On the onset of relativistic instability in highly centrally condensed stars

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Summary. It is shown that along the polytropic sequence, the criterion for the onset of the instability of relativistic origin by radial perturbations is to a sufficient degree of accuracy (< 3 per cent) given by

\[ R < 0.226 \left( \frac{\rho_c}{\bar{\rho}} \right)^{\frac{1}{3}} \frac{2GM}{c^2} \frac{1}{\Gamma_1 - 4/3} \] (p_c > 10^6 \rho),

where \( 2GM/c^2 \) is the Schwarzschild radius of the star, \( \rho_c \) and \( \bar{\rho} \) are the central and the mean density, respectively, and \( \Gamma_1 \) is the adiabatic exponent assumed to be constant and close to 4/3. It is suggested, on the basis of this result, that the relativistic instability may be more relevant than presently appreciated, in considerations relative to the late stages in the evolution of stars when nuclear synthesis of the elements of higher atomic numbers takes place.

1 Introduction

It is known that in the framework of general relativity, dynamical instability by radial perturbations can set in, no matter how high the adiabatic exponent \( \Gamma_1 \) (or, more precisely, some average of it) may be, provided the radius of the equilibrium configuration is less than a certain determinate multiple of the Schwarzschild radius (Chandrasekhar 1964). In particular, if \( \Gamma_1 \) (assuming it to be a constant for the sake of simplicity) differs from 4/3 by a small positive constant, the criterion for instability takes the form

\[ R < \frac{2GM}{c^2} \frac{K}{\Gamma_1 - 4/3}, \] (1)

where \( G \) is the constant of gravitation, \( c \) is the velocity of light, \( M \) is the mass of the star, and \( K \) is a constant which depends on the entire march of the pressure and the density distribution in the star. It is clear from the form of the inequality (1) that this instability of relativistic origin will set in long before the effects of general relativity on the structure of the equilibrium configuration make themselves felt. On this account, the constant \( K \) in the inequality (1) is determined by the structure of the equilibrium configuration in the New-
tonian framework. For polytropic distributions the constant $K$ is given by [Chandrasekhar 1965, equation (77)]

$$K = \frac{5 - n}{18} \left[ \frac{2(11 - n)}{(n + 1)} \xi_1^4 \right] \int_0^{\xi_1} \theta \left( \frac{d\theta}{d\xi} \right)^2 \xi^2 d\xi + 1,$$

where $n$ is the polytropic index, $\theta$ and $\xi$ are the Emden variables in their standard normalizations (cf. Chandrasekhar 1939, chapter 4), $\xi_1$ is the value of $\xi$ where $\theta$ vanishes, and $\theta'_1$ is the derivative of $\theta$ at $\xi = \xi_1$.

Values of $K$ for $n$ in the range $0 < n < 3.5$ were tabulated in the paper referred to (Chandrasekhar 1965, p. 1530); and in this range $K$ varied between 0.4524 for $n = 0$ and 1.500 for $n = 3.5$. However, one of us (SC) recently had the occasion to examine the values of $K$ for $n > 3.5$, and it was found that (Chandrasekhar 1984)

$$K = 22.91 \text{ and } 45.94 \quad \text{for } n = 4.5 \text{ and } 4.95, \text{ respectively.}$$

The constant $K$ thus appeared to diverge for $n \rightarrow 5 - 0$ and the polytropic distribution tends towards infinite central condensation. This fact seemed to be of some importance since the models currently constructed for stars in their late stages of evolution, when successive nuclear ignitions of elements such as carbon, neon, etc, take place, are extremely centrally condensed with central densities as high as $10^9$ times the mean densities. For this reason, it seemed useful to determine the behaviour of $K$ for $n \rightarrow 5 - 0$. It is the object of this note to provide this required behaviour.

2 The limiting form of $K$ for $n \rightarrow 5 - 0$

Rewriting equation (2) in the form

$$K = \frac{5 - n}{\left| \theta'_1 \right|} \left[ \frac{2(11 - n)}{(n + 1)} \xi_1^4 \left( \frac{\xi_1}{\theta'_1} \right)^2 \int_0^{\xi_1} \theta \left( \frac{d\theta}{d\xi} \right)^2 \xi^2 d\xi + 1 \right],$$

we observe that the quantity in square brackets on the right-hand side remains finite for $n = 5$. For, when $n = 5$, we have Schuster’s solution,

$$\theta = \frac{1}{(1 + \xi^2/3)^{1/2}},$$

for the Lane–Emden function. For this solution

$$\theta' \rightarrow 0 \text{ and } \xi^2 \theta' \rightarrow -\sqrt{3} \text{ as } \xi \rightarrow \infty$$

and

$$\int_0^\infty \theta \left( \frac{d\theta}{d\xi} \right)^2 \xi^2 d\xi = \frac{1}{9} \int_0^\infty \frac{\xi^4}{(1 + \xi^2/3)^{7/2}} d\xi = \frac{\sqrt{3}}{5}.$$

Accordingly, we may conclude that the asymptotic behaviour of $K$ is given by

$$K = \left[ \lim_{n \rightarrow 5 - 0} \left( \frac{5 - n}{\left| \theta'_1 \right|} \right) \right] \left( \frac{1 \ 2 \ \sqrt{3}}{18 \ 3 \ 5} \right) = \frac{1}{45\sqrt{3}} \lim_{n \rightarrow 5 - 0} \left( \frac{5 - n}{\left| \theta'_1 \right|} \right).$$

It remains to determine $\theta'_1$ as $n \rightarrow 5 - 0$. 

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3 The asymptotic behaviour of $\xi_1(n)$ and $\theta'_1(n)$ as $n \to 5 - 0$

We start with the integral formula

$$
\int_0^1 \xi \xi_1^2 \theta^{n+1} d\xi = \frac{n+1}{5-n} \frac{1}{\xi_1} |\xi_1^2 \theta'_1|^2,
$$

(9)

as given by Milne* (1929). This formula is, in fact, equivalent to the well-known expression (cf. Chandrasekhar 1939, p. 101),

$$
\Omega = \frac{3}{5-n} \frac{GM}{R},
$$

(10)

for the gravitational potential energy of a polytrope of index $n$. Rewriting the formula (9) in the form

$$
\xi_1(n) = \frac{n+1}{5-n} \left[ \frac{|\xi_1^2 \theta'_1|^2}{\int_0^1 \xi \xi_1^2 \theta^{n+1} d\xi} \right],
$$

(11)

we observe that the quantity in square brackets on the right-hand side of equation (11), again, remains finite for $n = 5$. For, with Schuster's solution (5) for $\theta$,

$$
|\xi_1^2 \theta'_1|^2 \to 3 \quad \text{for} \quad \xi \to \infty,
$$

(12)

and

$$
\int_0^\infty \xi^2 \theta^6 d\xi = \int_0^\infty \frac{\xi^2}{(1+\xi^2/3)^3} d\xi = \frac{\pi}{16} 3^{3/2}.
$$

(13)

We conclude, then, that

$$
\xi_1(n) \to \frac{32\sqrt{3}}{\pi} \frac{1}{5-n} \quad \text{for} \quad n \to 5 - 0.
$$

(14)

From equation (6) it now follows:

$$
\theta'_1 \to -\frac{n^2}{1024\sqrt{3}} (5-n)^2 \quad (n \to 5 - 0).
$$

(15)

A further quantity of interest is the ratio of the central to the mean density given by

$$
\frac{\rho_c}{\bar{\rho}} = -\frac{\xi_1}{3\theta'_1}.
$$

(16)

By equations (14) and (15) the asymptotic behaviour of this ratio is

$$
\frac{\rho_c}{\bar{\rho}} \to \left(\frac{32}{\pi}\right)^3 \frac{1}{(5-n)^3} \quad (n \to 5 - 0).
$$

(17)

* Actually, Milne considered the more general integral,

$$
\int_0^1 \xi \xi_1^s \theta^{n+1} d\xi,
$$

and showed that an explicit evaluation is possible for the cases $s = 1, 2, \text{and} 3$. 

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4 The asymptotic behaviour of $K$ for $n \to 5 - 0$

Returning to equation (8), we now find by making use of the relation (15),

$$K \to \frac{1024}{45\pi^2} \frac{1}{5-n} = \frac{2.30562}{5-n} \quad (n \to 5-0). \quad (18)$$

For $n = 4.9$ and $4.95$, equation (18) gives

$$K = 23.06 \text{ and } 46.11, \text{ respectively,} \quad (19)$$

and these values agree to well within a per cent of the 'exact' values given in equation (3). Even for $n = 3$, the asymptotic relation gives for $K$ a value (= 1.153) which differs from the exact value (1.125) by no more than 3 per cent.

Eliminating $(5-n)$ between the relations (17) and (18), we can write, alternatively,

$$K \to \frac{32}{45\pi} \left(\frac{\rho_c}{\bar{\rho}}\right)^{1/3} = 0.22645 \left(\frac{\rho_c}{\bar{\rho}}\right)^{1/3} \quad (\rho_c/\bar{\rho} \to \infty). \quad (20)$$

We may expect this last relation to give us an indication of how $K$ increases with increasing central condensation of a star.

5 Concluding remarks

We have found that along the polytropic sequence, the criterion for the relativistic instability can, with sufficient accuracy (< 3 per cent), be written in the form,

$$R < 0.226 \left(\frac{\rho_c}{\bar{\rho}}\right)^{1/3} \frac{2GM}{c^2} \frac{1}{\Gamma_1 - 4/3} \quad (\rho_c > 10^6 \bar{\rho}), \quad (21)$$

for $\Gamma_1$ close to $4/3$. On general grounds we may expect this criterion to give us an indication of when the relativistic instability will become relevant for highly centralised condensed stars. Since the stellar models currently considered for the late stages of stellar evolution, when the elements of the higher atomic numbers are being synthesized, are very centrally condensed, it would appear that the role of the relativistic instability may be more significant than presently appreciated.

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References