THE SCATTERING OF ENERGETIC PARTICLES BY WAVES IN A FINITE $\beta$ PLASMA

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ABSTRACT

Solutions to the dispersion relation for waves propagating parallel to the ambient magnetic field in a plasma of arbitrary $\beta$ are obtained, where $\beta$ is the ratio of the thermal pressure to the magnetic pressure. The results of previous authors which have dealt exclusively with wave frequencies, $\omega$, much less than the ion cyclotron frequency, $\Omega_i$, are extended to include all frequencies less than the electron cyclotron frequency, $\Omega_e$.

It is found that the cyclotron turnovers, which occur at $\Omega_i(\Omega_e)$ for the right-hand (left-hand) circularly polarized waves in a cold plasma, occur at significantly lower frequencies in a finite $\beta$ plasma. The lightly damped modes of a $\beta > 40$ plasma are confined to frequencies $\omega < \Omega_i$. For plasmas of intermediate $\beta$ ($0.1 \leq \beta \leq 10$) the dispersion relation for propagating waves is appreciably modified in the frequency range $\Omega_i \leq \omega \leq \Omega_e$. In addition cyclotron damping is significant for small wavelength waves.

In the low-frequency range $\omega < \Omega_i$, the transition between low and high $\beta$ wave propagation essentially occurs in the range $1 < \beta < 40$. Waves in plasmas with $\beta$ below this range are very well approximated by cold plasma modes, and waves in plasmas with $\beta > 40$ are well approximated by the high $\beta$ modes.

These results show that when one considers scattering of energetic particles by electromagnetic turbulence in a $\beta > 40$ plasma, scattering by waves with $\omega > \Omega_i$ can be neglected. This condition on $\beta$ is satisfied in many astrophysical plasmas of interest.

Subject headings: particle acceleration — plasmas

I. INTRODUCTION

Particle scattering in electromagnetic turbulence is a process which appears to operate in many astrophysical environments. For example, cosmic ray and solar flare particles propagating through the interplanetary medium are known to exhibit the diffusive behavior characteristic of strong scattering, and the observed isotropy of cosmic rays seems to imply the existence of an efficient scattering mechanism in the interstellar medium. Recent work by Holman, Ionson, and Scott (1979) has emphasized the importance of collisionless damping in the scattering process. Basically, they showed that when $\beta \gg 1$ (where $\beta$ is the ratio of thermal to magnetic pressure) the isotropization time for the particle distribution can be much longer than standard theories which neglect damping would predict.

The purpose of this paper is to examine the propagation and damping of wave modes in a finite $\beta$ plasma with particular emphasis on those modes which could result in significant particle scattering.

This problem has been discussed before. Foote and Kulsrud (1979) considered the wave modes of a $\beta > 1$ magnetized plasma. They obtained numerical solutions to the dispersion relation for waves having $\omega \ll \Omega_i$, where $\omega$ is the wave frequency and $\Omega_i$ is the nonrelativistic ion gyrofrequency. They demonstrated that waves which do not propagate nearly parallel to the ambient magnetic field are strongly damped.

Formally, the result of Foote and Kulsrud demonstrated only that wave damping in the frequency range $\omega \ll \Omega_i$ is strong. However, it is easy to see that the damping of off-axis waves in the frequency range $\omega > \Omega_i$ should be strong also if one considers the physical mechanism responsible for the wave damping. Off-axis waves cause local first-order increases in the magnetic field strength, $|B|$, which propagate at essentially the phase speed of the wave. Thermal particles in the background plasma collide with these magnetic mirrors. Since the thermal particles have an isotropic velocity distribution, there are slightly more head-on collisions than there are overtaking collisions, and there is a net transfer of energy from the waves to the particles. This results in severe damping of off-axis waves. From this physical argument and the results of Foote and Kulsrud one can conclude that scattering by off-axis waves in a high $\beta$ plasma can be ignored.

Achterberg (1981) obtained analytic solutions to the dispersion relation for $\omega \ll \Omega_i$. He obtained expressions for the wave frequency $\omega$ and the damping rate $\Gamma$ in the limits of $\beta \gg 1$ and $\beta \ll 1$. He assumed parallel propagation since off-axis waves are strongly damped. Achterberg applied his results to several problems of astrophysical interest where one expects the approximation $\beta \gg 1$ to be valid. In general, his more detailed calculations verified the basic picture proposed by Holman, Ionson, and Scott (1979).

All three of these analyses, however, considered only scattering by waves having $\omega \ll \Omega_i$. It is reasonable to ask whether waves having $\omega \gg \Omega_i$ can induce sufficient particle scattering to invalidate the results of Holman et al. and Achterberg. In addition, since wave propagation was considered only in the limits of large and small $\beta$, the details of wave propagation in
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plasmas with intermediate values of $\beta$ have not been examined. Therefore the precise limits of the "high $\beta$" approximation have not been established. In this paper both of these questions are addressed.

II. BASIC EQUATIONS

The dielectric for plane waves propagating in a homogeneous, magnetized plasma has been obtained from linearized Vlasov theory by many previous authors (e.g., Stix 1962). It has been shown that two linearly independent, circularly polarized transverse wave modes exist with dielectric functions given by

$$\epsilon(k, \omega) = 1 - \frac{\omega_p^2}{\omega_{ki}^2} f_0(Z^2) - \frac{\omega_i^2}{\omega_{ki}^2} f_0(Z^2) ,$$

(1)

where the upper (lower) sign refers to right-(left-)hand circularly polarized waves. In this relation $v_{ti} = (K T / m_j)^{1/2}$ is the thermal speed of particles of type $j$ with mass $m_j$. The plasma frequency for particles of type $j$ is represented by $\omega_j = (4\pi n_j e_j^2/m_j)^{1/2}$. The argument $Z$ is given by $Z = (\omega / \omega_{ij})$ where $m = +1 \ (-1)$ for RH (LH) polarized waves. The wave frequency $\omega$ and the wave number $k$ are related through the dispersion relation

$$k^2 c^2 - \epsilon(k, \omega) = 0 .$$

(2)

Analytic expressions are not available which describe the function $f_0$ for all values of $Z$; however, series expansions can be obtained which are applicable in the following ranges of $Z$.

For $Z_m^+ \gg \sqrt{2}$,

$$f_0(Z_m^+) = \frac{1}{Z_m^+} \left( 1 + \frac{1}{Z_m^+} \right) - i \frac{\pi k}{\sqrt{2} |k|} \exp \left[ \frac{1}{2} Z_m^+ \right]$$

(3)

and for $Z_m^- \ll \sqrt{2}$,

$$f_0(Z_m^-) = Z_m^- - i \frac{\pi k}{\sqrt{2} |k|} .$$

(4)

To use these expressions, we must divide the $\omega, k$ plane into four distinct regions as shown in Figures 1 and 2 for RH and LH wave polarizations.

Region I is defined by the inequalities $Z_+^+ \gg \sqrt{2}$ and $Z_+^- \gg \sqrt{2}$. In this region waves are least affected by thermal properties of the plasma and damping is usually small.

Region I has been subdivided into low-frequency (IA) and high-frequency (IB) regions. In region IA, damping is predominantly due to thermal protons, and in region IB damping is due primarily to thermal electrons. This division is chosen to make the dispersion relations derived below take on simpler forms.

Region II is defined by the inequalities $Z_+^- \ll \sqrt{2}$ and $Z_+^+ \gg \sqrt{2}$. In this region the wavelength of the plasma oscillation is small compared to the gyroradius of a typical thermal particle and significant wave damping results (Davila 1982).

Region III is defined by the opposite inequalities $Z_+^+ \ll \sqrt{2}$ and $Z_+^- \gg \sqrt{2}$. In this region significant damping is caused by thermal electrons in the plasma. In addition the inequalities defining the boundaries of this region can only be satisfied for waves having $\omega > \Omega_e$ which means that this region need be considered only when the higher frequency waves are being investigated.

Finally, in region IV $(Z_+^+ \ll \sqrt{2}, Z_+^- \ll \sqrt{2})$ the wave motion is almost completely dominated by thermal motions in the plasma. All waves are heavily damped.

Only waves in region I are considered in this paper. Davila (1982) has discussed waves in the other regions of the $\omega, k$ plane and demonstrated that they are all strongly damped, and hence one would not expect them to cause significant particle scattering.

III. RESULTS

In region I waves are lightly damped and an approximate solution to the dispersion relation can be obtained by first dividing the dielectric into real and imaginary parts,

$$\epsilon(k, \omega) = \epsilon_1(k, \omega) + i \epsilon_2(k, \omega) .$$

(5)

Fig. 1.—Regions of approximation for the RH wave dispersion relation. The numerical values presented assume an electron-proton plasma.

Fig. 2.—Regions of approximation for the LH wave dispersion relation. The numerical values assume an electron-proton plasma.

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The solution of the real part gives the wave frequency using
\[ \frac{k^2 c^2}{\omega^2} - \epsilon_1(k, \omega) = 0, \tag{6} \]
and the damping rate is obtained by Taylor expansion in the usual way as
\[ \Gamma(k) = -\frac{\epsilon_2(k, \omega)}{(1/\omega^2) (\partial^2/\partial \omega^2) \epsilon_1(k, \omega)}, \tag{7} \]
We will now consider the dispersion relation for waves in region I in two frequency intervals.

**a) Low-Frequency Waves**

For frequencies \( \omega \ll \omega_i \ll \Omega_i \), where \( \omega_0 = \Omega_i (m_i T_i/m_e T_e)^{1/2} > \Omega_i \), the dominant wave damping mechanism is ion cyclotron damping. In this frequency range the dielectric (valid for the Alfvén speed \( v_A < c \)) can be written as
\[ \epsilon_1(k, \omega) = \frac{e^2}{\epsilon_0} \frac{1}{(1 \pm \tilde{\omega}_i)^3} \left( \frac{k_i^2}{\tilde{\omega}_i (1 \pm \tilde{\omega}_i)^3} \right) \tag{8} \]
and
\[ \epsilon_2(k, \omega) = \frac{e^2}{\epsilon_0} \frac{1}{\sqrt{2}} \frac{1}{\tilde{\omega}_i k_i} \exp - \frac{1}{2} \left( \frac{1 \pm \tilde{\omega}_i}{k_i} \right)^2. \tag{9} \]

The dimensionless variables used in the dielectric are defined by \( \tilde{\omega}_i = \omega_i / \Omega_i \) and \( k_i = kv_i / \Omega_i \).

Using these relations the dispersion relation can be solved for \( k_i(\tilde{\omega}_i) \) and \( \Gamma(\omega) \):
\[ k_i^2 = \frac{\beta}{2} \left( \frac{\tilde{\omega}_i (1 \pm \tilde{\omega}_i)^2}{2 (1 \pm \tilde{\omega}_i)^3} + \frac{1/2 \tilde{\omega}_i}{k_i} \right) \tag{10} \]
and
\[ \Gamma(\omega) = -\left( \frac{\pi}{8} \right)^{1/2} \exp - \frac{1}{2} \left[ (1 \pm \tilde{\omega}_i) \right]^2 \left[ (1 \pm \tilde{\omega}_i) \right]^2 \left[ (1 \pm \tilde{\omega}_i) \right]^3 + [3k_i^2 / (2 (1 \pm \tilde{\omega}_i)^2)] \right) \tag{11} \]

![Fig. 3.—Dispersion relation for RH wave for several values of \( \beta \)](image1)

![Fig. 4.—Damping rate for RH wave for several values of \( \beta \). The damping rates given in this figure correspond to the wave frequencies given in Fig. 3. Damping in a \( \beta = 1 \) plasma is too small to appear at this scale.](image2)

In the expressions above only terms of order \( \omega / \Omega_i \) have been neglected, so they correctly describe the propagation of waves with frequencies near the ion gyrofrequency. The results of previous authors can be obtained by taking the limit \( \tilde{\omega}_i \ll 1 \).

Graphs of the wave frequency and the damping rate are shown in Figures 3-6.

This dispersion relation exhibits several interesting differences from the cold plasma result. These can be easily seen by considering the denominator of equation (10) and are summarized as follows

1. For \( \beta \tilde{\omega}_i < (1 \pm \tilde{\omega}_i)^3 \) one obtains the usual cold plasma result \( \omega = kv_A \) for both RH and LH polarized waves when \( \tilde{\omega}_i < 1 \). For \( \tilde{\omega}_i > 1 \) the LH wave does not propagate and the RH wave becomes the whistler mode with \( \omega \propto k^2 \).

2. For \( \beta \tilde{\omega}_i \sim (1 \pm \tilde{\omega}_i)^3 \) the LH wave dispersion relation turns over so that the frequency is no longer a function of \( k \).
This is analogous to the ion-cyclotron turnover at \( \omega = 1 \) in a cold plasma, but it occurs at a significantly lower frequency in a finite \( \beta \) plasma; i.e., finite \( \beta \) tends to depress the cyclotron turnover frequency.

3. For \( \beta > 2(1 + \beta_i)^2 \) the RH wave has a whistler-like dispersion relation, \( \omega \propto k^3 \). If \( \beta \) is large enough the transition to whistler-like waves can occur far below the ion cyclotron frequency.

Finally for \( \beta > 40 \) there are no waves in region IB and no lightly damped wave modes with \( \omega > \Omega_i \). This result, obtained by numerical evaluation of the dispersion relation (10) and (11) can be understood analytically in the following way. For \( \omega \sim \omega_i > \Omega_i \), the dispersion relation for RH waves can be approximated as

\[
\hat{k}_e^2 = \frac{1}{2} \beta \omega_i \Omega_i. \tag{12}
\]

The assumption that \( \omega \) and \( k \) lie in region IB requires \( \hat{k}_e \leq \omega_i / \Omega_i \). \tag{13}

When this condition is combined with equation (12) above, it is easy to show that this inequality is violated whenever

\[
\beta \geq \left( \frac{m_i T_i}{m_e T_e} \right)^{1/2} \approx 40. \tag{14}
\]

Therefore whenever one considers particle scattering in a \( \beta \geq 40 \) plasma, only scattering by waves having \( \omega < \Omega_i \) will be significant. All other waves with \( \omega < \Omega_i \) are strongly damped by the thermal particles of the background plasma (see also Davila 1982).

b) High-Frequency Waves

In this interval the dielectric can be approximated as

\[
\epsilon_\infty(k, \omega) = \frac{Z e^2 \omega_i^2}{\omega_\infty v_A^2} \left[ \frac{1}{1 - \omega_e} + \frac{\hat{k}_e^2}{(1 - \omega_e)^2} \right]. \tag{15}
\]

\[
\epsilon_\infty(k, \omega) = \frac{Z e^2 \omega_i^2}{\omega_\infty v_A^2} \left( \frac{1}{\sqrt{2 \omega_e |k_e|}} \exp\left\{ -\frac{1}{2} \left( \frac{1 - \omega_e}{k_e^2} \right)^2 \right\} \right). \tag{16}
\]

To obtain this result it has been assumed that \( \omega \gg \omega_e \) and \( v_A^2/\omega_i^2 \ll 1 \), where \( \alpha = m_e/m_i \). The dimensionless variables are \( \omega_e = \omega / \Omega_e \) and \( k_e = k v_i / \Omega_e \).

Using these expressions for the dielectric, the dispersion relation can be solved to obtain

\[
\hat{k}_e^2 = \frac{1}{2} \beta \frac{T_i}{T_e} \left[ \left( 1 - \omega_e \right)^2 - (1/2) \beta (T_e/T_i) \omega_e \right]. \tag{17}
\]

\[
\Gamma/\omega = -\frac{(\pi/2)^{1/2}}{k_e} \exp \left\{ -\frac{1}{2} \left[ (1 - \omega_e) \hat{k}_e^2 / \omega_e \right] \right\} \times \left( \frac{\hat{k}_e^2 / \omega_e^2}{1 - \omega_i} \right)^{(1/2)} + \left[ 3 \hat{k}_e^2 / (1 - \omega_i) \right]. \tag{18}
\]

Graphs of these functions are presented in Figures 7 and 8.

Again the characteristics of the dispersion relation can be most easily seen by considering the terms in the denominator of equation 14.

The effects of finite plasma temperature, i.e., finite \( \beta \), are negligible whenever the inequalities \( \omega \gg \omega_e \) and \( \beta \omega_e \ll 2(T_e/T_i)(1 - \omega_i)^3 \) can be simultaneously satisfied. For these frequencies the dispersion relation can be approximated by the usual cold plasma result.

![Fig. 7.—Dispersion relation for the RH wave in the high-frequency portion of region I.](image-url)

![Fig. 8.—Damping rate for the RH wave in the high-frequency portion of region I.](image-url)
As larger and larger frequencies are considered the denominator becomes smaller until at a frequency defined by \( \beta \omega_c \sim 2(T_i/T_e)(1 - \omega_c) \) the wave frequency becomes independent of \( k \). This is analogous to the electron cyclotron turnover in a cold plasma. For sufficiently high \( \beta \) this turnover occurs for \( \omega_c \ll 1 \), and then the turnover frequency is given by

\[
\omega_t = \left( \frac{2}{\beta} \frac{T_i}{T_e} \right) \Omega_e. \tag{19}
\]

For \( \beta \gg 1 \) this can occur at a frequency considerably below the electron cyclotron frequency. There are no solutions to the dispersion relation for \( \omega \) greater than the turnover frequency.

Finally, whenever the turnover frequency, \( \omega_t, \omega_d \) there are no solutions to the dispersion relation in the frequency range \( \omega \gg \omega_d \). By equating \( \omega_d \) and \( \omega_t \), it can be shown that this occurs whenever

\[
\beta \gtrsim 2(m_e T_i/m_i T_e)^{1/2} \approx 40. \tag{20}
\]

This is consistent with the results obtained in the previous section for low frequency waves in this region of the \( \omega, k \) plane. There it was found that for \( \beta > 40 \), the frequency of the RH wave remained below \( (m_e T_i/m_i T_e)^{1/2} \Omega_e \) until \( k \) became large enough so that the assumption \( \omega + Q > \sqrt{2k v_i} \) was violated.

From this one must conclude that in a high \( \beta \) plasma there are no solutions to the dispersion relation with \( \Omega_e < \omega < \Omega_i \). This conclusion has been confirmed by considering waves with larger values of \( k \) (i.e., \( \omega \pm Q > \sqrt{2k v_i} \)) by Davila (1982).

IV. CONCLUSIONS

In this paper solutions to the dispersion relation for waves propagating parallel to the static magnetic field in a plasma of arbitrary \( \beta \) were obtained. It was found that the cyclotron turnovers, which occur at \( \Omega_e(\Omega_i) \) for the RH (LH) circularly polarized waves in a cold plasma, occur at significantly lower frequencies in a finite \( \beta \) plasma. The lightly damped modes of a high \( \beta \) (\( \beta > 40 \)) plasma are confined to frequencies \( \omega < \Omega_i \). From this it is concluded that for \( \beta > 40 \) only scattering by low frequency waves (\( \omega < \Omega_i \)) is significant, and therefore the neglect of scattering by high-frequency waves in previous work was justified.

For plasmas of intermediate \( \beta \), the situation is not as clear. Examination of Figure 7 reveals that even for \( 0.1 \leq \beta \leq 10 \) the dispersion relation for the propagating waves is appreciably modified in the high-frequency portion of region I. In addition cyclotron damping is significant for large values of \( k \). Values of \( \beta \) in this range are common in cooler astrophysical plasmas like the interplanetary medium. Evaluation of the effects of collisionless damping on particle propagation in the interplanetary medium will be the subject of a future paper.

In the low-frequency range \( \omega \leq \Omega_i \), the transition between low and high \( \beta \) wave propagation essentially occurs in the range \( 1 \leq \beta \leq 40 \). Waves in plasmas with \( \beta \) below this range are very well approximated by cold plasma modes, and waves in plasmas with \( \beta \geq 40 \) are well approximated by the high \( \beta \) wave modes. In either case the appropriate relation for the damping rate, \( \Gamma \), should also be used.

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