficiently strong to permit the neglect of both thermal conduction and convective diffusion; thus energy inside the flux tube is transported only by radiation.

The flux tube is assumed to be thin, i.e., its radius is small compared with the photon mean free path inside the tube. Internal quantities such as temperature, density, and gas pressure are then nearly uniform across the tube. The influence of the flux tube upon the external atmosphere is ignored.

We determine the structure of the internal atmosphere from the conditions of hydrostatic and radiative equilibrium of the tube embedded in an external atmosphere. In the construction of the flux-tube atmosphere we compute the gas temperature along the axis of the tube, taking the contribution by internal gas into account but neglecting the variation of the temperature with radial position. We determine the geometry of the flux tube from the conservation of magnetic flux and from the equilibrium of total pressure in horizontal layers.

Subject headings: hydromagnetics — stars: atmospheres

I. INTRODUCTION

Much recent attention has focused on the dynamics of magnetic flux in the solar convection zone (e.g., Parker 1979; Spruit 1976; van Ballegooijen 1982), as well as in possibly related physical systems such as accretion disks (e.g., Stella and Rosner 1984). One major issue is the physical state of such flux tubes; this problem lies at the heart of stability analyses of buoyant physical systems such as accretion disks (e.g., Stella and Rosner 1984). One major issue is the physical state of such flux tubes; this problem lies at the heart of stability analyses of buoyant magnetic flux, and of wave propagation in flux tubes, for it is necessary in all these cases to determine the equilibrium state (if it exists).

Previous work on the flux-tube equilibrium problem can be divided into two distinct types of studies. In the first (and best known), one attempts to calculate the stratification of large flux tubes in the context of the diffusive approximation by assuming that the tube dimensions are large compared both with the photon mean free path and with the local mixing length; one then determines the stratification within the flux tube, for example, by fixing the required energy transport rate—which is an observable for flux tubes with dimensions typical of sunspots—and using standard mixing-length theory (e.g., Danielson 1961). Usually one does not include all effects of the magnetic field on the energy transport but rather tries to take them into account by using an effective mixing-length parameter.

In the second type of calculation one focuses on thin flux tubes (i.e., those with transverse dimensions sufficiently small that radiative transport from the flux-tube exterior into the interior must be explicitly included), and solves the stratification problem by assuming an anisotropic thermal eddy diffusivity and diffusive radiative transport (e.g., Spruit 1976, 1977).

In this case there is no available theory for modeling the change in the eddy diffusivity in the presence of magnetic fields, and so such models have been parameterized in some convenient manner in order to study the structuring as the parameters are varied.

It is important to point out that, in either case, the physics assumed in the calculation may not be fully appropriate. In order to appreciate these limitations, it is useful to describe briefly the physics which we believe does determine the flux-tube structure (although computational limitations may prevent us from actually carrying out these calculations). Far from the flux tube we expect the atmosphere to be unperturbed by its presence. Within the flux tube proper, we expect energy transport by advection to be severely inhibited by the magnetic field (and radiative energy transport to play a far more important role); indeed, for sufficiently thin tubes (such as we shall consider), radiative transport may completely dominate over advection. The two domains (i.e., flux-tube interior and exterior) are separated by a boundary layer formed because the external medium is perturbed by the presence of the flux tube. In general, we can distinguish two layers within this boundary layer: (1) a sublayer A of thickness order of the photon mean free path in the external medium (Δext), from which the main contribution to the radiation field within the flux tube comes, and whose structure must be determined by solving the exact transfer equation because τ ~ 1 in this sublayer, and (2) a
sublayer B (for which the diffusion approximation for the radiative transfer equation may be valid), whose structure depends on the details of the form of the convective flux. More specifically, the temperature structure in the boundary layer is fixed by the energy flux conservation equation

\[ \mathbf{V} \cdot \mathbf{F}_R = - \mathbf{V} \cdot \mathbf{F}_{\text{conv}}. \]  

The essential point is that no matter how we determine \( \mathbf{F}_{\text{conv}} \) in equation (1.1), only a knowledge of the right-hand side of equation (1.1) allows us to solve for the boundary-layer structure. Given our inexact knowledge of the convection, a variety of distinct approaches are available. For example, Spruit (1976) assumes that horizontal convective transport occurs via turbulent advection, and that the corresponding transport coefficient scales as the ratio of the distance from the tube wall to the local mixing length. The thickness of the resulting boundary layer is therefore determined by the ratio between the radiative and the unperturbed convective diffusion coefficients, and deep in the convection zone this thickness turns out to be much larger than \( \lambda_{\text{ext}} \). This approach is quite reasonable as long as the integral scale length of the turbulent fluid is small compared with the characteristic scale length of the obstacle wall (cf. Townsend 1980). However, the flux tube we consider does not satisfy this constraint; indeed, we would not expect it to influence the external medium at distances much larger than its radius as long as the tube radius is much smaller than the mixing length (this is, as we shall show, the case we consider).

In this paper we assume that the convective diffusion coefficient is not significantly changed by the presence of the tube up to a distance \( \sim \lambda_{\text{ext}} \) from the wall, and therefore the structure of sublayer B is isothermal in the horizontal direction. This neglect of the effect of the tube on its surroundings places an upper bound on the interior tube temperature (because it maximizes the wall boundary temperature). Given our ignorance of the effects of a local small-scale obstacle on the convective flow structure, our calculation therefore serves to define the other extreme of convective inhibition to that studied by Spruit (1976), whose model assumes the maximal inhibition appropriate for large-scale boundaries.

Throughout this paper we assume that magnetic flux trapped within a steadily convecting atmosphere is not diffuse but rather is concentrated in bundles of flux (so-called flux tubes); we do not justify this assumption beyond noting that—at least in the solar case—there is no good evidence for diffuse magnetic flux, so that (at least) simplicity leads one to assume that spatial intermittency characterizes magnetic flux throughout the convection zone.

The paper is structured as follows. In § II we sketch the structure of a flux-tube atmosphere. In § III we construct the model atmosphere in the outside medium, and in § IV we solve for the gas temperature inside the flux tube. We discuss the magnetic structure of the tube in § V and summarize the results in § VI.

II. SKETCH OF A FLUX-TUBE ATMOSPHERE

The main features of a thin flux-tube atmosphere, namely, its temperature and pressure structure, can be described without detailed numerical calculations if some simplifying assumptions are made. Thus, one might suppose that the flux tube is very thin, i.e., its radius is small compared with the photon mean free path in the surrounding gas, and that the opacity is gray (the latter assumption applies also to our numerical solutions).

Since the flux tube is very thin, the intensity of the radiation field inside the tube is equal to the intensity outside, and since the opacity is gray, the Planck function and the intensity in the tube have the same value. Thus the relation between inside and outside gas temperatures depends on the relation between the intensity and the Planck function in the ambient atmosphere. In most instances, the inside and outside temperatures will be nearly equal (see Kalkofen et al. 1985).

If we assume that the ratio of the pressures exerted by the gas and by the magnetic field, i.e., the plasma \( \beta \) (cf. eqs. [4.2] and [4.4]), is constant and that the inside and outside temperatures are the same, the inside gas density is a fixed fraction \((<1)\) of the outside density. Optical path lengths inside the tube are then increased, and the inside temperature gradient, while the same as the outside temperature gradient relative to geometrical depth, is much steeper relative to optical depth. Hence, optical depth unity in the tube occurs at a much higher temperature, and, consequently, the flux tube transports more flux and appears brighter than the surrounding medium.

This picture describes the main features of a magnetic flux tube. The details are modified mainly by temperature effects. These are present, first, if the outside gas temperature is different from the radiation temperature and, second, if radiative transfer takes place over longer distances inside the flux tube; the latter effect enters if the tube radius is not small compared with the outside photon mean free path.

We can easily estimate in which sense the outside gas and radiation temperatures differ from each other in two limiting cases: for layers of purely radiative transport in the outside medium, and for the region where the transition between radiative and convective transport occurs.

For a gray gas in radiative equilibrium throughout, the gas temperature is equal to the radiation temperature. Thus the gas temperature inside the flux tube is the same as that outside. This is also true for nongray opacity provided that the inside and outside opacities have a similar frequency dependence—in a real atmosphere it would be possible to have a higher temperature inside merely because of a shift of the dominant absorption to "bluer" frequencies.

If energy transport in the outside medium is mainly by convection, we may distinguish two special cases: if the contribution of the radiative flux to the total flux is fixed, it is as though the atmosphere were in radiative equilibrium, i.e., the radiative flux gradient is zero and the gas and radiation temperatures agree; if, on the other hand, the temperature gradient is equal to the adiabatic gradient (cf. eq. [3.11]), the mean intensity may be larger or smaller than the Planck function. Thus deep in the convection zone the temperature may be higher or lower in the flux tube than outside.

In the transition region between radiative and convective transport, where the value of the radiative flux decreases from that of the total flux to a much smaller value, the intensity must be less than the Planck function in accordance with the transfer equation, \( dH_\gamma/dt = J - B \), where \( H_\gamma \), \( J \), \( B \), and \( \tau \) are, respectively, the net radiative flux, the mean integrated intensity, the integrated Planck function, and the optical depth. Since \( dH_\gamma/dt < 0 \) at the top of the convection zone, the radiation temperature is less than the outside gas temperature and hence the inside gas is cooler than the outside gas. Note, however, that this relation holds at a given physical depth; at a given optical depth we still expect the inside gas to be much hotter than the outside gas.

As for the dependence of the inside temperature on the radius of the flux tube, we note that, when the optical path
length inside the tube becomes comparable to the scale height of the opacity (which is of the order of the density scale height; it is equal to about half the density scale height when H− dominates the opacity), the excess of the intensity of radiation entering the flux tube from deeper layers may be larger than the deficit of the intensity from shallower layers, leading to an increase of the mean intensity in the flux tube and, hence, to an increase in the inside temperature. Thus, in the radiative atmosphere, the inside temperature is equal to the outside temperature for zero tube radius, and increases with radius. If the layers of the outside atmosphere that contribute to the radiation field in the tube are in equilibrium, the temperature difference between the flux tube and the outside medium cannot be predicted with confidence even for a gray atmosphere, since it depends on the balance of two effects: the rate of increase with optical depth of the outside source function, which can be faster or slower than linear, and the rate of increase with geometrical depth of the optical path length, which depends on the exponential density stratification.

Finally, we note that the presence of the flux tube will perturb the structure of the external atmosphere in a boundary layer on the tube wall, so that in principle the assumption of a plane-parallel stratified external atmosphere, while an excellent approximation in the case of very thin flux tubes, can fail when the tube is thin, especially in the deep layers of the atmosphere. An exact approach to the problem requires a two-dimensional calculation. In the following we will neglect the perturbation introduced by the tube itself on the external atmosphere, as discussed in § I.

In summary, the dominant features of the flux-tube atmosphere compared with the surrounding gas are the pressure and density reduction and the near-equality of inside and outside temperatures. The difference of the temperatures at a given physical depth depends on the energy transport in the outside medium; on the opacity inside and outside, especially the frequency dependence; and on the tube radius. For larger tube radii the geometry plays a role as well, since the tube walls are not vertical (cf. § V).

III. MODEL EXTERNAL ATMOSPHERE

The atmosphere external to the flux tube is assumed to be stratified in plane-parallel layers, to be in hydrostatic equilibrium, and, depending on the Schwarzschild criterion, to be in radiative or convective equilibrium. The presence of the flux tube is assumed not to be felt by the gas in the external atmosphere.

The gas is partially ionized. The electrons contributed by metals are assumed to have a fixed abundance relative to hydrogen, but hydrogen ionization is taken into account. The opacity is due to H− ions and to H atoms in the third level (as radiation is due to H atoms in the third level at 5000 Å; Z is the abundance of hydrogen atoms, and nH− that of H− ions. The density of hydrogen atoms in the third bound level is given by

\[ n_3 = 9n_{\text{H}}(1 - x) \exp \left( -E_3/k_{\text{B}}T \right) . \]  

(3.2c)

The Saha function for H− is

\[ \Phi_{\text{H}}(T) = 2.07 \times 10^{-16}T^{-3/2} \exp (8617/T) . \]  

(3.3)

The ionization fraction \( x = \alpha(n_{\text{H}}; T) = n_{\text{H}^-}/n_{\text{H}} \) is given by

\[ x = -\frac{1}{2} \left( Z + \frac{1}{\Phi_{\text{H}}n_{\text{H}}} \right) + \frac{1}{2} \left[ \frac{4}{\Phi_{\text{H}}n_{\text{H}}} + \left( Z + \frac{1}{\Phi_{\text{H}}n_{\text{H}}} \right)^2 \right]^{1/2} , \]  

(3.4)

where \( \Phi_{\text{H}} \) is the Saha function for hydrogen,

\[ \Phi_{\text{H}}(T) = 2.07 \times 10^{-16}T^{-3/2} \exp (1.578 \times 10^5/T) . \]  

(3.5)

The set of equations defining the external equilibrium in the case of purely radiative transport is given by the transfer equation,

\[ T^4 = T_0^4(1 + \tau/\mu_0) , \]  

(3.6a)

which we have written in the two-stream approximation, with \( \mu_0 = 3 - 1/2 \); by the equation of state,

\[ p = n_{\text{tot}}k_{\text{B}}T = n_{\text{H}}(1 + Y + 2Z + x)k_{\text{B}}T ; \]  

(3.6b)

by the hydrostatic equilibrium equation,

\[ dp = m_p(1 + 4Y + AZ)n_{\text{H}}dz , \]  

(3.6c)

with the helium abundance \( Y \) by number (\( Y = 0.08 \)) and the mass ratio \( A = M_{\text{H}}/m_p(A = 30) \), where \( M_{\text{H}} \) is the mass of the metal atoms and \( m_p \) is the proton mass; and, finally, by the relation between geometrical and optical depths,

\[ d\tau = \kappa dz . \]  

(3.6d)

Note that equations (3.6c) and (3.6d) imply the convention that the z-coordinate is measured positive inward, into the atmosphere (and \( \tau = 0 \) at \( z = 0 \)).

In the equations of state and hydrostatic equilibrium we have assumed that the number of hydrogen atoms in excited states is negligible compared with that in the ground state, so that the hydrogen density in the ground state can be identified with the total neutral hydrogen density. This approximation is adequate in the radiative zone and in the shallower layers of the convection zone that are of interest to us. This set of equations must be solved for the hydrogen density \( n_{\text{H}} \), the gas temperature \( T \), and the optical depth \( \tau \) as functions of \( z \).

In layers in which the Schwarzschild criterion for convective instability is satisfied, the energy is partly transported by convection. The radiative transfer equation is then replaced by a different energy equation, requiring that the total energy flux, by radiation and convection, be constant and equal to \( \pi F = \sigma_5 T_{\text{eff}}^4 \), where \( \sigma_5 \) is the Stefan-Boltzmann constant and \( T_{\text{eff}} \) the effective temperature of the atmosphere. Thus the energy equation is

\[ F_{\text{conv}} + F_R = F , \]  

(3.7)

where the convective flux may be written as

\[ F_{\text{conv}} = \rho C_p v_c \delta T ; \]  

(3.8)
\( C_p \) is the specific heat at constant pressure, \( \delta T \) is the temperature difference between the convective element and the surrounding medium, and \( v_c \) is the velocity of the convective elements (cf. eq. 5.7).

We recall that the atmosphere becomes convectively unstable when the radiative temperature gradient exceeds the adiabatic gradient, i.e., when the Schwarzschild criterion is satisfied:

\[
\nabla_R > \nabla_A, \quad (3.9)
\]

where the gradients are defined by

\[
\nabla_R = \left( \frac{d \ln T}{d \ln p} \right)_R = \frac{1 + Y + 2Z + \frac{\kappa}{4}}{1 + 4Y + AZ} \frac{v}{4 \mu_0 m_p g T^3} \kappa F_0,
\]

\[
\nabla_A = \left( \frac{d \ln T}{d \ln p} \right)_A = \frac{\Gamma - 1}{\Gamma}_2, \quad (3.10a)
\]

with the flux constant \( \pi F_0 = \sigma_T T_0^4 \), and

\[
\nabla_A = \frac{1 + \frac{1}{2}x(1 - x)(x/k_B T + 5/2)}{5/2 + \frac{1}{2}x(1 - x)(x/k_B T + 5/2)^2}, \quad (3.11)
\]

The energy equation (3.7) must be solved for the actual temperature gradient. This can be accomplished very easily if the convective instability occurs sufficiently deep in the atmosphere for the diffusion approximation to be valid (see Mihalas 1970). In this case the relation between the radiative flux and the total flux is

\[
F_R/F = \nabla/\nabla_R. \quad (3.12)
\]

We note that the requirement for the instability to occur in deep layers may not be satisfied in our calculations, where the instability begins at \( \tau = 1 \). In addition, the linear depth dependence (eq. [3.6a]) of the temperature is valid only if the atmosphere is in radiative equilibrium throughout. We shall assume, however, that the linear law is valid and that the Schwarzschild criterion is first satisfied in sufficiently deep layers. We then solve the transfer equation for the temperature structure that we have obtained from these assumptions in order to estimate the error incurred.

From equations (3.7) and (3.12) we obtain for the convective flux

\[
\pi F_{\text{conv}} = \sigma_T T_0^4(1 - \nabla/\nabla_R), \quad (3.13)
\]

which, together with expression (5.7) for the convective velocity, can be brought into the form

\[
\mathcal{A} \chi + \mathcal{B} x = \nabla_R - \nabla_A, \quad (3.14)
\]

where

\[
x = (\nabla - \nabla_B)^{1/2}, \quad \mathcal{A} = \frac{a^{1/2} \sqrt{\epsilon} / (l/H)^{1/2}}{2 \sigma_T T_{\text{eff}}},
\]

\[
\mathcal{B} = \frac{\nabla_E - \nabla_A}{(\nabla - \nabla_B)^{1/2}} = \frac{8 \sigma_T T^3}{a^{1/2} \rho C_p (gH)^{1/2} (l/H) (1 + 1/2 \tau_e^2)}, \quad (3.14)
\]

\( \tau_e = kl \), and \( l \) is the mixing length (see § V). Using the solution of the algebraic equation (3.14) for \( \mathcal{V} \), we can write a differential equation for the temperature:

\[
\frac{dT}{dz} = \frac{T}{p} \frac{dp}{dz} \mathcal{V}. \quad (3.15)
\]

Equations (3.6b)-(3.6d) together with equation (3.15) are a complete set of equations for determining the particle density \( n_i \), the gas temperature \( T \), and the optical depth \( \tau \) as functions of the \( z \)-coordinate.

We have solved the equations for radiative and for convective energy transport, using a Runge-Kutta method of integration. The resulting atmosphere is shown in Figure 1 (thin line); the error due to the approximations (3.6a) and (3.12), which reaches a maximum of about 9% at the top of the convection zone, is shown in Figure 2. In Figure 1 the temperature is plotted against \( \tau \) and against geometrical depth \((=z - z_{\text{ext}})\) for the boundary temperature \( T_0 = 5000 \) K. The depth where the energy transport changes from radiative to convective is marked by an arrow. In Figure 1 we show also, as a comparison, the linear fit (in \( \ln \tau \)) to the temperature structure obtained by Allen (1976) for \( \tau \geq 1 \) (heavy line); the extension to \( \tau < 1 \) is assumed to be still linear (see also Avrett 1977).

In connection with the problem of the influence of the tube on the external medium, discussed in § I, we present in Figure 3 the ratio of the external mean free path \( \lambda_{\text{ext}} \) to the tube radius \( R \) (curve i) and to the mixing length \( l \) (curve ii). We see that at small external optical depths (i.e., at geometrical depths \( <0 \) for the flux-tube radius considered in this paper), we have \( \lambda_{\text{ext}}/R > 1 \); this ensures that the external radiation field is essentially unperturbed by the intruding flux tube in this region. At large external optical depths, deep in the convection zone, this may no longer be true; for this case we address the reader to the discussion in § I.

IV. EQUILIBRIUM ATMOSPHERE IN A FLUX TUBE

We determine the equilibrium structure of the atmosphere in a magnetic flux tube by solving the radiative transfer equation numerically in the two-stream approximation for gray opacity. Within the tube we neglect both thermal conduction and convective diffusion and consider energy transport by radiation only. Our analysis is restricted to thin tubes, whose radius is much smaller than the photon mean free path inside the flux tube. If the flux tube is very thin, i.e., its radius is small also relative to the photon mean free path in the atmosphere outside the flux tube, all points in a horizontal layer within the tube are bathed in the same radiation field coming from the external atmosphere and are therefore at the same temperature, which may be different from the external temperature; in fact, in the convection zone, it will be different. For thin tubes this need not be true.

The tube is embedded in the atmosphere described in the previous section, and its geometrical structure is obtained by requiring both conservation of magnetic flux and pressure balance in horizontal layers with the external medium. The hydrostatic equilibrium condition, \( dp_T = \rho g dz \), where \( p_T \) and \( \rho_T \) are the internal thermal pressure and mass density, is also satisfied inside the tube. The transfer equation is then solved given the resulting shape of the tube, and the structure of the internal atmosphere is obtained by imposing energy equilibrium. We now examine in detail the structure of the tube and the radiative transfer problem.
The magnetic field vector in the vertical flux tube is assumed to be of the form

$$B = B_z(z)\hat{z} + B_r(r, z)\hat{r},$$

(4.1)

with the radial component small compared with the vertical component, i.e., $B_r^2 \ll B_z^2$, so that the magnetic field is mainly vertical, i.e., $|B| \approx B_z$; $\hat{z}$ is a unit vector along the vertical axis and is directed downward (i.e., into the medium). The radius $R_0$ of the tube and the value of the plasma $\beta$ at $z = 0$, where $\beta(z) = 8\pi p/\beta^2$, are prescribed. The tube is assumed to be in pressure equilibrium with its surroundings, i.e.,

$$p_i + \frac{B^2}{8\pi} = p,$$

(4.2)

where $p$ is the external thermal pressure. In order to determine the geometrical structure of the tube, we use the equation for the conservation of magnetic flux, from which we estimate the radius as a function of depth, $R = R(z)$, with the radius at $z = 0$ being $R(0) = R_0$:

$$R = R_0\left(\frac{B_z(0)}{B_z(z)}\right)^{1/2}.$$  

(4.3)

The $z$-component of the magnetic field is given by

$$B_z \approx |B| = (8\pi p/\beta)^{1/2}.$$

(4.4)

Recall that for thin tubes we can extend the axial value of $B_z(z)$ throughout across the tube. In Figure 4 we show the variation of tube radius $R$ with depth $z$ for $\beta(0) = 0.10$, corresponding to $\beta_{ext} (\equiv 8\pi p/B^2) = 1.10$ (this variation of $R$ with depth is almost independent of $\beta$). From Figure 3 we see that the radius, at $\tau = 1$, is reduced by a factor of about 2. We assume that the magnetic field of the tube is strong enough to prevent convective flow both across and along the field lines, so that energy exchange with outside plasma is due to radiative transfer only (Meyer, Schmidt, and Weiss 1976). Within the tube we impose energy equilibrium

$$\mathbf{V} \cdot F = 0.$$

(4.5)

Balancing the radiative heating of the gas by the radiation from the external atmosphere against the radiative cooling of the gas, the radiation transport equation may be written as

$$\mathbf{V} \cdot F = 4\sigma_s \kappa(T_g^4 - T^4).$$

(4.6)
where $T_i$ is the internal gas temperature and $T_R$ the radiation temperature describing the radiation field from the external medium; $F$ is the radiative flux measured positive in the outward direction (recall that $z$ is measured positive in the inward direction). The contribution of the (classical) thermal conduction has been neglected in equation (4.6) because the temperature gradients are too small to drive a significant thermal flux.

We solve the transfer equation numerically in the two-stream approximation for a gray atmosphere:

$$ \pm \mu_0 \frac{dI^\pm}{dz} = \kappa[I^\pm - S(T)] , \quad (4.7) $$

where $I^\pm$ is the specific intensity in the outward and inward directions, respectively, and $S(T)$ is the source function (assumed to be the Planck function). The overall atmospheric structure of the flux tube is described by the hydrostatic equilibrium equation (eq. [3.6c]) and the transfer equation (4.7). All quantities are evaluated on the axis of the flux tube; furthermore, in this plane-parallel geometry, we assume that these quantities are constant across the tube. We remark that we do not solve a two-dimensional problem, but simply take into account the shape of the flux tube by varying the internal optical path accordingly.

We determine the internal temperature, $T_i(z)$, typically by assuming initially that its value at a given depth is the same as that of the external medium. The integration of the transfer equation then gives an estimate of the intensity and, therefore, of the radiation temperature $T_R(z)$ inside the tube. We then set the internal gas temperature equal to the radiation temperature (see eqs. [4.5] and [4.6]). Usually less than 10 iterations suffice to reduce the temperature corrections to a fraction of a degree.

In Figure 5 the internal temperatures are plotted against $\tau$. The external temperature is also plotted for comparison (curve c). The initial radius of the tube is assumed to be $R_0 = 80$ km, the external density at $\tau = 0$ is $n_0 = 10^{16}$ cm$^{-3}$, and the plasma parameter has the values $\beta(0) = 0.10$ (curve a) and 0.25 (curve b).

In Figure 6 we show the behavior of $\beta$ as a function of $t_{\text{ext}}$ for $\beta(0) = 0.25$. From this curve we see that, at least for thin tubes, the approximation $\beta = \text{constant}$ is fully justified. In Figure 7 the fractional difference (in %) between internal and external temperatures is plotted against $t_{\text{ext}}$ for $\beta(0) = 0.10$.

V. MAGNETIC STRUCTURE OF THE TUBE

We now determine the magnetic structure of the tube. We already know the values of the tube radius $R(z)$ and of the vertical component of the magnetic field $B_z(z)$ from the equations for the conservation of magnetic flux from pressure balance (eqs. [4.2]–[4.4]). We estimate the radial component $B_r(r, z)$ from the equation

$$ \mathbf{V} \cdot \mathbf{B} = 0 . \quad (5.1) $$
For axially symmetric tubes, $\partial/\partial \phi = 0$; in cylindrical coordinates this equation may be written

$$ \frac{1}{r} \frac{\partial}{\partial r} \left(r B_r \right) + \frac{\partial B_z}{\partial z} = 0. \quad (5.2) $$

In the thin flux tube approximation, where the vertical component of the magnetic field is constant across horizontal planes, we may set

$$ \frac{\partial B_z}{\partial z} = f(z). \quad (5.3) $$

The function $f(z)$ is obtained by numerical integration of equation (5.3) for the vertical component of the magnetic field, which is equated to the total field intensity $|B|$ by approximation (4.4); and $|B|$ is found from the hydrostatic equilibrium equation (3.6c) for the external atmosphere and the analogous equation for the flux-tube atmosphere. The solution of equation (5.1) then yields the radial component $B_r(r, z)$,

$$ B_r(r, z) = -\int f(z) r, \quad (5.4) $$

where we have required that $B_r \to 0$ when $r \to 0$, i.e., on the tube axis the field is purely vertical. In Figure 8 we show the dependence of $B_r(R, z)$, the maximum of $B_r(r, z)$, and of $B_z(z)$ on geometrical depth for the case $\beta(0) = 0.10$. The magnetic field vector is therefore given by

$$ B = -\frac{1}{2} f(z) r \hat{r} + B_z(z) \hat{z}. \quad (5.5) $$

Finally, we want to make sure that the field inside the tube is strong enough to prevent convective diffusion. For this purpose we estimate the equipartition value of the magnetic field $B_c$ from the equation

$$ \rho v_c^2 \sim \frac{B_c^2}{8\pi}, \quad (5.6) $$

where $v_c$ is a typical velocity of the convective elements,

$$ v_c^2 = a \frac{g}{H} \lambda^2 (\nabla T - \nabla T_e), \quad (5.7) $$

with the pressure scale height $H$ defined by

$$ \frac{1}{H} = \frac{d \ln p}{dz} = \frac{g \rho}{\rho}. \quad (5.8) $$

The parameter $a$ has the value $a = \frac{1}{H}$, $V$ is the actual temperature gradient, $\nabla T_e$ is the temperature gradient inside the convective elements, and $l$ is the mixing length. The ratio of the mixing length to the scale height, $a = l/H$, is assumed to be $\tilde{a} = 2$. An equipartition convective pressure is obtained by solving equations (5.6) and (5.7); the corresponding ratio
Fig. 8.—Axial field $B_z$ and maximum radial field $B_r(R)$ vs. geometrical depth (the scaling factor $h$ is 10 and $\beta(0) = 0.10$).

between the internal magnetic pressure [for $\beta(0) = 0.25$] and the convective pressure (5.6) is shown in Figure 9 as a function of the depth $\bar{z} = z - z_c$, where $z_c$ is the depth where convection starts. We see that this ratio reaches a minimum at $\bar{z} = 60$ km, where the magnetic pressure is still about 6 times the convective pressure.

VI. CONCLUSIONS

We have determined the equilibrium structure of the atmosphere inside an axially symmetric magnetic flux tube by solving the radiative transfer equation for a gray atmosphere. We recall that the calculations have been carried out for a tube geometry that satisfies the conservation equation for magnetic flux. The results show that the internal temperature at a given physical depth is approximately equal to the external temperature, the largest difference being a few hundred degrees, with only a weak dependence on the plasma $\beta$ and on the tube radius. The reduction of the internal thermal pressure affects mainly the particle density, with a consequent reduction of the internal absorption coefficient, which depends on the square of the density in the layers where the H$^-$ opacity dominates. Figure 6 also shows that the approximation of constant $\beta$, while it does not satisfy the hydrostatic equilibrium equation exactly (because the internal and external temperatures are only approximately equal), is excellent.

The expected behavior of the temperatures is shown in Figure 5, where the internal temperature at a given optical depth is always higher than the external temperature. Thus, thin flux tubes should appear brighter than the external medium. The calculations reported here have been carried out for a flux-tube diameter (at $r_{\text{ext}} = 1$) of about 80 km, which corresponds to the resolution limit of about 0\textdegree 1.

We now compare our calculations with those of Spruit (1976), who studied the same problem in the context of the solar atmosphere. He considered a cylindrical flux tube, using a diffusion approximation for the radiative energy transport, and assumed an additional turbulent energy transport process within the tube which operates along the magnetic field only. This turbulent transport process is attributed to convective motions within the tube and—in light of one's ignorance of the details of magnetohydrodynamic turbulence—is left parameterized. Thus, his approach differs from ours in that we do not make the diffusion approximation (but rather solve the radiative transfer equation), and we assume that there is no additional energy transport other than radiation within the tube.

Furthermore, our boundary conditions and constraints differ substantially. For a flux tube of specified radius, Spruit (1976) prescribes (1) the depth of the Wilson depression (equivalent to specifying the gas-pressure deficit within the flux tube at the height corresponding to $\tau = 1$ in the external medium), (2) the depth of the lower boundary, and (3) the total energy flux entering the tube from below. Only one of these input parameters is observationally constrained (namely, the first), and that only for flux tubes that are resolved. In contrast, we have only one constraint for a tube of specified radius: the plasma $\beta$ within the tube at $\tau = 0$; in principle, this parameter can be constrained for spatially resolved flux tubes by observations.

Because the physics considered here is rather different from Spruit's (1976), a detailed comparison between our results and his is not possible. Nevertheless, we do study the same physical system (i.e., thin flux tubes), and it is therefore worth noting the differences between the two approaches. Since we are here solving the complete transfer equation, our analysis is likely to be more accurate in the vicinity of boundaries and, in general, in the regions $\tau \leq 1$. On the other hand, we are limited by a one-dimensional model and therefore cannot address the problem of the horizontal structure of the atmospheres (such as the effect of the flux tube on the external medium) as Spruit (1977) does. The two models can be regarded as complementary for understanding the physics of slender flux tubes.

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