CONTINUOUS SPECTRA OF OSCILLATION FREQUENCIES OF AN AXISYMMETRIC INCOMPRESSIBLE EQUILIBRIUM WITH A POLOIDAL MAGNETIC FIELD

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1. Introduction

The study of the spectrum of the linearized equations of ideal Magnetohydrodynamics (MHD) is a powerful approach to a theoretical understanding of the behaviour of plasma near a state of static equilibrium. Investigation of the spectrum of the linear diffusion pinch revealed singularities that were identified as a manifestation of two continuous parts of the spectrum, an Alfvén continuum and a slow or cusp continuum (see e.g. Grad, 1969; Appert et al., 1974). The continuous spectrum arises from the existence of characteristic surfaces in the linearized equations and from the non-homogenities in the equilibrium state and is a relatively invariant feature of the spectrum of the linearized equations of ideal MHD.

The continuous spectrum is fundamental for the understanding of the structure of the spectrum of the linearized equations of ideal MHD (see e.g. Goedbloed, 1975), and it has been investigated for a variety of magnetohydrodynamic equilibria over the last decade in the plasma physics literature. In all the cases known to the present authors, gravity has been neglected, which is justified under laboratory conditions but probably not for a wide range of astrophysical conditions. Hence there is a clear need for the derivation of the continuous spectrum of a magnetohydrodynamic equilibrium with gravity included.

The present paper concerns the continuous spectrum of incompressible linear motions in a static self-gravitating equilibrium with a poloidal magnetic field.

2. Equilibrium and linearized equations

We consider a static and axisymmetric equilibrium and view all equilibrium quantities as functions of the cylindrical coordinates \( \omega \) and \( z \) but not of the angle \( \varphi \). The magnetic field \( \mathbf{B} \) is taken to be poloidal and, for an axially symmetric equilibrium, it can be written in terms of a magnetic flux function \( \psi(\omega, z) \) as

\[
\mathbf{B} = -\nabla \psi(\omega, z) \times \hat{\varphi},
\]

with \( \hat{\varphi} \) being the unit vector in the azimuthal direction. We then define a local orthogonal system of flux coordinates \((\psi, \chi, \varphi)\) with \( \chi \) a poloidal angle-like variable. We view all equilibrium quantities as functions of \( \psi \) and \( \chi \) and write the components of the force balance equation and the equation of Poisson as

\[
\frac{\partial p}{\partial \psi} + \rho \frac{\partial \phi}{\partial \psi} + \frac{1}{J} \frac{\partial}{\partial \psi} \left( \frac{J B^2}{4 \pi} \right) = 0,
\]

\[
\frac{\partial p}{\partial \chi} + \rho \frac{\partial \phi}{\partial \chi} = 0,
\]

\[
\frac{1}{J} \frac{\partial}{\partial \psi} \left[ J B^2 \frac{\partial \phi}{\partial \psi} \right] + \frac{1}{J} \frac{\partial}{\partial \chi} \left[ \frac{1}{J B^2} \frac{\partial \phi}{\partial \chi} \right] = 4 \pi G \rho,
\]

where \( \rho, p, \) and \( \phi \) are the density, pressure, and gravitational potential.

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respectively. \( J \) is the Jacobian of the flux coordinate system. The equations that describe the linear incompressible motions about a static equilibrium are

\[ -\rho \partial^2 \xi = \nabla p' + \rho \nabla \phi + \rho \nabla \phi' + \frac{1}{4\pi} \left[ (\nabla \times \vec{B}) \times \vec{B}^\prime + (\nabla \times \vec{B}^\prime) \times \vec{B} \right], \]  

(5)

\[ \nabla \cdot \xi = 0, \]  

(6)

\[ \rho' = -\xi \cdot \nabla \rho, \]  

(7)

\[ \nabla^2 \phi' = 4\pi G \rho', \]  

(8)

\[ \vec{B}' = \nabla \times (\xi \times \vec{B}). \]  

(9)

\( \xi \) is the Lagrangian displacement, and \( f' \) denotes the Eulerian perturbation of \( f \).

We have Fourier-decomposed all perturbed quantities with respect to time \( t \) and the ignorable angle \( \varphi \) by making them proportional to \( \exp[i(\sigma t + m \varphi)] \).

3. **The continuous spectrum**

Modes belonging to the continuous spectrum are recognized by the their non-square integrable singularities at a flux surface \( \psi = \psi_0 \) (Pao, 1975), so we have to write equations (5)-(9), giving special attention to the derivatives across the flux-surface, i.e., \( \psi \)-derivatives. Further, the solutions that correspond to the Alfvén-continuum and the slow continuum in the linear diffuse pinch are characterized by motions in the flux surfaces that are polarized purely perpendicular or parallel to the magnetic field lines. Although, this polarization property does not hold for a general axisymmetric equilibrium, it is still convenient to have vectors in the flux surface decomposed in components parallel and perpendicular to the magnetic field lines. In the case of a poloidal magnetic field \( \xi_\chi, \xi_\varphi, B_\chi \) and \( B_\varphi \) are such components. We follow Goedbloed (1975) and use \( X, \xi_\chi, \) and \( \xi_\varphi \) as unknown dependent variables with \( X = \int_0^\infty \! \! F \xi_\psi \), and introduce the differential operators \( D \) and \( F \) as

\[ D = \frac{1}{J} \frac{\partial}{\partial \psi}, \quad F = -\frac{i}{J} \frac{\partial}{\partial \chi}. \]  

(10)

\( D \) is a differential operator perpendicular to a flux surface, and \( F \) is a differential operator tangent to the flux surface and along the magnetic field line. Goedbloed (1975) also defined a differential operator \( G \) tangent to the flux surface and perpendicular to the magnetic field lines, but this operator is algebraic for a poloidal magnetic field. In addition to \( X, \xi_\chi, \) and \( \xi_\varphi \), we have also \( p' \) and \( \phi' \) as unknown dependent variables. For the sake of compactness, the full set of linearized equations in terms of the variables \( X, \xi_\chi, \xi_\varphi, p', \) and \( \phi' \) and the operators \( D \) and \( F \) are not given here.

The continuous spectrum arises due to perturbations that exhibit large derivatives at a single flux surface and is obtained by taking the limit \( D \to \infty \) and neglecting any perturbed quantity \( Q \) that is differentiated with respect to \( \psi \) compared to \( \partial Q/\partial \psi \). The reduced set of equations for finite values of \( \sigma^2 \) is

\[ DX + \frac{im}{\omega} \xi_\varphi + iF \left( \frac{\xi_\chi}{B} \right) = 0, \]  

(11)

\[ D \left[ \frac{B^2}{4\pi} DX + \frac{B^2}{4\pi \omega} \xi_\varphi - p' - \rho \phi' \right] = 0, \]  

(12)
\[
\frac{m}{\omega} \left[ \frac{B^2}{4\pi} \left( DX - p' - \rho \phi' \right) \right] + \frac{i}{4\pi \omega} F \left[ \omega^2 F \left( \frac{\xi_\varphi}{\omega} \right) \right] + \frac{im^2B^2}{4\pi \omega^2} \xi_\varphi - i\rho \sigma^2 \xi_\varphi = 0 ,
\]
(13)

\[
i\phi (\sigma^2 - N^2) \frac{B\xi}{\chi} + Fp' + \rho F\phi' = 0 ,
\]
(14)

\[
J^2 B \omega^2 D^2 \phi' = - 4\pi G \frac{1}{J} \frac{\partial \rho}{\partial \chi} \xi_\chi ,
\]
(15)

where

\[
N^2_\chi = - \frac{1}{J^2 B} \frac{1}{\rho} \frac{\partial \rho}{\partial \chi} \frac{\partial \phi}{\partial \chi}
\]
(16)

can be considered as the square of the Brunt-Väisälä frequency along the magnetic field lines.

Equations (15) and (11) imply that \(D^2 \phi'\) is of the same order as \(DX\), so \(\phi'\) can be neglected relative to \(DX\). This has the important consequence that the perturbation of the gravitational potential has no effect on the continuous spectrum. From (11) it follows that

\[
DX = - \frac{im}{\omega} \xi_\varphi - i F \left( \frac{\xi_\chi}{B} \right) .
\]
(17)

Integration of (12) yields

\[
p' = - \frac{i}{4\pi} B^2 F \left( \frac{\xi_\chi}{B} \right) .
\]
(18)

Equations (13) and (14) can then be cast in the form of

\[
\sigma^2 \xi_\varphi = \frac{1}{4\pi \rho} F \left[ \omega^2 F \left( \frac{\xi_\varphi}{\omega} \right) \right] ,
\]
(19)

\[
\sigma^2 \xi_\chi = N^2_\chi \xi_\chi + \frac{1}{4\pi \rho B} \left[ B^2 F \left( \frac{\xi_\chi}{B} \right) \right] .
\]
(20)

Equations (19) and (20) determine the continuous spectrum. They are two uncoupled second-order differential equations not involving derivatives across the magnetic surfaces and can be written in compact form as follows:

\[
\sigma^2 \hat{\vec{\psi}} = L(\psi) \hat{\vec{\psi}} ,
\]
(21)

with

\[
\hat{\vec{\psi}} = \left[ \xi_\varphi , \xi_\chi \right]^t .
\]
(22)

The operator \(L(\psi)\) has the essential property of being a differential operator in \(\chi\) but algebraic in \(\psi\). Because of this property, we can separate the improper normal dependence from the proper tangential dependence in the solutions \(\hat{\vec{\psi}}\). The solution \(\hat{\vec{\psi}}_0(\chi)\) to the non-singular eigenvalue problem

\[
\sigma^2 \hat{\vec{\psi}}_0(\chi) = L(\psi_0) \hat{\vec{\psi}}_0(\chi)
\]
(23)
obtained under proper boundary conditions corresponds to an improper solution of the linearized equations of MHD

$$\tilde{V}(\psi, \chi) = \delta(\psi - \psi_0) \tilde{V}_0(\chi).$$

(24)

The continuous spectrum can then be found in the following way (Goedbloed, 1975). The non-singular eigenvalue problem (23) is solved at each magnetic surface. For a fixed value of \( m \), this leads to two discrete sets of eigenvalues \( \alpha_A^2(\psi_0) \) \(_k \) and \( \sigma_C^2(\psi_0) \) \(_k \) where the label \( k \) refers to the different oscillatory solutions that will be found. The values of \( \alpha_A^2(\psi_0) \) \(_k \) and \( \sigma_C^2(\psi_0) \) \(_k \) are discrete on any particular flux surface \( \psi = \psi_0 \), but they sweep out two continuous spectra as the flux surface \( \psi = \psi_0 \) varies.

The equations (19) and (20) are uncoupled, and we have solutions to these equations with \( \xi_\phi \neq 0, \xi_\chi = 0, \) and \( \xi_\phi = 0, \xi_\chi \neq 0 \) respectively. The solutions have motions in the flux surfaces that are polarized purely perpendicular or purely parallel to the magnetic field lines. The polarization properties of the solutions found in the classical linear pinch are recovered here, and the Alfvén continuum and the cusp continuum are decoupled. The Alfvén continuum is determined from Equation (19), the cusp continuum from Equation (20).

It is instructive here to consider an equilibrium where gravity is absent and all equilibrium quantities do not depend on \( \chi \). \( F \) is then an algebraic operator and we find that

$$\sigma^2 \xi_\phi = \frac{F^2}{4 \pi \rho} \xi_\phi = \frac{k B^2}{4 \pi \rho} \xi_\phi,$$

(25)

$$\sigma^2 \xi_\chi = \frac{F^2}{4 \pi \rho} \xi_\chi = \frac{k B^2}{4 \pi \rho} \xi_\chi,$$

(26)

with

$$k = \frac{F}{B} = -\frac{i}{J B} \frac{3}{\partial \chi}.$$

(27)

The two continuous spectra in the present case are affected by curvature and toroidicity, but only the cusp continuum is affected by gravity.

The operator \( L(\psi_0) \) is Hermitian in the Hilbert-space of the one-dimensional functions \( \tilde{V}(\chi) \) with respect to the inner product

$$\langle \tilde{V}_1, \tilde{V}_2 \rangle = \pi \int \rho \tilde{V}_1 \cdot \tilde{V}_2 J d\chi.$$

(28)

From this Hermitian property of \( L \) we can obtain a variational principle for the determination of the continuum frequencies \( \sigma^2(\psi) \)

$$\sigma^2(\psi) = \frac{\int \rho \left| \frac{\xi_\phi}{\psi_0} \right|^2 + \frac{\rho N^2}{4 \pi} \left| \xi_\chi \right|^2 + \frac{B^2}{4 \pi} \left| F \left( \frac{\xi_\chi}{B} \right) \right|^2 J d\chi}{\int \rho \left( \left| \xi_\phi \right|^2 + \left| \xi_\chi \right|^2 \right) J d\chi}.$$

(29)

The Alfvén continuum and the cusp continuum are uncoupled in the case considered here, and there is a variational expression for each of them. For the frequencies in the Alfvén continuum we have, with \( \xi_\phi \neq 0, \xi_\chi = 0, \)

$$\sigma_A^2(\psi) = \frac{\int \omega^2 \left| F \left( \frac{\xi_\phi}{\psi_0} \right) \right|^2 J d\chi}{\int \rho \left| \xi_\phi \right|^2 J d\chi}.$$

(30)
and for the cusp continuum, we find, with $\xi_\phi=0, \xi_\chi \neq 0$,

$$\sigma_C^2(\psi) = \frac{\left(\rho N_X^2 |\xi_\chi|^2 + \frac{B^2}{4\pi} |F(\frac{\xi_\chi}{B})|^2\right)}{\int \rho |\xi_\chi|^2 Jd\chi} \cdot$$ (31)

We can conclude that the Alfvén continuum is on the stable side, but the cusp continuum can attain negative values when $N_X^2 < 0$. Negative values of $\sigma_C^2(\psi)$ are found for trial functions $\xi_\chi$ that satisfy $F(\xi_\chi/B)=0$ when $N_X^2 < 0$. The instability of the cusp continuum is due to an unstable stratification of the density, pressure, and gravity along a magnetic field line.

4. Conclusion

We have discussed the continuous spectrum of incompressible linear motions in an axisymmetric self-gravitating equilibrium with a poloidal magnetic field. We have shown that there exist two uncoupled continuous spectra that are each determined by an ordinary second-order differential equation. The solutions that correspond to the continuum frequencies have motions in flux surfaces that are polarized purely perpendicular and purely parallel to the magnetic field lines, and the two continuous spectra can be considered as an Alfvén continuum and a cusp continuum. The Alfvén and cusp continuum are both affected by curvature and toroidicity but only the cusp continuum is influenced by the equilibrium gravity. Variational expressions have been derived for each of the continuum frequencies, and it is shown that the Alfvén continuum is on the stable side of the spectrum but the cusp continuum may attain negative values when there is an unstable stratification of pressure, density, and gravity along the magnetic field lines.

References

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