INFLUENCE OF ROTATION AND MAGNETIC FIELDS ON
STELLAR OSCILLATION EIGENFREQUENCIES

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ABSTRACT

A formalism to calculate rotational and magnetic splitting of
otherwise degenerate high-order oscillation p modes of low degree is
outlined. For the purposes of illustration, attention is restricted to
axisymmetric toroidal magnetic fields, with axes of symmetry that are
permitted to be different from the rotation axis. We have in mind
investigating the effects of a potential magnetic field in the interior of
the sun, with intensities no greater than about 10^7 G. We find that
advection is the most important contributor to rotational splitting; in
general, magnetic perturbations add to the complexity of the frequency
spectrum when the oscillations are viewed from an inertial frame. The
combined effect of the two agents is to split the degeneracy into (2l+1)^2
components, (l+1)(2l +1) of which should, at least in principle, be
detectable in whole-disk observations from Earth.

We consider briefly the effect of a field whose axis lies close
to the sun's equatorial plane. Such a field has been proposed by Dicke
(1976) to explain his oblateness data. We find that amongst these special
field configurations, none is able to to split five-minute dipole modes
into observable triplets. We therefore deduce that the splitting reported
by Claverie et al. (1981) precludes such fields.
INTRODUCTION

This study was motivated by recent observations of frequency splitting in Ap stars and in the sun. Kurtz (1982, 1983) has reported measurements of high-frequency oscillations of rotating Ap stars that appear to be consistent with wave patterns of low degree aligned with an observed large-scale magnetic field. This has immediately raised the theoretical problem of explaining why the oscillations and the magnetic field should rotate in synchronism (Dolez and Gough, 1982). Claverie et al. (1981) have reported splitting of the frequencies of high-order solar p modes of low degree. These observations have revealed more components than would have been expected from rotational perturbations alone (Isaak, 1982; Gough, 1982a), and Isaak (1982) has suggested that a buried magnetic field is responsible. In this case there is no direct evidence for the field; multiple frequency splitting requires only that the sun is not symmetrical about its axis of rotation. Nevertheless, the idea merits serious study.

Preliminary theoretical investigations of the combined influence of rotation and magnetic fields on oscillation eigenfrequencies have been reported by Gough (1982a) and Dicke (1982a) in connection with the splitting of high-order solar p modes. In both cases simplifying assumptions were made: Gough (1982a) presumed that the small frequency perturbations produced by the Lorentz force and by rotation could merely be added together, and Dicke (1982a) modelled the Lorentz force as simply an anisotropic contribution to the scalar sound speed. In both cases it was possible to reproduce the number of split components in the frequency spectrum observed by Claverie et al. (1981), provided certain apparently chance relations between field and angular velocity were satisfied. That conclusion is not negated here.

Estimates of constraints imposed on the sun’s magnetic field by the rotational splitting of g and f modes claimed by Hill et al. (1982) have been made by Gough (1982b). A more thorough investigation is reported by Dziembowski and Goode (1983; also these proceedings). The mean field intensity that was deduced is lower than that required to reproduce the p-mode splitting. However, since the p-, f- and g-mode frequencies depend on differently weighted averages of the field, the observations do not obviously contradict one another.

A proper assessment of the implications of the observations requires a consistent theoretical treatment. Here we outline how that can be accomplished, and apply it to the five-minute solar modes of low degree. We consider, in particular, an axisymmetric magnetic field whose axis lies close to the equatorial plane, in the spirit of Dicke's (1976, 1978, 1982b) discussions of the solar oblateness, and find that that is unlikely to account for the splitting.

PERTURBATIONS INDUCED BY SLOW ROTATION AND BY WEAK MAGNETIC FIELDS

We presume the star to have a unique axis of rotation, and assume the existence of a frame of reference S, rotating with angular velocity \( \omega_c \), in which the structure of the star, aside from the oscillations, is
Steady. Thus, in particular, we assume a rigid equilibrium magnetic field, rotating with angular velocity \( \Omega_\ast \). Any differential motion of the stellar material is described by a velocity \( \mathbf{v} \) in \( S \), which later we shall assume to result solely from nonuniform rotation.

Lorentz forces are assumed to be weak compared with pressure gradients, and the centripetal acceleration associated with the rotation is much less than the acceleration due to gravity. Therefore the problem may be solved by expanding about the nonrotating nonmagnetic state. For this purpose it is convenient to work with dimensionless variables.

In \( S \), the linearized wave equation describing nonradial adiabatic oscillations can be written (cf. Lynden-Bell and Ostriker, 1967)

\[
\ddot{\xi} + \rho \omega^2 \xi = \epsilon \omega \eta \xi + \epsilon^2 \eta \xi + \delta \mathbf{B} \cdot \nabla \xi
\]  

(1)

for the amplitude \( \xi \) of the oscillation displacement \( \text{Re}[\xi(\tau) \exp(i\omega t)] \), where \( \tau \) is the position vector and

\[
\ddot{\xi} \equiv \nabla \left( \mathbf{\rho} \cdot \mathbf{v} \right) \nabla \xi - \xi \cdot \nabla \mathbf{v} + \mathbf{p} \nabla \cdot \nabla \xi - \mathbf{p} \cdot \nabla \xi \cdot \nabla \mathbf{v},
\]

\[
\eta \xi \equiv 2i \rho \left( \mathbf{\Omega} \times \mathbf{x} + (\nabla \cdot \mathbf{x}) \right),
\]

\[
\eta \xi \equiv \rho \left( \mathbf{\Omega} \cdot \mathbf{x} \right) + 2 \mathbf{\Omega} \cdot (\nabla \cdot \mathbf{x}) - (\nabla \cdot \mathbf{x}) \mathbf{\Omega} - (\mathbf{x} \cdot \mathbf{\Omega}) \mathbf{\Omega} + (\mathbf{\Omega} \cdot \mathbf{x} \cdot \nabla \mathbf{v}),
\]

\[
\mathbf{B} \cdot \nabla \xi \equiv - \left( \mathbf{\nabla} \times \mathbf{B'} \right) \cdot \mathbf{B} + (\mathbf{\nabla} \cdot \mathbf{B}) \mathbf{B'} / 4\pi; \quad \mathbf{B}' \equiv \mathbf{\nabla} \times (\mathbf{\xi} \times \mathbf{B}).
\]

(2)

(3)

(4)

(5)

Here \( \rho, \rho, c \) and \( \mathbf{B} \) are pressure, density, sound speed and magnetic field of the equilibrium state, measured in units of \( G M R^{-\frac{3}{2}} M R^{-\frac{3}{2}} \) and \( \mathbf{B} \) respectively, where \( G \) is the gravitational constant, \( M \) and \( R \) are the mass and radius of the star, and \( \mathbf{B} \) characterizes the magnitude of the equilibrium magnetic field. Time \( t \) and frequency \( \omega \) are measured in units of \( (GM/R^3)^{-\frac{1}{2}} \) and \( (GM/R^3)^{\frac{1}{2}} \), and \( \Omega_\ast \) is measured in units of, say, the surface equatorial angular velocity \( \Omega_\ast \). Equations (5) depend on the assumption of infinite electrical conductivity. Also, perturbations to the gravitation potential induced by the oscillations have been ignored; this is a good approximation for solar five-minute modes.

The dimensionless parameters \( \epsilon^2 = \Omega_\ast^2 R^3 G^{-1} M^{-1} \) and \( \delta^2 = B^2 R^4 G^{-1} M^{-2} \) are both small. In the sun, for example, \( \epsilon = 5 \times 10^{-3} \) and \( \delta = 9 \times 10^{-3} \) if \( \mathbf{B} = 10^6 \mathbf{G} \). Thus, for this choice of \( \mathbf{B} \), these parameters are of the same order, and we shall write \( \delta = \epsilon \lambda \) and take \( \lambda \) to be of order unity.

The eigenvalue problem may be solved by developing the solution in powers of \( \epsilon \) thus:

\[
\xi = \epsilon \xi_0 + \epsilon^2 \xi_2 + \epsilon^2 \xi_2 + \cdots,
\]

(6)

with a similar expansion for \( \omega \). The equilibrium state is perturbed only by centrifugal and Lorentz forces, so, for example,

\[
\rho = \rho_0 + \epsilon^2 \rho_2 + \cdots.
\]

(7)

Substituting these expansions into Equation (1) and equating powers of \( \epsilon \).
Yields, in an obvious notation,

\[ \mathcal{L}_0^x + \omega^2 \rho_0 \xi_0 = 0, \quad (8) \]

\[ \mathcal{L}_{0-o}^x + \omega^2 \rho_0 \xi_{0-o} = - \omega (2 \omega \rho_0 \xi_0 - \mathcal{M}_0 \xi_0), \quad (9) \]

\[ \mathcal{L}_{-2-o}^x + \omega^2 \rho \xi_{-2-o} = - \mathcal{L}_{-2-o}^x - \omega^2 \rho \xi_{-2-o} - (\omega^2 + 2 \omega_0 \omega) \rho_0 \xi_{0-o} \\
+ \omega_1 \mathcal{M}_1 \xi_{-1-o} + \omega \mathcal{M}_1 \xi_{0-o} + \mathcal{N}_1 \xi_{-1-o} + \mathcal{N}_1 \xi_{0-o}, \quad (10) \]

and so on. Equation (8) is the usual wave equation for a nonrotating nonmagnetic spherically symmetrical star. The operator \( \mathcal{L}_0 \) is self-adjoint, so that for some \( \tilde{n} \) and \( \tilde{\xi} \), if angular brackets denote integration over the volume of the star, \( \langle \tilde{n} \xi_0 \rangle = \langle \tilde{n} \xi_0 \rangle \), provided certain surface integrals vanish (e.g. Lynden-Bell and Ostriker 1967). This is quite accurately the case if both \( \tilde{n} \) and \( \tilde{\xi} \) satisfy the boundary conditions appropriate for stellar pulsation.

In view of the self-adjointness of \( \xi_0 \), a necessary condition for the existence of a solution \( \xi_0 \) to Equation (9) is that the right-hand side be orthogonal to \( \xi_0^* \), where the asterisk denotes complex conjugate. Thus

\[ \omega_1 = (2\pi)^{-1} \langle \xi_0^* \cdot \mathcal{M}_0 \xi_0 \rangle, \quad (11) \]

where \( I = \langle \rho_0 \xi_0^* \xi_0 \rangle \). In the case of pure rotation this reduces to the results of Hansen et al. (1977) and Gough (1981), and further reduces to the well known formulae of Cowling and Newing (1949) and Ledoux (1951) when the rotation is uniform. Similarly, the second-order contribution is

\[ \omega_2 = (2\pi)^{-1} \{ - \langle \xi_0^* \cdot (\mathcal{L}_{-2-o}^x + \omega^2 \rho \xi_{-2-o} - \omega_1^2 I) \\
+ \langle \xi_0^* \cdot (\omega \mathcal{M}_1 \xi_{0-o} + \omega \mathcal{M}_1 \xi_{-1-o} + \mathcal{N}_1 \xi_{0-o} + \mathcal{N}_1 \xi_{-1-o}) \rangle + \mathcal{N}_1^2 \langle \xi_0^* \cdot \mathcal{B}_1 \xi_0 \rangle \}. \quad (12) \]

We now estimate the magnitudes of the contributions to \( \omega_2 \), to justify a subsequent simplification of our discussion. We restrict attention to five-minute modes of low degree, which have order \( n \approx 25 \) and cyclic frequencies of about 3000 \( \mu \)Hz. For these modes \( \omega_0 = 0(n) \), and \( \omega_1 = 0(1) \), yielding rotational splitting of order 3000 \( \epsilon n^4 \mu \)Hz = 0.6 \( \mu \)Hz.

The first integral in the curly brackets in Equation (12) comes from the distortion of the equilibrium state by both rotation and the magnetic field. It provides a contribution of order \( \epsilon^2 \omega_0 \) to \( \omega_2 \). The next two terms arise from the advection terms in the equation of motion, and contribute no more than about \( \epsilon^2 \omega_0 \omega \). The last term is the direct effect of the perturbed Lorentz force; if \( \mathcal{B} \) is significant throughout a substantial fraction of the radiative interior, it contributes an amount similar to that of the magnetic distortion of the equilibrium state (e.g. Gough, 1982a). Thus it would appear that we need consider only the first and last terms in (12). We have confirmed by numerical integration that that is indeed so.

Finally we note that the contribution \( \epsilon^2 \omega_0 \), corresponding to about 0.08 \( \mu \)Hz, is roughly one-tenth of the total rotational splitting, and

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that a mean magnetic field of some 3 MG would be required to perturb $\omega$ by an amount comparable with rotation.

To illustrate the influence of Lorentz forces we consider explicitly an equilibrium axisymmetrical toroidal magnetic field of the type $B = (0, 0, B(r) \frac{dP_k}{d\vartheta'})$ with respect to spherical polar co-ordinates $(r', \vartheta', \varphi')$ about its axis of symmetry, where $P_k(\cos \vartheta')$ is a Legendre polynomial. These fields fall into the class considered by Dicke (1978, 1982b), though they are simpler than the field Dicke requires to explain his oblateness data completely. The Lorentz forces they produce have the form

$$F = \Sigma (F_{\lambda r}(r) P_{\lambda}(\cos \vartheta'), F_{\lambda \vartheta'}(r) P_{\lambda}(\cos \vartheta'), 0),$$  

(13)

where $\Sigma$ denotes the sum over even values of $\lambda$ from 0 to $2k$, and $F_{\lambda \vartheta'} = 0$.

The evaluation of the first integral in Equation (12) deserves some comment. Unlike the other integrals encountered here, it must be evaluated over the perturbed volume of the star. This requires some care. The method we have adopted is to use a co-ordinate system $(r')$ that is distorted, in an appropriate sense, with the star. A similar procedure has been used by Simon (1969) and Goossens (1972). Since only the leading contributions to the distortion are considered, they may be computed separately and subsequently added together. The evaluation of the distortion is discussed by Lebovitz (1970). We have solved the equilibrium equations to order $\epsilon^2$, without making Dicke's (1982b) approximation of ignoring the variation of the gravitational acceleration on level surfaces. The radial co-ordinate $r$ was replaced by a $\vartheta'$-dependent variable $r' = [1 + \epsilon^2 \Sigma h_\lambda(r) P_{\lambda}(\cos \vartheta')] r$, with $h_\lambda$ defined so that $r'$ is constant on surfaces of constant pressure. Thus, in particular, $r' = R$ at the surface of the star. The modifications to $\rho_\vartheta$ and $c_\varphi$ are then of the form $\Sigma h_\lambda (r') P_{\lambda}(\cos \vartheta')$, etc., and the equations for the $r'$ dependence each involve only a single value of $\lambda$. It is with respect to the co-ordinates $(r', \vartheta', \varphi')$ that the expansions (6) - (10) were carried out, and therefore $F_{\vartheta'}$ and the variables describing the equilibrium state are considered to be functions of these co-ordinates, and not $(r, \vartheta, \varphi)$. By adopting this approach one can avoid near divergences of integrands in the contribution to $\omega_\vartheta$. The centrifugal force is also of the form (13), having a single term with $\lambda = 2$ when expressed with respect to spherical polar co-ordinates about the axis of rotation. It can therefore be treated in the same way.

The integrals in Equations (11) and (12) can be estimated using the asymptotic expansion for $F_{\vartheta'}$ at large $n$ (e.g. Vandukurov, 1967, Tassoul, 1980). We assume, for simplicity, that rotation is constant on spheres (generalization to a latitudinally dependent angular velocity is straightforward), so that $\lambda(r) \equiv r \Delta \Omega_{r'} (r)$, and we consider only functions $B$ and $\tilde{\Omega}_1 \equiv \Omega_1 (r) k$, where $k$ is constant, that vary on a scale much larger than $nR$. Then the perturbation from rotation to the eigenfrequency of a mode of degree $\lambda$ and azimuthal order $m$ relative to the axis of rotation is

$$\omega_{1m} = \epsilon \omega_1 + \epsilon^2 \omega_{2\lambda}$$

where

$$\omega_1 (m) \sim m^{-1} \int_0^a \Omega_1 c^{-1} \mathrm{d}r \equiv m I_1,$$  

(14)
\[ \omega_{2\Omega}(\lambda,m) \sim \omega^{-1} Q_{2\Omega}\int_0^a (h_\Omega + dh_\Omega/d\delta\nu r' + c^{-1}c_{2\Omega})^{-1} dr' = I_2 Q_{2\Omega}, \]  
\[ Q_{\lambda \pm m} = Q_{\Omega m} = K_{3\Omega} \int_1^{\lambda} (P_{\lambda}(\mu) [P_{\pm m}(\mu)]^2 d\mu, \]  
\[ K_{3\Omega} = \left\{1(2\lambda+1)(\lambda-|m|)!/(\lambda+|m|)!, \quad \tau = \int_0^a c^{-1} dr. \]  

\(Q_{\Omega m}\) is the Gaunt integral, and \(h_\Omega\) and \(c_{2\Omega}\) are the rotationally induced distortion amplitudes analogous to the amplitudes \(h_2\) and \(c_{22}\) that arise from the Lorentz force. The convention that \(m > 0\) corresponds to waves travelling in the same sense as \(\Omega\) has been adopted. The integrals have been evaluated up to \(r = a\), or \(r' = a\), the level above which the modes become evanescent. The criterion \(\omega = c(2\Omega)^{-1} \sqrt{(1 + 2 \text{ dH}/\text{d}r)}\) with \(\omega/2\pi = 3\text{ mHz}\) was used to estimate \(a\), where \(H\) is density scale height (Deubner and Gough, 1984). The magnetic perturbation for a mode of degree \(\lambda\) and azimuthal order \(m\) with respect to the magnetic axis of symmetry is \(\omega_{B\lambda m} = \Omega_{2\lambda} + \omega_d\), where (cf. Gough, 1982a) 
\[ \omega_{2\lambda}(\lambda,m') \sim \omega^{-1} Q_{\lambda \pm m'} \int_0^a (h_\lambda + dh_\lambda/d\delta\nu r' + c^{-1}c_{2\lambda})^{-1} dr' = I_{3\lambda} Q_{\lambda \pm m'}, \]  
\[ \omega_d \sim S_{k\pm m'} \int_0^a (\nu_A/c)^2 c^{-1} dr = I_4 k_{k \pm m'}, \]  
\[ S_{k\pm m'} = K_{3\Omega} \int_1^{\lambda} (1-\mu^2)[dP_{k}(\mu)/d\mu]^2 [P_{\pm m'}(\mu)]^2 d\mu. \]  

and \(\nu_A^2 = \rho^{-1}\beta^2\) measures the square of the Alfvén speed in the equilibrium state. 

The form of the distortion terms merits some comment. Eigenfrequencies of the unperturbed star, which are given roughly by \(\omega \sim (n + \lambda/2)\tau^{-1}\), depend on the structure of the star via the acoustic radius \(\tau\). It is simply that factor that dominates in the perturbed case too. Indeed, the perturbation to this expression for \(\omega_d\) arising from the change in both \(c\) and the dimensions of the star is just the distortion term on the right-hand side of Equation (18) without the factor \(Q_{\lambda \pm m'}\). The geometrical factor \(Q_{\lambda \pm m'}\) arises because different modes sample the distortion differently, and its form (17) is precisely what one would expect from the unperturbed solution once one notices that the latter stems from the asymptotic expansion of \(\omega_{2\Omega} = 1^{-1}(\lambda \times \lambda - \lambda \times \lambda_0)/10\). Rotation, for example, causes the star to be distended near the equator; the acoustic travel time is greater in the equatorial plane than along the axis. 

\[ \begin{array}{cccc}
\lambda & m & Q_{\lambda \pm m} & S_{2\pm m} \\
\hline
1 & 0 & 4/5 & 0 & 108/35 \\
1 & 1 & -2/5 & 0 & 72/35 \\
2 & 0 & 4/7 & 4/7 & 12/7 \\
2 & 1 & 2/7 & -8/21 & 24/7 \\
2 & 2 & -4/7 & 2/21 & 12/7 \\
\end{array} \]  

Table 1. The geometrical factors defined by Equations (16) and (19).
Consequently the distortion lowers the frequencies of the equatorially concentrated sectoral \((m = \pm \ell)\) modes relative to those of the zonal \((m = 0)\) modes, which exist preferentially in the polar regions. This phenomenon is evident in the examples listed in Tables 1 and 2.

A NUMERICAL EXAMPLE

We illustrate the behaviour of the eigenfrequencies with a simple example applied to the sun. In the spirit of Dicke (1978, 1982b) we imagine the magnetic field \(B\) to be confined within a spherical core of radius \(r_c\), and uniformly rotating with angular velocity \(\Omega\). However, outside the core we take an angular velocity to be \(\Omega = (R/r)^2\Omega_c\) in the inertial frame, and we choose \(\Omega_c\) to make \(\Omega\) continuous. We consider only quadrupole fields \((k = 2)\), so that the representation (13) of the Lorentz force has two nonspherical components: \(\lambda = 2\) and \(\lambda = 4\). We approximate Dicke's (1982b) magnetic field by

\[
B(r) = (1 + \alpha) \left(1 - \frac{r}{r_c}\right)^\alpha B_0 (\frac{r}{r_c})^2 \left[1 - \left(\frac{r}{r_c}\right)^2\right]^\alpha
\]

(21)

choosing \(\alpha(r_c)\) such that the position \(r_m\) of the maximum field \(B_0\) agrees approximately with the value in Dicke's Figure 1.

The integrals \(I_1 - I_4\) have been evaluated using the solar Model A of Christensen-Dalsgaard et al. (1979). They are listed in Table 2 for various values of \(r_c > 0.3\). Smaller values of \(r_c\) were not considered because the asymptotic approximation to the displacement eigenfunction that we have used is inaccurate near the centre of the sun. Also listed are the amplitudes \(h_1(R)\) and \(h_2(R)\) of the distortion of the sun's surface. In Table 1 we list some examples of the geometrical factors \(Q_{\lambda m}\) and \(S_{2\ell m}\).

To obtain some indication of the uncertainty in the asymptotic procedure, the upper limit of integration in \(I_1 - I_4\) was moved to the surface to \(r = R\), with a consequent increase of less than 2 per cent in

<table>
<thead>
<tr>
<th>(r_c/R)</th>
<th>(\alpha)</th>
<th>(I_1)</th>
<th>(I_2)</th>
<th>(10^3h_{1}(R))</th>
<th>(10^3h_{2}(R))</th>
<th>(10^3h_{3}(R))</th>
<th>(10^3h_{4}(R))</th>
<th>(10^3I_{14})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>4</td>
<td>1.52</td>
<td>0.058</td>
<td>8.3</td>
<td>4.75</td>
<td>6.9</td>
<td>-0.98</td>
<td>0.94</td>
</tr>
<tr>
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<td>5</td>
<td>1.19</td>
<td>0.054</td>
<td>7.8</td>
<td>4.53</td>
<td>5.4</td>
<td>2.37</td>
<td>2.7</td>
</tr>
<tr>
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<td>0.051</td>
<td>7.5</td>
<td>2.52</td>
<td>0.9</td>
<td>9.14</td>
<td>6.4</td>
</tr>
<tr>
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<td>0.046</td>
<td>7.3</td>
<td>-5.65</td>
<td>-6.2</td>
<td>22.1</td>
<td>14.0</td>
</tr>
<tr>
<td>0.7</td>
<td>8</td>
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<td>0.042</td>
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<td>-14</td>
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<td>-</td>
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</tr>
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</table>

Table 2. Integrals \(I_1 - I_4\) determining the frequency perturbations resulting from the angular velocity \(\Omega(r)\) described in the text and the magnetic field \(B(r)\) defined by Equation (20) with \(B_0 = 10^7 G\). The quantities \(h_1(R), h_2(R)\) and \(h_4(R)\) measure the distortion of the solar surface. All entries in the last 5 columns are proportional to \(B_0^2\).
all cases. We have also evaluated the corresponding integrals in Equations (11) and (12) using the vertical component of those true (numerically computed) adiabatic displacement eigenfunctions of degrees 1 and 2 with frequencies closest to 3 mHz, though we omitted the horizontal component. The frequency perturbations so obtained differ from those implied by Table 2 by up to about 10 per cent.

FREQUENCY SPLITTING DUE TO THE COMBINED INFLUENCE OF ROTATION AND MAGNETIC FIELDS

Rotation and magnetic fields split the degeneracy of the $2l+1$ independent modes of like order and degree. In the case of pure rotation the perturbed eigenmodes correspond to the unperturbed modes of different azimuthal order $m$ defined with respect to the rotation axis. In the rotating frame the advection term is linear in $m$, whereas the distortion term, which cannot distinguish between eastward and westward propagating waves, is a function of $m^2$. The result is a $(2l+1)$-fold nonuniform splitting, as illustrated in Figure 1. Notice, however, that modes with odd values of $l+m$ cannot be detected in whole-disk observations such as those made by Claverie et al. (1981); therefore only $l+1$ should be visible.
Figure 2: Diagram corresponding to Figure 1 with the addition of a nonalined rigidly rotating magnetic field. Once again the spherically symmetrical perturbation to the equilibrium state has been ignored. In the rotating frame the motion can be represented by three separate orthogonally orientated standing dipole oscillations with different frequencies, which are not aligned parallel and perpendicular to the rotation axis. Dashed lines represent frequencies of modes in the inertial frame that have amplitudes $O(\psi)$ when the angle $\psi$ between the solar equator and the axis of symmetry of the magnetic field is small. Dotted lines represent modes with odd $\ell + m$, as in Figure 1.

A magnetic field that is axisymmetrical about the rotation axis does not alter the qualitative nature of this result. But if the magnetic axis is inclined to the rotation axis, the multiplicity of frequencies observed increases. In the rotating frame in which $B$ is steady there are still $2\ell + 1$ eigenfrequencies, but the eigenfuntions no longer have a pure sinusoidal dependence on $\varphi$. Consequently each mode appears to have more than one frequency when viewed from an inertial frame.

In the rotating frame the eigenfunctions can be computed in the manner indicated by Dicke (1982a). They are represented approximately as a linear combination $a_{\ell n m}^{\ell^* n^* m^*}$ of the $2\ell + 1$ degenerate zero-order basis eigenfunctions $\xi_{\ell n m}$ of order $\ell$, degree $n$ and azimuthal order $m$ with respect to the rotation axis. Here we use the summation convention and adopt the normalization $\langle \xi_{\ell n m}^{\ell^* n^* m^*} | \xi_{\ell p q} \rangle = \delta_{\ell \ell^*} \delta_{n n^*} \delta_{m m^*}$, $a_{\ell n m}^{\ell n m} = 1$, where $\delta_{ij}$ is the Kronecker delta. Each basis function can also be expressed as a linear combination of modes $\xi_{\ell^* n^* m^*}$ of like order and degree relative to the magnetic axis: $\xi_{\ell n m} = R_{\ell^* n^* m^*}^{\ell n m}$, where $R_{\ell n m}$ is simply the appropriate
rotation matrix for spherical harmonics. Then if $U_{mm'}, V_{mm'}$ are the diagonal matrices with $m, m'$ components $\omega_{2\ell}(\ell, m)$, \( \omega_{2\ell}^{\pm}(\ell, m') \) for fixed $\ell$, the factors $a_m$ defining the oscillation eigenfunctions are determined by the eigenvectors of the matrix
\[
\Omega_{mm'} = \delta_{mm'} + U_{mm'} + R_{mr} V_{rs} R_{s'm'} ;
\]
the frequency perturbations are its eigenvalues. The latter are illustrated in Figure 2. Note that now, however, as Dicke (1982a) has pointed out, each eigenfunction is a combination of $2\ell + 1$ spherical harmonics $P_n^m(\cos \theta)$ $\exp(-im\varphi)$, which suffer different frequency increments $m\Omega$ when viewed from an inertial frame. Thus one would expect many more peaks in the power spectrum of the oscillations than Claverie et al. have found.

**Dipole Splitting When the Magnetic Axis is Close to the Equatorial Plane**

The matrices $\omega_1 \delta_{mm'} + U_{mm'}$ and $V_{mm'}$ are of the form
\[
\begin{pmatrix}
  u_1 & 0 & 0 \\
  0 & u_2 & 0 \\
  0 & 0 & u_3
\end{pmatrix}
\quad
\begin{pmatrix}
  v_1 & 0 & 0 \\
  0 & v_1 - v_2 & 0 \\
  0 & 0 & v_1
\end{pmatrix}
\]
(23)
where $u_1 + u_2 + u_3 = 0$ (see Table 1) and $u_1 - u_2 = 2\Omega_c$. If the angle subtended between the magnetic and rotation axes is close to $\pi/2$, principal axes of $\Omega_{mm'}$ lie close to the rotation axis and to the magnetic axis.

The dipole component whose axis lies close to the rotation axis cannot be detected by whole-disk observations from the ground. It can be shown that the two modes that remain have eigenfrequencies that are roughly $\omega_0 + (v_2 - u_2)/2 \pm \omega$, where $\omega^2 = v_0^2/4 + \Omega_c^2$. Let their actual amplitudes be $A_+$ and $A_-$. The question we now ask is whether it is likely that just three of them should be seen from the earth, as Claverie et al. report.

To compute what would be observed, we transform to the inertial frame (or, more precisely, to a frame rotating with the orbital angular velocity of the earth about the sun). The dislocations from their mean of the frequencies of the oscillations that would be observed, and the squares of their relative amplitudes are:
\[
\begin{align*}
w &+ \Omega_c & A_+^2(1 + Z_+^2)^{-1} \\
w &- \Omega_c & Z_+^2 A_+^2(1 + Z_+^2)^{-1} \\
-w &+ \Omega_c & A_-^2(1 + Z_-^2)^{-1} \\
-w &- \Omega_c & Z_-^2 A_-^2(1 + Z_-^2)^{-1}
\end{align*}
\]
Where \( Z = 2(\Omega + \omega)/v_2 \). We can now see that all four oscillations have similar amplitudes and different frequencies unless either one of \( A_z \) is small or \( v_2 < \Omega \). However, it is straightforward to verify that in either exceptional case only two visible oscillations remain. In the first case there is only one eigenmode, which is split into two visible oscillations; in the second the magnetic contribution is nearly isotropic, and the situation is qualitatively no different from that of rotation alone.

The preceding discussion is exact if the magnetic axis lies in the equatorial plane. If the angle \( \Psi \) subtended by the magnetic axis with the equatorial plane is nonzero but small, then the frequency splitting deviates from that quoted above by only \( O(\Psi^2) \); the amplitudes deviate by \( O(\Psi) \). Thus if \( \Psi \) varies slowly with time, as Dicke (1982b) has suggested, the effect on the observations is predominantly a variation of the amplitudes, accompanied by relatively small frequency shifts.

**CONCLUDING REMARKS**

The splitting reported by Claverie et al. (1981) of the degeneracy of solar dipole oscillations into three components that are detectable in whole-disk observations from Earth has forced us to contemplate the possibility of large-scale asymmetries in the structure of the sun. Isaak (1982) has suggested the explanation is a magnetic field in the core, of the kind proposed by Dicke (1982b). However, to explain his oblateness data Dicke requires a predominantly toroidal axisymmetric magnetic field whose axis lies within about \( 5^\circ \) of the equatorial plane. On the other hand, Dicke's (1982a) attempt to reproduce the oscillation data yielded a field inclined at an angle of about \( 45^\circ \).

We have investigated with some care the nature and magnitude of degeneracy splitting by rotation and a magnetic field. We have considered a class of simple fields not very different from those invoked to explain the oblateness. Then we asked whether they are consistent with the oscillation data; the fact that the field which reproduces the Birmingham oscillation data the most accurately is rather different from the field Dicke constructed to explain his oblateness does not immediately preclude the possibility that the latter might produce acceptable oscillation frequencies. However, we have failed to reconcile the two observations in an obvious way. Perhaps we have restricted ourselves to too simple a class of fields. Alternatively, maybe, the multiplicity of the nearly degenerate observable dipole modes is actually four, as Dicke (1982a) has conjectured, and the observations have not yet succeeded in resolving it.
REFERENCES

Dicke, R.H. 1982b Solar Phys. 78, 3-16.