The transfer of line radiation. II. Approximate solutions for semi-infinite atmospheres

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An atmosphere scattering radiation with complete frequency redistribution may be treated as an amplifier filtering primarily the low spatial frequencies from the primary-source distribution. The corresponding passband width is closely related to the customary thermalization length \( \tau_e \). Two approximations for the line source function in a semi-infinite atmosphere (Paper I), the L2 (Ivanov) and the F (U. and H. Frisch), are studied in detail. From these approximations the resolvent function \( \Phi \) as well as the source function in an isothermal atmosphere and in atmospheres with exponential primary-source distributions are calculated and are compared with the corresponding numerically exact functions, evaluated by using a sum of exponentials to approximate the kernel of the integral equation for the source function. For any value of the Voigt parameter \( \sigma \), the error in L2 never exceeds 40%, whereas at large \( \tau \) the F approximation gives the correct order of magnitude. The coefficients \( \kappa_n, L_n \) in the approximation \( \Phi(\tau) = \sum_{n=1}^{\infty} L_n \exp(-\kappa_n \tau) \) of the conservative resolvent are tabulated for \( \sigma = 0, 10^{-3}, 10^{-5} \); the fit is accurate to better than 1%.

1. INTRODUCTION

In Paper I, published in the last issue,\(^1\) we have explained the interrelations among various approximate solutions to the problem of line-radiation transfer for a two-level atom, and have offered a general critical analysis of those solutions. A comparison was made between the first-order\(^2\) and second-order\(^3\) escape probability approximations (see also Transfer,\(^4\) Secs. 5.5, 6.3, 8.11), which we designated\(^5\) L1, L2, respectively. We also discussed\(^6\) and, H. Frisch's approximation (the F approximation, analogous to the L2) for a semi-infinite atmosphere, as well as its generalization by Puetter et al.\(^8\) (the C\(^\text{a}\) approximation) to an atmosphere of arbitrary optical thickness in the form proposed by Hummer and Rybicki.\(^9\) We focused on the "standard" problem of a homogeneous, isothermal layer, either finite or semi-infinite.

This second paper presents a more detailed analysis of the L2 and F approximations. Although the atmosphere will be regarded as semi-infinite and homogeneous, we shall allow the power of the primary sources exciting the upper level (and accordingly the initial source function) to vary arbitrarily with depth. It will be shown that in every case the L2 approximation provides high accuracy, whereas the F approximation, if the same amount of calculating is done, may give huge errors. We base this conclusion both on extensive numerical material and on a comparison against the exact asymptotic relations.

The notation adopted in Paper I and in the book on line transfer\(^10\) will be retained.

2. BASIC EQUATIONS

For a semi-infinite atmosphere, in the approximation of complete frequency redistribution with continuum absorption neglected, the source function \( S(\tau) \) in a spectral line will be given by the solution of the integral equation (see Transfer,\(^\text{a}\), Chapter 2, or Mihalas,\(^11\) Chapter 11)

\[
S(\tau) = \int_{-\infty}^{\tau} K_{1}(\tau') S(\tau') d\tau' + S(\tau),
\]

in which the kernel function

\[
K_{1}(\tau) = A \int_{-\infty}^{\tau} \alpha(x) E_{1}(\alpha(x), \tau) dx
\]

is normalized to unity:

\[
\int_{-\infty}^{\tau} K_{1}(\tau) d\tau = 1;
\]

\( \alpha(x) \) represents the line absorption-coefficient profile,\n
\[
A \int_{-\infty}^{\infty} \alpha(x) dx = 1, \quad \text{optical depths} \ \tau \ \text{are measured at the line center, and the initial (primary) source function} \ S(\tau) \ \text{is regarded as specified in advance.}
\]

The source function is expressed in terms of the radiative intensity in customary fashion:

\[
S(\tau) = \frac{1}{2} A \int_{-\infty}^{\tau} \alpha(x') \int_{-\infty}^{\tau'} I(\tau, \mu, \tau') d\mu' d\tau' + S(\tau),
\]

where \( I \) is the solution of the radiative-transfer equation

\[
\mu \frac{\partial I}{\partial \tau} = -\alpha(x) (I - S)
\]

with the boundary condition \( I(0, \mu, \tau) = 0 \) for \( \mu > 0 \) and with some auxiliary condition to stipulate the behavior of \( I \) as \( \tau \to \infty \).

Both the kernel function \( K_{1}(\tau) \) and the related function

\[
K_{2}(\tau) = \int_{-\infty}^{\tau} K_{1}(\tau') d\tau' = A \int_{-\infty}^{\tau} \alpha(x) E_{2}(\alpha(x), \tau) dx
\]

have been thoroughly studied for the most important types of absorption coefficients: Doppler, Voigt, and Lorentz profiles. In particular, approximations are available in terms of elementary functions, well adapted to machine computations. Henceforth we shall consider \( K_{1}, K_{2} \) to be known.

We preface our analysis of the approximate solutions to Eq. (1) for the source function in half-space by discussing the combined effect of multiple scatterings in a slightly different manner from the traditional approach as ex-
3. FILTRATION OF SPATIAL FREQUENCIES FOR MULTIPLE SCATTERINGS

Consider a plane but infinite homogeneous medium with an arbitrary depth distribution $S^*(\tau)$ for the primary source function. This function may be expanded as a Fourier integral

$$S'(\tau) = \int a'(\omega) e^{i\omega\tau} d\omega$$  \hspace{1cm} (7)

so it is natural to test the response of the infinite medium to a harmonic disturbance $S_0(\tau)$; thus

$$S_0(\tau) = \frac{\nu}{2} \int K_1(\tau' \tau) S_0(\tau') d\tau' + e^{i\omega_{\tau}}.$$  \hspace{1cm} (8)

Since the problem is linear, the source function $S(\tau)$ in an infinite medium with an arbitrary source term $S^*(\tau)$ may then be represented in the form

$$S(\tau) = \int a'(\omega) S_0(\tau) d\omega.$$  \hspace{1cm} (9)

We seek an expression of the type

$$S(\tau) = a(\omega) e^{i\omega_{\tau}}.$$  \hspace{1cm} (10)

On substituting this expression into Eq. (8) we at once obtain

$$a(\omega) = \frac{1}{\lambda V(\omega)} V(\omega)^{-1},$$  \hspace{1cm} (11)

where $V(\omega)$ is the Fourier transform of the function $K_1(\tau)/2$:

$$V(\omega) = \frac{1}{2} \int K_1(\tau) e^{i\omega\tau} d\tau$$  \hspace{1cm} (12)

$$= \int_{0}^{\infty} K_1(\tau) \cos \omega \tau d\tau.$$

Let us assume that the function $V(\omega)$ falls off monotonically with $\omega$ then $a(\omega)$ will decrease from $a(0) = (1 - \lambda)^{-1}$ to $a(\infty) = 1$, as illustrated in Fig. 1 (these curves refer to the Doppler-profile case).

The function $a(\omega)$, which describes how the medium will respond to a harmonic disturbance with spatial frequency $\omega$, represents a very deep property of the multiple-scattering process. The effect of repeated scatterings will be tantamount to an amplification of the initial disturbance. Since high-frequency spatial harmonics will not be amplified so efficiently as low-frequency ones, they will ultimately be quenched. The multiply-scattering medium will act as an amplifier which filters out only the low spatial frequencies.

To measure the passband width it is convenient to adopt the spatial frequency $\omega_{\lambda}$ such that $1 - V(\omega_{\lambda}) = 1 - \lambda$. Figure 1 clarifies this definition. The reciprocal of $\omega_{\lambda}$ may then be taken as the familiar thermalization length $\tau_1$; thus

$$1 - V(1/\tau_1) = 1 - \lambda.$$  \hspace{1cm} (13)

Such a definition of $\omega_{\lambda}$ seems perfectly natural; it has been introduced previously on other grounds.\(^4\)\(^\text{12}\) Equation (13) gives values for the thermalization length similar to those deduced from other, more common definitions of $\tau_1$ (see below).

Of special interest are strongly scattering, that is, nearly conservative, media ($1 - \lambda \ll 1$). One can then evaluate the thermalization length simply from the behavior of $V(\omega)$ for small $\omega$. We shall assume that

$$1 - V(\omega) \sim v(\omega) \omega^\gamma, \quad \omega \to 0,$$  \hspace{1cm} (14)

where $\gamma$ is some constant ($0 \leq \gamma \leq 1$) and $v(\omega)$ is a slowly varying function at zero — a function such that for any $a(\omega) = 0$

$$v(\omega) / v(\omega) \to 1, \quad \omega \to 0.$$  \hspace{1cm} (15)

In the great majority of cases of practical interest, $v(\omega)$ will degenerate to a constant. But the Doppler profile is an important exception, having $v(\omega) = (\pi/4) \ln (1/\omega)^{-1/2}$.

The asymptotic constant $\gamma$ is called the characteristic index of the kernel $K_1(\tau)$ for of the profile $a(\omega)$. One can show that $\gamma \ll 1/2$ for lines with wings of unbounded extent. The more slowly $K_1(\tau)$ approaches 0 as $\tau \to \infty$, the smaller $\gamma$ will be. For monochromatic scattering, when $K_1(\tau) \equiv E_1(\tau)$, we have $\gamma = 1$.

The characteristic index $\gamma$ represents a defining asymptotic constant. The two quantities $\gamma$ and $v(\omega)$ will completely determine the form of the asymptotes, both spatial (in stationary problems) and temporal (nonstationary problems). Table I indicates the values of $\gamma$ and $v(\omega)$ for several profiles (see also Transfer,\(^6\) pp. 83–86, and Nagirner's article,\(^13\) pp. 229–230).

For lines with unbounded wings the thermalization length is often defined by the equation

$$\left(\frac{\lambda}{2}\right)K_1(\tau_1) = 1 - \lambda.$$  \hspace{1cm} (15)

This definition was used, in particular, in Eq. (12) of Paper I. If $1 - \lambda \ll 1$, the $\tau_1$ values given by Eqs. (13), (15) will differ by an insignificant factor. Indeed, for large $\tau$ we may write, to the accuracy given by the leading term of the expansion (Transfer,\(^6\) p. 86),

$$K_1(\tau) = \frac{2}{\pi\gamma} \Gamma(2\gamma) \sin \pi \gamma(1/\tau) \tau^{-\gamma} = \frac{2}{\pi\gamma} \Gamma(2\gamma) \sin \pi \gamma[1 - V(1/\tau)].$$  \hspace{1cm} (16)
Here \( \Gamma(x) \) represents the gamma function. If the wings of the line are infinitely broad, then \( \gamma \ll 1/2 \) and the factor \( (2/\gamma) \Gamma(2\gamma) \sin \pi \gamma \) will be close to 1.

We would emphasize that the definition (13) is more general than Eq. (15), as it holds for all \( \gamma \) including \( \gamma = 1 \). In particular, if \( K_0(\tau) = E_1(\tau) \) (monochromatic scattering, with a square profile), then for near-conservative media \( (1 - \lambda \ll 1) \) Eq. (13) will yield the classical diffraction result, \( \tau_\lambda = (3(1 - \lambda))^{-1/2} \).

4. FUNDAMENTAL AND STANDARD PROBLEMS

The "standard" problem of line-formation theory is the customary term for the task of calculating the radiation field in a homogeneous, isothermal atmosphere. In this event the source function will satisfy Eq. (1) with \( S^* = \text{const.} \)

It also is an interesting problem, however, to calculate the radiation field for a function \( S^*(\tau) \) that depends arbitrarily on depth. Sobolev showed in the 1950s (see for example, his textbook, Sec. 3, or Transfer, Secs. 5.1, 5.2) that it suffices to consider the case \( S^*(\tau) = (\lambda/2)K_1(\tau) \). Physically this case corresponds to a shell source at the boundary, emitting photons isotropically in the line with their frequency distribution proportional to the absorption profile \( \alpha(\rho) \). The corresponding source function (the "resonant" function) is customarily denoted \( \Phi(\tau) \); it represents the solution, bounded at infinity, of the equation

\[
\Phi(\tau) = \frac{\lambda}{2} \int \Phi(\tau')d\tau' + \frac{\lambda}{2} K_1(\tau).
\]

In half-space with constant \( \lambda \) and arbitrary \( S^*(\tau) \) the source function is expressed by integrals of \( \Phi(\tau) \).

We shall call the task of determining the function \( \Phi(\tau) \) the "fundamental" problem. Apart from a constant factor the resonant function \( \Phi \) coincides with the derivative of the source function for the standard problem. Indeed, let \( S_0(\tau) \) denote the solution of Eq. (1) for \( S^* = 1 \); then (see Transfer, Sec. 6.1)

\[
S_0(\tau) = (1 - \lambda) \int \Phi(t) dt,
\]

so that

\[
\Phi(\tau) = (1 - \lambda)^{-1} S_0'(\tau).
\]

Since the source function, for an arbitrary source term \( S^*(\tau) \), is expressed in terms of integrals of the resonant function, the problem of obtaining approximate solutions to Eq. (1) that have some specified accuracy can be handled by finding an adequate approximation to the resonant \( \Phi(\tau) \).

In Sec. 3 of Paper I we have discussed in some detail the approximation earlier introduced by one of us,\(^5\) (again see Transfer, Secs. 5.5, 6.3, 8.11), called the L2 approximation in Paper I:

\[
L_2: \quad S_0(\tau) = (1 - \lambda)^{-1} D^{1/2}(\tau),
\]

where

\[
D(\tau) = 1 - \lambda + \lambda K_1(\tau).
\]

The expression (20) possesses the property we require it provides a good fit not only to the function \( S_0(\tau) \) itself but also to its derivative. Combining Eqs. (20) and (19), we arrive at an L2 approximation to the resonant:

\[
L_2: \quad \Phi(\tau) = (1 - \lambda)^{-1} D^{1/2}(\tau).
\]

When the approximation (22) was originally introduced,\(^5\) comparison with the following exact results (Transfer, Sec. 6.5) had indicated that it was quite accurate. At depths large compared with the thermalization length and for \( \gamma < 1 \), the resonant is expressed, as far as the leading term in the asymptotic, by

\[
\Phi(\tau) = (1 - \lambda)^{-1} K_1(\tau), \quad \tau \gg \tau_\lambda.
\]

and similarly in the kernel \( K_1(\tau) \) the leading term in the asymptotic expansion is to be taken for large \( \tau \). At depths considerably smaller than \( \tau_\lambda \) but large compared with 1, one may write, to within the leading terms in the asymptotic of the functions appearing in the expression given below, the asymptotic relation

\[
\Phi(\tau) = (1 - \lambda)^{-1} K_1(\tau) + \ldots, \quad \tau \gg 1,
\]

where

\[
\gamma = \frac{2}{\pi} \Gamma(2\gamma) \sin \pi \gamma
\]

and \( \gamma \) is the characteristic index of the kernel [the asymptotic solution (24) holds for \( \gamma < 1 \)]. The vital fact here is that in all cases of practical interest (lines with unbounded wings, \( \gamma \leq 1/2 \)) the factor \( C \) is close to 1. If \( \gamma = 1/2 \) (for a Doppler profile, in particular) \( C = 0.800 \); as \( \gamma \) diminishes to 0, the quantity \( C \) tends monotonically to 1, so that when \( \gamma = 1/4 \) (Voigt and Lorentz profiles) we have \( C = 0.984 \).

Furthermore, Eq. (1) implies directly that as \( \tau \to 0 \) we may write, as far as the leading term,

\[
\Phi(\tau) = (1 - \lambda)^{-1} K_1(\tau) + \ldots, \quad \tau \to 0.
\]

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And finally, the resolvent satisfies the exact integral relations

\[
\int \Phi(\tau) d\tau = (1-\lambda)^{-\nu} - 1, \tag{27}
\]

\[
\int K_i(\tau) \Phi(\tau) d\tau = (2/\lambda) (1 - \sqrt{1-\lambda}) - 1. \tag{28}
\]

The L2 approximation to the function \( \Phi \) is so constructed that in the corresponding regions of \( \tau \) it reproduces the expressions (23) and (26) exactly, while Eq. (24) is also satisfied apart from the near-unity factor \( C \). Moreover this approximation ensures that the relations (27), (28) will be strictly satisfied. Accordingly the only places where the values of \( \Phi(\tau) \) in the L2 approximation for profiles with unlimited wings (provided \( \gamma = 1/2 \)) might be in error by appreciably more than \( 10\% \) are in the two transition regions from Eq. (23) to Eq. (24) (that is, when \( \tau \) is comparable with the thermalization length) and from Eq. (24) to Eq. (26) (at optical depths of order unity). In the next section, however, we will show that the L2 approximation possesses high accuracy in these regions as well.

Along with L2, let us consider the F approximation to the resolvent. From the Frisch's approximate differential equation for the source function in a line \( \gamma \) [see also Eq. (21) of Paper II],

\[
F: \quad D^\nu(\tau) \frac{d}{d\tau}D^\nu(\tau)S(\tau) = -\frac{dS(\tau)}{d\tau}, \tag{29}
\]

with \( S^\nu(\tau) = (\lambda/2)K_\nu(\tau) \), we obtain the following expression for the resolvent in the F approximation:

\[
F: \quad \Phi(\tau) = -D^{-\nu}(\tau) \frac{\lambda}{2} K_\nu'(\tau) D^{-\nu}(\tau) \frac{d\tau}{d\tau}. \tag{30}
\]

This formula does not give a good approximation at all. In the range \( \tau_0 > \tau > 1 \), it yields an asymptotic \( \tau \)-dependence quite different from Eq. (24). For \( \tau > \tau_0 \), despite a correct functional form for the asymptote, the proportionality factor is \( (1-\lambda)^{-1/2} \) times smaller than the exact value, a very important defect when \( 1 - \lambda < 1 \). The consequences are far-reaching: the F approximation actually has a highly restricted range of applicability, and its accuracy is low.

5. SOLUTION BY KERNEL APPROXIMATION

For a semi-infinite atmosphere with \( \lambda = \text{const} \) the line source function can be derived explicitly (see Transfer\textsuperscript{16,6} Chapter 5). But even with very simple expressions for \( S^\nu(\tau) \), say \( S^\nu = \text{const} \) or \( S^\nu = \exp(\tau/\nu) \), and certainly when \( S^\nu(\tau) \) is an arbitrary function, the corresponding formulas are so cumbersome that they are no longer expedient for numerical calculations (although these expressions are indeed very useful for analytic studies, especially for investigating various limiting cases).

The numerically precise (to \( 1\% \) or better) values we shall henceforth adopt for the source functions to estimate the accuracy of the L2 and F approximations have been obtained by direct numerical solution of the integral equations (1) for \( S(\tau) \), using the accepted methods\textsuperscript{15,16} of approximating the kernel by a sum of exponentials:

\[
K_i(\tau) = \sum_{\nu=1}^{\infty} a_i \exp(-b_i \tau). \tag{31}
\]

With this representation for \( K_i \), the expressions for \( \Phi \) and \( S_1 \), as one can easily verify, will become

\[
\Phi(\tau) = \sum_{\nu=1}^{\infty} \frac{\lambda}{2} \left( \sum_{\nu=1}^{\infty} \frac{L_{\nu}}{k_{\nu}} \right) \left( 1 - \lambda \right)^{-1 - \nu}, \tag{32}
\]

\[
(1-\lambda)^{-\nu} S_i(\tau) = 1 + \sum_{\nu=1}^{\infty} \frac{L_{\nu}}{k_{\nu} \lambda} \left( 1 - \lambda \right)^{-1 - \nu}, \tag{33}
\]

where the \( k_{\nu} \) (\( \nu = 1, 2, \ldots, N \)) are the roots of the characteristic equation

\[
1 - \lambda \sum_{\nu=1}^{\infty} \frac{a_{b_i}}{b_i - k_{\nu} \lambda} = 0, \tag{34}
\]

while the constants \( L_{\nu} \) represent the solution of the linear system

\[
\sum_{\nu=1}^{\infty} \frac{L_{\nu}}{k_{\nu} \lambda} = 1, \quad i=1, 2, \ldots, N. \tag{35}
\]

Incidentally, since \( S_1(\tau) = (1-\lambda)^{-\nu} \), Eq. (33) yields the relation

\[
1 + \sum_{\nu=1}^{\infty} \frac{L_{\nu}}{k_{\nu} \lambda} = (1-\lambda)^{-\nu}, \tag{36}
\]

which is convenient for checking on the accuracy of the calculations.

In addition to the resolvent \( \Phi(\tau) \), an important role in the theory of equations of the type (1) is played by the so-called H-function, essentially the Laplace transform of \( \Phi \):

\[
H(z) = 1 + \sum_{\nu=1}^{\infty} \Phi(\nu) e^{-\nu z}. \tag{37}
\]

When the kernel is approximated by the expression (31), the H-function is given, according to Eqs. (37) and (32), by the following expansion in common fractions:

\[
H(z) = 1 + z \sum_{\nu=1}^{\infty} \frac{L_{\nu}}{1 + k_{\nu} z}. \tag{38}
\]

The method just described for finding the resolvent has recently been used at Tartu\textsuperscript{17,18} for the case of monochromatic scattering. In line-transfer problems, however, the function \( \Phi(\tau) \) seems not previously to have been calculated, either by our method or by any other.

We here give our results for the very important case of conservative scattering \( (\lambda = 1) \). Table II contains the values of \( k_{\nu} \), \( L_{\nu} \) for a Doppler profile; Tables III and IV, for Voigt profiles with \( a = 10^{-3} \) and \( 10^{-2} \). In these calculations we have used published coefficients \( a_1, b_1 \) for the Doppler and Voigt profiles.\textsuperscript{15,16} With our tabulated \( k_{\nu} \) and \( L_{\nu} \) the representation (32) is valid all the way from \( \tau = 10^{-2} \) to \( \tau = 6 \times 10^{6}, 10^{4}, 2 \times 10^{4} \) for \( a = 0, 10^{-3}, 10^{-2} \), respectively, furnishing \( 1\% \) accuracy or better. This statement relies on a comparison against the leading term of the asymptotic expression (24) for \( \Phi \) (large \( \tau \)) as well as the more accurate asymptotic expansions given in Transfer\textsuperscript{16,6} (Secs. 5,6,5,7) which include higher-order terms. When
TABLE II. Constants $k_{a}$, $L_{a}$ for Conservative Scattering: Doppler Profile

<table>
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<th>$a$</th>
<th>$k_{a}$</th>
<th>$L_{a}$</th>
<th>$a$</th>
<th>$k_{a}$</th>
<th>$L_{a}$</th>
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TABLE III. Constants $k_{a}$, $L_{a}$ for Conservative Scattering: Voigt Profile, $a=10^{-4}$

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<th>$L_{a}$</th>
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<td>1.589×10⁻⁴</td>
<td>21</td>
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<td>9.190×10⁻³</td>
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<td>1.214×10⁻³</td>
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<td>4.177×10⁻⁴</td>
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</table>

$\tau$ exceeds the upper limits quoted above, one should use the asymptote (24), as it then will definitely be accurate to better than 1%.

We wish to point out that the coefficients $a_1$, $b_1$ published by Avrett and Hummer appear to contain some errors; in particular, they fail to satisfy the normalization criterion $\sum_{n=1}^{\infty} a_n b_{n-1} = 1$. We have therefore multiplied the coefficients $a_n b_{n-1}$ by a constant factor in order to normalize them. Conceivably the nonmonotonic $\alpha$-dependence of the $L_{\alpha}$ values given in Table II might reflect the inaccuracies in Avrett and Hummer's Table 8, but a careful analysis shows that such errors could significantly have affected the $\Phi_{\alpha}$ values only in the large-$\tau$ range, which is not the case, because a comparison with the asymptotic expressions indicates that the approximation (32) using the coefficients in Table II is in error by no more than 1% (as far as $\tau = 6 \cdot 10^5$). 

6. ACCURACY OF L2 AND F APPROXIMATIONS

We turn now to the accuracy achieved by the $L_2$ and $F$ approximations for the fundamental and standard problems, beginning with the case of conservative atmospheres ($\lambda = 1$).

In the conservative case we may write the exact solution of the fundamental problem in the form

$$\Phi(\tau) = \frac{1}{2} K_1(\tau) - \frac{\Phi(\tau)}{K_2(\tau)}.$$

Here $\Phi(\tau)$ is a correction factor which explicitly describes the departure of the $L_2$ approximation from the exact solution, as given by Eq. (32). A comparison of Eq.
with Eqs. (26), (24) indicates that \( \phi(0) = 1, \phi(\infty) = C \). Remarkably, whatever the value of the Voigt parameter, from \( a = 0 \) (the Doppler case) to \( a = \infty \) (the Lorentz case), the values of \( \phi(\tau) \) are within 35% of unity for all \( \tau \). The Doppler case gives the least accurate fit; as \( a \) increases, the departures of \( \phi(\tau) \) from unity diminish. In Figs. 2, curves 1 plot the factor \( \phi(\tau) \) for the case \( a = 0 \). When \( \tau \) is large, the function \( \phi(\tau) \) for a Doppler profile can be evaluated from the asymptotic expansion
\[
\phi(\tau) = 0.9003 \left(1 - 0.1534 \frac{\ln \tau}{\ln^2 \tau} + 0.0618 \frac{\ln \tau}{\ln^2 \tau} + \ldots \right),
\]
which is readily obtained from the results given in Secs. 2.7 and 5.6 of Transfer.\(^{16,6}\)

By analogy to the standard problem in the conservative case one may seek a solution to the homogeneous equation
\[
\tilde{S}(\tau) - \frac{1}{2} \int K_i(|\tau - \tau'|)\tilde{S}(\tau') d\tau' = 0, \quad \tilde{S}(0) = 1.
\]
(41)
The function \( \tilde{S}(\tau) \) may be interpreted as the limiting value of \((1 - \lambda)^{1/2} S_i(\tau) \) and \( \lambda \rightarrow 1 \). Equation (18) therefore gives
\[
\tilde{S}(\tau) = 1 + \int \Phi(\tau') d\tau',
\]
(42)

The resolvent \( \Phi(\tau) \) here referring to the conservative case. First introduced by one of us\(^{29\,a} \) in 1962, the function \( \tilde{S}(\tau) \) has subsequently been studied in detail,\(^{21-23} \) most recently by H. and U. Frisch\(^{24} \); a summary of the results will be found in Transfer\(^{16,8} \) (Chapters 5, 6). Strangely enough, however, no comprehensive numerical values of \( \tilde{S}(\tau) \) have been published – nothing but a short table of \( \tilde{S}(\tau) \) for the Doppler profile. Nearly all its values refer to large \( \tau \) and can be computed from the asymptotic formula \( \tilde{S}(\tau) = 4\pi^{3/4} (\ln \tau)^{-1/4} t^{1/4} \), a special case (for Doppler profiles) of the general asymptotic expression
\[
\tilde{S}(\tau) = CK_{\tau}^{-1/2}(\tau), \quad \tau \rightarrow \infty,
\]
valid for \( \gamma < 1 \); it can be derived from Eqs. (24) and (41).

The solution of the homogeneous equation (41) also may conveniently be written in a form analogous to Eq. (39):
\[
\tilde{S}(\tau) = [K_i(\tau)]^{-1/2} \tilde{S}(\tau).
\]
(44)
The factor \( \tilde{S}(\tau) \) corrects the L2 approximation, converting it to the exact solution. Curve 2 in Fig. 2a plots \( \tilde{S}(\tau) \) for a Doppler profile. The limiting values \( \tilde{S}(0) = 1, \tilde{S}(\infty) = C \). In the Doppler case the asymptotic value is reached slowly: one can show that
\[
\tilde{S}(\tau) \approx 0.9003 \left(1 - 0.1534 \frac{\ln \tau}{\ln^2 \tau} - 0.2450 \frac{\ln \tau}{\ln^2 \tau} + \ldots \right).
\]
As the Voigt parameter \( a \) increases, the departures of \( \tilde{S}(\tau) \) from 1 diminish, the asymptotic value \( \tilde{S}(\infty) = C = 0.984 \) being approached for small \( \tau \) (of order \( \tau^{-4} \)). In summary, for any Voigt profile, from the Doppler to the Lorentz limit, the L2 approximation gives a solution to the homogeneous equation accurate to 20% or better.

We would emphasize that since the values of \( K_{\tau}, L_{\tau} \) given in Tables II–IV are available, we are justified in regarding \( \tilde{S}(\tau) \) and the conservative \( \Phi(\tau) \) as known and in using them to approximate the solutions of other problems, just as is true when the L2 approximation is adopted and the kernel functions \( K_i, L_i \) are considered known (even though they are nonelementary).

As for the F approximation to the function \( \Phi(\tau) \) for the conservative case \( \lambda = 1 \), its error grows rapidly with \( \tau \). In fact, the leading term in the asymptotic expression for the F approximation to the conservative \( \Phi(\tau) \), as Eq. (30) indicates, will be
\[
F: \Phi(\tau) = (\lambda^2) K_i(\tau)/K_i(\tau),
\]
(45)
which differs from the exact result (24) by \( CK_{\tau}^{-1/2}(\tau) \), a factor that increases indefinitely with \( \tau \). In this case, then, the F approximation is certainly not applicable.

Now let us consider nonconservative atmospheres. As mentioned in Sec. 4, the departures of the L2 approximation from the exact solution might exceed 10% in two regions: \( \tau \sim 1 \) and \( \tau \sim \tau_i \). Figure 3 shows a typical plot of the relative error of the L2 approximation against depth in a nonconservative atmosphere. These curves refer to the resolvent, and have been computed for a Doppler profile. The errors are largest in the Doppler case and diminish in absolute value as the Voigt parameter \( a \) increases. We see clearly from Fig. 3 that in a weakly absorbing atmosphere \((1 - \lambda \ll 1)\) the errors become independent of \( \lambda \) in the surface layer; they stabilize there. In this region the behavior of the errors with depth is described by the correction factor \( \phi(\tau) \). Moreover, the error curves all have a smooth bump which shifts inward as \( \lambda \) approaches unity; the maximum occurs at \( \tau \sim \tau_i \approx (1 - \lambda)^{-1} \). Notably, the height of the maximum is no greater than the depth of the well found at optical depths of order 1.

We may conclude that the relative error of the L2 approximation [Eq. (22)] to the resolvent, for all \( 0 \leq \lambda \leq 1 \), for any value of the Voigt parameter \( a \) \((0 \leq a \leq \infty)\), and for arbitrary \( \tau \approx 0 \), is no more than 37% (just as in the conservative case).

For the standard problem the relative-error curves of the L2 approximation are similar to the \( \tilde{S}(\tau) \) curves (such as curve 2 in Fig. 2a), except that deep in the atmosphere \((\tau \approx \tau_i)\) the error tends to zero. Its maximum value, about 20%, is reached at \( \tau \approx 3 \).

The F approximation to the resolvent is given by Eq. (30), and in Fig. 4 it is compared (dashed curves) against the exact values of \( \Phi(\tau) \) (solid curves) for several values of

![Graphs showing the correction factor and error curves for different parameters.](image-url)
of $1 - \lambda$, again assuming a Doppler profile. Notice that the scale is logarithmic along both axes; thus at the scale of Fig. 4 the curves for the L2 approximation would be indistinguishable from the exact values of $\Phi(\tau)$. Clearly it would make no real sense to employ the F approximation for evaluating the resolvent.

7. EXPONENTIAL SOURCES

Taking a very simple example, let us demonstrate how the L2 approximation can be used when a variable source term $S^*(\tau)$ is present. We shall consider the case of exponentially distributed primary sources:

$$S(\tau) = \exp(-\tau/z),$$  \hspace{1cm} (46)

where $z$ serves as a parameter ($0 < z < \infty$); its value represents the average depth at which photons are produced in the atmosphere.

In this event (see, for example, Transfer,\textsuperscript{10,16} Sec. 5.1) the source function will satisfy the following exact differential equation:

$$\frac{\partial S}{\partial \tau} = -S + zH(z) \Phi(\tau)$$  \hspace{1cm} (47)

with the boundary condition

$$S(0) = H(z),$$  \hspace{1cm} (48)

where the function $H$ is defined by Eq. (37). Accordingly the source function will be given in terms of the integral of the resolvent by

$$S(\tau) = H(z) \left[ \exp(-\tau/z) + \int_S \Phi(\tau') \exp(-\tau-\tau/z) d\tau' \right].$$  \hspace{1cm} (49)

From these expressions one can easily calculate the source function in the L2 approximation. First, taking the L2 approximation (22) for $\Phi(\tau)$, we enter Eq. (37) to find by numerical quadratures the corresponding approximate value of $H(z)$, and then calculate $S(\tau)$ by solving the differential equation (47) numerically, say by the Runge-Kutta method. $S(\tau)$ can also be determined from Eq. (49). The amount of calculation proves to be comparable with that involved in using the F approximation to evaluate the source function; in that case a numerical integration of the (approximate) differential equation (29) is required. But the results differ strikingly in accuracy (Fig. 5). At the scale of this diagram the L2 approximation is indistinguishable from the exact solution, whereas at large depths ($\tau > z$) the F approximation gives results having the wrong order of magnitude. That is true even for large $z$, when the power of the primary sources varies slowly with depth.

Since we see that the F approximation works badly even for constant $\lambda$, the optimistic view\textsuperscript{1,8} that it might be applicable for $\tau$-dependent $\lambda$ is surely unjustified.

One final point: the L2 approximation remains applicable when continuum absorption is introduced. If the corresponding kernel $K_2$ is normalized by the condition (3), if $K_2$ is defined by the first of Eqs. (6), and if $\lambda$ is suitably redefined, then Eq. (23) will, as before, provide a highly accurate approximation to the resolvent. A detailed discussion of L2, allowing not only for the continuum but also for macroscopic motions in the atmosphere, will be given in forthcoming papers.

\textsuperscript{1} V. V. Ivanov and V. M. Serbin, Astron. Zh. 81, 691 (1984) [Sov. Astron. 28, No. 4 (1985)].
\textsuperscript{2} V. V. Sobolev, "The radiative equilibrium of planetary nebulae" [in Russian], Candidate's dissertation, Leningrad Univ. (1941).
\textsuperscript{5} V. V. Ivanov, Astron. Zh. 49, 115 (1972) [Sov. Astron. 16, 91 (1971)].
Signal shape recovery in gravitational-wave experiments

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A computer experiment demonstrates how one can recover the shape and parameters of an external pulse signal disturbing a gravitational detector equipped with a relaxation-type sensor circuit. A mathematical procedure is developed and solutions of this inverse problem are presented for some typical gravitational-wave bursts, allowing for the antenna noise fluctuations. If the pulse structure is fairly simple, it can be restored despite a low signal/noise ratio (~1). This advantage will be invaluable for future gravitational-wave and other delicate physical experiments.

1. INTRODUCTION

Gravitational-wave experiments are currently in a stage where new observing programs are being organized in hopes of detecting bursts of gravitational radiation from cosmic sources.1,2 Experimenters are putting much effort into the design of second-generation gravitational antennas sensitive enough to record the pulses signals which reasonable astrophysical predictions indicate should accompany the dynamic processes at work in superdense stars, clusters of such stars, active galactic nuclei, and quasars. Generally speaking, a gravitational-wave pulse is predicted3 to carry an energy density $10^{-16}$ erg/cm$^2$, to last for $10^{-4}$–$10^{-3}$ sec, and to have a central frequency $\nu$ in its spectrum such that $\nu T \sim 1$. Detectable events would be expected to occur at least once a month.

Careful analysis has shown1–3 that it is a very complex engineering problem to develop gravitational antennas whose sensitivity can meet even the optimum, upper-limit forecasts. Under these circumstances it would be natural for such experiments in their initial phase to aim merely at recording some gravitational burst and confidently demonstrating that it is indeed of gravitational origin (by wholly eliminating all possible sources of nongravitational noise). Once this barrier has successfully been overcome, however, the dawn of gravitational-wave astronomy4 will call for more detailed analysis of the signals being received, including measurement of their parameters and recovery of their shape.

In this paper we inquire into the chances for reconstructing the profile and parameters of a pulsed signal activating a Weber antenna. As a first approximation we consider the simplest type of antenna, a gravitational detector with a nonparametric recording circuit. (This model encompasses antennas having piezocrystal and magnetostriuctive transformers as well as constant-displacement capacitive and inductive sensors.) We have carried out a numerical experiment, simulating an antenna and a gravitational signal on the computer via certain basic equations. To ascertain the burst structure from the antenna response we have employed standard algorithms from the theory of inverse ill-posed problems, as modified to reflect the special features of gravitational-wave experiments. This procedure for recovering the signal that has triggered a Weber antenna gives satisfactory results for pulses at the threshold of sensitivity.

In order to simulate the effect of recording a gravitational signal one also must adopt some plausible structure for the bursts of gravitational waves. A good deal of work has now been done in calculating the dynamics of gravitational radiation, based on various models of cosmic catastrophes in superdense stars—collisions, collisions, and so on. Comprehensive references may be found in various conference proceedings.5,6 For the most part a numerical approach is taken: authors will specify certain parameters for the radiating system, and numerous constraints will limit the generality of the results. Nevertheless, from an inspection of the literature one