ON THE TOPOLOGICAL STABILITY OF MAGNETOSTATIC EQUILIBRIA

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ABSTRACT

An important question in the study of magnetic field-dominated plasmas in both the astrophysical and laboratory settings is whether magnetostatic equilibrium (which is generally mathematically tractable) can be regarded as an appropriate description, at least to lowest order; Parker, for example, has argued that magnetostatic equilibria are exceptional states, and that one ought to regard departures from static equilibrium as the common state of affairs. We address this problem by exploring the formal analogy, in the ideal MHD limit, between the topology of magnetic lines of force in coordinate space and the topology of integral surfaces of one and two-dimensional Hamiltonian systems in phase space. The correspondence between ideal MHD and dynamical theory allows us to prove that if a symmetric magnetostatic equilibrium is subjected to a perturbation of arbitrary symmetry, then for a finite range of pressure values, two-dimensional constant-pressure surfaces no longer necessarily coincide with magnetic flux surfaces, so that magnetostatic equilibrium is no longer satisfied everywhere. We conclude that spatially symmetric ideal magnetostatic equilibria are topologically unstable to finite-amplitude perturbations which do not share their symmetry properties. This result extends Parker's principal conjecture, and extends his necessary conditions for magnetostatic equilibrium to include finite-amplitude perturbations of arbitrary (or no) symmetry properties. It therefore appears that symmetric magnetostatic equilibria are indeed exceptional states; furthermore, the possible resulting ergodicity of perturbed symmetric equilibria has important consequences for heat dispersal in magnetically dominated plasmas, which are discussed.

Subject headings: hydromagnetics — Sun: magnetic fields

I. INTRODUCTION

A striking feature of magnetic fields in solar and other astrophysical systems is their often intimate phenomenological connection with “activity,” that is, with (often violent) departure from quiescent conditions. It has been suggested that this phenomenological relation reflects a fundamental aspect of magnetized fluids, namely that steady equilibria are either unstable or physically inaccessible (Parker 1979). In this connection, Grad (1967) conjectured, and Parker (1972) and Yu (1973) demonstrated, that ideal magnetostatic equilibria exhibiting certain spatial symmetries (such as translational or rotational invariance) are topologically unstable to infinitesimal perturbations (analytic in the expansion parameter) which do not share their symmetry properties; that is, the perturbed nonsymmetric system no longer satisfies the equilibrium equations. This result appears to have even more general validity, as it has recently been extended to the magnetohydrodynamic domain by Tsinganos (1982h). If all steady equilibrium solutions to the ideal MHD equations exhibit some spatial symmetry, and if physically plausible perturbations can all be subsumed in the class of analytic infinitesimal perturbations, then these results force one to conclude that the existence of steady MHD equilibria in nature is exceptional. However, a demonstration that steady MHD equilibria must necessarily exhibit some spatial symmetry is at present lacking; the above sought-for conclusion thus cannot be as yet drawn. Furthermore, Rosner and Knobloch (1982) have shown that perturbation calculations have limited applicability if the radius of convergence of the perturbation expansion is finite; and that, in addition, symmetry-breaking transitions to new steady configurations can occur which are not accessible to perturbation theory.

In this paper, we investigate the topological stability (cf. Parker 1972, 1979) of MHD equilibria by exploiting the formal similarity between the above problem and the problem of magnetic flux surface destruction extensively studied in laboratory plasmas (cf. Hénon and Heiles 1964; Morozov and Solov’ev 1966; Rosenbluth et al. 1966; Filonenko, Sagdeev, and Zaslavsky 1967; Hamada 1972) and, in particular, by further developing the topological analogy between magnetic flux surfaces in physical space and integral surfaces of Hamiltonian systems in some appropriate phase space. Our aim is to demonstrate that in an astrophysical setting, symmetric magnetostatic equilibria satisfying the ideal MHD equations must indeed be exceptional, a result derived without resort to standard MHD perturbation theory. Our principal result is that previous infinitesimal perturbation theory calculations can indeed be generalized to include finite-amplitude and symmetry-breaking effects.

Our paper is structured as follows. In the next section (§ II) we develop the mathematical tools for our analysis, explicitly demonstrate the formal analogy between classical MHD and Hamiltonian dynamics (with specific examples of such analogies

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3 Classical stability theory addresses the question whether a given equilibrium configuration evolves away from (unstable) or back toward (stable) the equilibrium when perturbed. In the present context, “topological” stability refers to the question whether a given equilibrium configuration preserves its topological properties when subjected to perturbation; thus, the topologically unstable configurations may well be stable from the classical point of view.
presented in Appendix A). In § III we apply results from classical dynamics to the study of MHD equilibria. Our results are summarized with regard to their relation to astrophysical systems in § IV. A brief overview of the problem in classical dynamical theory which led to the recognition of the limitations of perturbation theory, and which motivated our present discussion, is presented in Appendix B in the context of our problem.

II. MATHEMATICAL DEVELOPMENT

We consider a magnetized fluid satisfying the ideal magnetohydrodynamic equations

\begin{align}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) &= -\nabla P + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}), \\
\nabla \cdot \mathbf{B} &= 0,
\end{align}

as well as an equation of state, where all variables have their usual meaning. For the moment, the formal problem we would like to address is whether the symmetric and static equilibrium solutions of equations (2.1) are topologically stable; more precisely, we would like to study the constraints on perturbations of symmetric magnetostatic equilibria such that the perturbed states still satisfy the magnetostatic \((\partial/\partial t = 0; \mathbf{v} = 0)\) ideal MHD equations. Therefore, the question we address can be rephrased as: Does there exist some nonsingular scalar function whose gradient balances the Lorentz force in the new perturbed state, and hence whose level surfaces coincide with magnetic surfaces? Before proceeding, note that an important restrictive assumption has been made in the above equations—namely, we do not consider the effect of gravity or other constraining forces on the system. This would seem to greatly restrict the construction of magnetostatic equilibria (Field 1982). However, all that will be important to us is that the Lorentz force is the only nonconservative force in the problem. In that case, the effect of the other forces, under fairly mild assumptions, is expressible as the gradient of a scalar. The important feature of the magnetostatic solutions to the above equations, namely that magnetic surfaces coincide with surfaces of constant pressure, ought thus in general be replaced by the statement that magnetic surfaces coincide with the level surfaces of a new scalar function which is the sum of the pressure and the scalar potential that gives rise to the constraining forces in the problem. A further discussion of other additional assumptions needed such that solutions of the above set of equations might be applicable to specific astrophysical environments such as the solar atmosphere shall be given in § IV. The constraints on perturbations of dynamical equilibria \((\partial/\partial t = 0; \mathbf{v} \neq 0)\), such that the perturbed equilibrium states satisfy the full MHD equations (2.1), will be studied in a subsequent article.

a) Two-dimensional MHD Equilibria and Hamiltonian Systems with One Degree of Freedom

Consider the equations governing the static equilibrium of magnetic fields in infinitely conducting fluids

\begin{align}
\nabla \cdot \mathbf{B} &= 0, \\
\nabla P &= \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}.
\end{align}

The only available exact, nonsingular solutions of this system of equations are symmetric, that is, correspond to equilibrium states for which a generalized orthogonal system \((x_1, x_2, x_3)\), with line elements \(h_1(x_1, x_2), h_2(x_1, x_2), h_3(x_1, x_2)\) can be found, such that there is at least one ignorable coordinate, say, \(x_3\) (Morozov and Solov'ev 1966; Edenstrasser 1980; Tsinganos 1982a, b). The magnetic field can then be expressed in terms of a flux function \(A(x_1, x_2)\) such that the continuity equation (2.2a) is satisfied identically

\begin{align}
B = \left( \frac{1}{h_2 h_3} \frac{\partial A}{\partial x_2}, \frac{1}{h_1 h_3} \frac{\partial A}{\partial x_1}, B_3 \right).
\end{align}

The momentum equation (2.2b) then yields a single, elliptic partial differential equation for \(A(x_1, x_2)\)

\begin{align}
\frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial x_1} \frac{\partial A}{\partial x_2} h_1 h_3 + \frac{\partial}{\partial x_2} \frac{\partial A}{\partial x_1} h_2 h_3 + \frac{\partial}{\partial x_2} \frac{\partial A}{\partial x_1} h_2 h_3 \right) + \frac{B_3}{h_3} \frac{dP}{dA} (h_3 B_3) + 4\pi \frac{dP}{dA} = 0.
\end{align}

Note that the plasma pressure \(P\) and \(h_1 B_3 = f\) are restricted by equation (2.2b) to be definite functions of \(A\). All translationally and rotationally symmetric magnetostatic equilibria governed by equations (2.2) correspond to the various solutions of equations (2.3) for all possible functional forms of \(f(A)\) and \(P(A)\), which in turn are related to the boundary conditions of the particular physical problem under consideration. Notice also that helically symmetric equilibria are governed by a similar set of equations (Morozov and Solov'ev 1966; Edenstrasser 1980; Tsinganos 1982a).

The solutions of equations (2.3) are completely characterized by the flux surfaces \(A(x_1, x_2) = \text{constant}\), which may be

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defined in this context as two-dimensional (2D) surfaces in (three-dimensional) configuration space on which the plasma pressure $P$ and $h_2 B_3$ are constants. Consider the case that the magnetic field $B$ is nonsingular in some volume of space $V$; in particular, consider the case that $B$ is nonvanishing in $V$. With these restrictions, and the assumption that the flux surfaces in $V$ have no edges, Kruskal and Kulsrud (1958) have pointed out that the only physically realizable topology for the flux surfaces is that of nested tori, on which the electric current and the plasma pressure are constants.

Our initial task is to construct a one-dimensional Hamiltonian $H^{(1)}(q, p)$ with the property (Kerst 1962) that the corresponding canonical equations of motion

$$\frac{dq}{dt} = \frac{\partial H^{(1)}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H^{(1)}}{\partial q},$$

(2.4)

are analogs of equations (2.2a). Consider the flux function $A(x_1, x_2)$, which is a solution of the partial differential equation (2.3b), and whose curl gives the two components $B_1$ and $B_2$ of the magnetic field $B(x_1, x_2)$, equation (2.3a). We define the new canonical coordinates

$$q = x_1, \quad p = x_2,$$

(2.5a)

and the “time parameter” $t$,

$$t = \frac{1}{f(A)} \int_{A=\text{constant}} \frac{h_3}{h_1 h_2} dx_3.$$

(2.5b)

Note that the integration in equation (2.5b) is carried out along a field line so that $x_1$ and $x_2$ are functions of $x_3$ alone on a given energy surface. Equation (2.5b) establishes the quantitative relationship between the ignorable coordinate $x_3$ and the Hamiltonian $H^{(1)}(q, p)$ which is then given simply by

$$H^{(1)}(q, p) = A(x_1, x_2).$$

(2.6)

It is now straightforward to write out the Lagrange equation of motion for this Hamiltonian; this is simply the momentum equation (2.3b):

$$\frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q} \frac{\partial H^{(1)}}{\partial h_3} + \frac{\partial}{\partial h_1} \frac{\partial H^{(1)}}{\partial \partial_h} + \frac{\partial}{\partial h_2} \frac{\partial H^{(1)}}{\partial \partial_p} \right] + \frac{1}{2h_3^2} \frac{d}{dH^{(1)}} \left[ (f(A))^2 + 4\pi \frac{dP}{dH^{(1)}} \right] = 0.$$

(2.7)

Similarly, one can write out the Hamilton’s equations (2.4) for $H^{(1)}$: these are in fact simply the classic generating equations for the magnetic field lines (viz., in Cartesian coordinates, $dx/B_n = dy/B_2 = dz/B_3$). This fact is well known in the plasma physics literature (cf. Kerst 1962), and the above represents some generalization of this previous work. Thus, a magnetic system characterized by a flux function $A(x_1, x_2)$ and the definite functions of $A, f(A)$, and $P(A)$ (which correspond to the magnetic field helical twist and the fluid pressure, respectively), is equivalent via equation (2.6) to a one-dimensional dynamical system characterized by a Hamiltonian $H^{(1)}(q, p)$ satisfying equation (2.7).

Now we can ask how the geometry of orbits in the phase space of $H^{(1)}$ is related to the geometry of the flux surfaces of $B$. This relationship is easily recovered by considering the intersections of the flux surfaces of $B$ with the surfaces $x_3 = \text{constant}$, and noticing that these (ID) curves are nothing else but the level curves $H^{(1)}(q, p) = \text{constant}$ in the phase space of $H^{(1)}$; some specific examples which illustrate this simple connection are given in Appendix A. Note that this Hamiltonian function $H^{(1)}(q, p)$ actually depends on the generalized coordinates $q$ and its conjugate momentum $p$, but not (explicitly) on $t$; our one-dimensional dynamical system is hence autonomous. This suggests a simple extension of our correspondence between the MHD system and one-dimensional Hamiltonian dynamics which we pursue in § IIIb below.

b) Two-dimensional MHD Equilibria and Hamiltonian Systems with Two Degrees of Freedom

We now develop an extension of the above analogy to two-dimensional dynamical systems. For simplicity, we shall first consider magnetostatic equilibria which contain one ignorable coordinate, i.e., such that $\partial_i B = 0$ for some $i$; we shall refer to this case as the strong symmetry constraint. For this case, it is evident that the mapping introduced in § IIa above has the property that $H^{(1)}$ is independent of $t$, so that the one-dimensional dynamical system is autonomous (and conservative).

Consider now an $n$-dimensional conservative dynamical system; that is, a system described by a Lagrangian $L^n$ which depends only on the generalized coordinates $q = (q_1, q_2, \ldots, q_n)$ and generalized velocities $\dot{q} = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$. Introducing the generalized momenta $p$,

$$p = (p_1, p_2, \ldots, p_n) \equiv \frac{\partial L^n(q, \dot{q})}{\partial \dot{q}},$$

(2.8)

we construct the Hamiltonian $H^n$ of the system,

$$H^n(q, p) \equiv p \cdot \dot{q} - L^n(q, \dot{q}),$$

(2.9)

where $q$ and $p$ now represent the $2n$ independent variables. Hamilton’s variational principle then states:

$$\delta \int_{t_1}^{t_2} L^n dt = \delta \int_{t_1}^{t_2} dt \left[ p \cdot \dot{q} - H^n(q, p) \right] = 0.$$

(2.10a)
If we now introduce, with Goldstein (1965), a parameter $\theta$ (which may, for example, be regarded as the arc length measured along the phase space trajectory), equation (2.10a) yields

$$\delta \int_{\theta_1}^{\theta_2} \frac{d\theta}{dt} \left[ p \cdot q' - H^{(0)}(q, p) \frac{dt}{d\theta} \right] = 0,$$

(2.10b)

where the prime denotes differentiation with respect to $\theta$. Thus, by introducing the definitions

$$q_{n+1} = t, \quad p_{n+1} = -H^{(0)},$$

(2.11)

we obtain the variational principle

$$\delta \int_{t_1}^{t_2} L^{(n)} dt = \delta \left[ \sum_{i=1}^{n+1} p_i \cdot q_i \right] \frac{dt}{d\theta} = \delta \int_{\theta_1}^{\theta_2} L^{(n+1)} d\theta = 0.$$  

(2.12)

Equation (2.12) is of course Hamilton's principle of least action for a dynamical system of $(n + 1)$ dimensions, whose phase space has $2(n + 1)$ dimensions. The Hamiltonian of that system is

$$H^{(n+1)}(q, p) \equiv q_{n+1} \left[ H^{(0)}(q, p) + p_{n+1} \right].$$

(2.13)

This well-known construction thus yields a new Hamiltonian with two properties of relevance to us: $H^{(2)}$ (by construction) is independent of time (i.e., is conservative), and has one ignorable coordinate (by construction, $q_{n+1}$). Note that $H^{(2)}$ must be conservative in order that the phase space flow $v$, which is to be identified with magnetic field lines in a manner to be specified, satisfies the condition $V \cdot v = 0$. In the particular case of the 1D system considered in § IIa, we have $n = 1$, so that $H^{(0)}$ above is just to be identified with $H^{(1)}(q, p)$. We thus have the assignments

$$x_1 \Rightarrow q_1, \quad (2.14a)$$
$$x_2 \Rightarrow p_1, \quad (2.14b)$$
$$x_3 \Rightarrow q_2 \left( = f^{-1}(A) \int dx_3 \frac{h_3}{h_1 h_2} \right), \quad (2.14c)$$
$$-A \Rightarrow p_2, \quad (2.14d)$$

and

$$t \Rightarrow \theta; \quad (2.14e)$$

Hamilton's equations for $H^{(2)}$ thus read

$$q' = \frac{\partial H^{(2)}}{\partial p}; \quad p' = -\frac{\partial H^{(2)}}{\partial q},$$

(2.15)

with $q = (q_1, q_2), \ p = (p_1, p_2)$. Note that $p_2' = 0$ (so that $q_2$ is ignorable), and that $H^{(2)}$ is not an explicit function of $\theta$. Hence there are two constants of the motion, and the dimensionality of the integral surfaces of $H^{(2)}$ (in four-dimensional phase space) is 2; indeed, by construction, because $q_2$ corresponds to the ignorable (MHD) coordinate $x_3$, these integral surfaces have identical form to the magnetic flux surfaces of $B$ in (3D) configuration space. That is, the flux surfaces $A(x_1, x_2) = \text{constant}$ correspond to the integral surfaces $H^{(2)}(q_1, p_1) = \text{constant}$. We can thus associate with every nonsingular, stationary magnetic flux geometry having at least one ignorable coordinate, a two-dimensional dynamical system which is integrable; and we shall identify the magnetic field lines associated with $B$ with the stream lines of the (incompressible) phase flow $v = (q_1', p_1', q_2', p_2') = \left( \frac{\partial H^{(2)}}{\partial p_1}, -\frac{\partial H^{(2)}}{\partial q_1}, \frac{\partial H^{(2)}}{\partial p_2}, -\frac{\partial H^{(2)}}{\partial q_2} \right)$, constant, 0.

(2.16)

which lies on the 2D integral surfaces of $H^{(2)}$. Note that as desired, $V \cdot B = 0$ translates directly into $V \cdot v = 0$.

From this derivation it would appear that $B_3$ is necessarily a constant independent of $x_1, x_2$. This is not an essential feature of the Hamiltonian formulation. If we are willing to slightly relax the form of $H^{(2)}$, we can generate more general magnetic field configurations. In the remainder of this section we will construct the Hamiltonian corresponding to a given magnetic field satisfying the strong symmetry constraint $\partial_3 B = 0$. In the next section we will generalize this construction to more general magnetic field configurations.

Consider a 2D Hamiltonian of the form

$$H^{(2)}(q_1, p_1, q_2, p_2) = a(q_1, p_1) + p_2 b(q_1, p_1),$$

(2.17)

where $a(q_1, p_1)$ and $b(q_1, p_1)$ are, as yet, unspecified functions of $q_1$ and $p_1$. The phase flow velocity is

$$v = \left( \frac{\partial H^{(2)}}{\partial p_1}, -\frac{\partial H^{(2)}}{\partial q_1}, \frac{\partial H^{(2)}}{\partial p_2}, -\frac{\partial H^{(2)}}{\partial q_2} \right) = \left( \frac{\partial a}{\partial p_1} + p_2 \frac{\partial b}{\partial p_1}, -\frac{\partial a}{\partial q_1} - p_2 \frac{\partial b}{\partial q_1}, b(q_1, p_1) \right).$$

(2.18)
Note that since $H^{(2)}$ is independent of $q_2$, the fourth component of $v$, $\dot{p}_2$, vanishes. As before, we would like to identify $B$ with the first three components of $v$. Here we notice a subtlety: $v$ is defined as a function of four coordinates $(q_1, p_1, q_2, p_2)$, but $B$ is a function of only three coordinates $(x_1, x_2, x_3)$. In order to make the correspondence between $v$ and $B$ precise, we must specify some 3D hypersurface in phase space on which we evaluate $v$. Fortunately there is a simple, natural way of doing this. Since $p_2 = 0$, the phase flow remains on hyperplanes of constant $p_2$. Let us pick one of these hyperplanes, say the $p_2 = 0$ hyperplane, and define $B$ to be equal to the first three components of $v$ evaluated on that hyperplane (as before, $x_1 \Rightarrow q_1, x_2 \Rightarrow p_1, x_3 \Rightarrow q_2$):

$$B(x_1, x_2, x_3) = B(q_1, p_1, q_2)$$

Then, using (2.17), we have

$$B = \left[ \frac{\partial a(q_1, p_1)}{\partial p_1}, -\frac{\partial a(q_1, p_1)}{\partial q_1}, b(q_1, p_1) \right].$$

Clearly, $a(q_1, p_1)$ can be identified with the flux function $A(x_1, x_2)$ (from eq. [2.3]), and $b(q_1, p_1)$, which gives the spatial dependence of $B_3$, can be related to the integral of the field line twist $f(A)$. Note again that, because $b$ is independent of $q_2$, $B = 0$ and, correspondingly, $\mathbf{V} \cdot B = 0$.

c) Construction of the Two-dimensional Hamiltonian under Weaker Symmetry Conditions

Suppose we are given a magnetic field $B(r) = (B_1, B_2, B_3)$ which satisfies the weaker symmetry condition $\partial_2 B = 0$, for some $i$ (say, $i = 3$), instead of the stronger symmetry condition $\partial_i B = 0$, as well as the constraint equation (2.2a). That is, we require only that $\partial_3 B_3 = 0$, but allow $B_1$ and $B_2$ to be $x_3$-dependent. We would now like to construct the corresponding 2D Hamiltonian, $H^{(2)}(q_1, p_1, q_2)$. To be precise, we identify the coordinates

$$x_1 = q_1, \quad x_2 = p_1, \quad x_3 = q_2.$$  

Then along some (3D) hypersurface $p_2 = f(q_1, p_1, q_2)$, we seek to identify the streamlines of the phase space flow defined by $H^{(2)}$ with the magnetic field $B$. In the previous section, we chose the hypersurface to be the $p_2 = 0$ hyperplane. This was a natural choice because, since $q_2$ was an ignorable coordinate, $p_2$ was a constant of the motion. In the present context we would also like to define our hypersurface with the aid of a constant of the motion. The only such constant which we know exists a priori is the energy (a fact which follows from our constraint that $\text{div} \ v = 0$), so that we are naturally led to choose the hypersurface $H^{(2)} = \text{constant} = 0$.

What then does the weak symmetry condition imply for the form of $H^{(2)}$? Clearly, this condition is equivalent to the statement that the mixed partial of $H^{(2)}$ vanish:

$$\frac{\partial^3 H^{(2)}}{\partial q_2 \partial p_1} = 0.$$

This then implies that $H^{(2)}$ must be of the form

$$H^{(2)} = f(q_1, p_1, q_2) + g(q_1, p_1, p_2),$$

note that the form of $H^{(2)}$ chosen in § IIb is but a special case of this more general form. Now, as before, we identify the components of $B$ with the first three components of the phase velocity $v$ on the hypersurface $H^{(2)} = 0$. Using Hamilton's equations, we find

$$B_1(x_1, x_2, x_3) = B_1(q_1, p_1, q_2) = \dot{q}_1 = \frac{\partial H^{(2)}(q_1, p_1, q_2, p_2)}{\partial p_1} \bigg|_{H^{(2)}(q_1, p_1, q_2, p_2) = 0} = \dot{q}_1;$$

$$B_2(x_1, x_2, x_3) = B_2(q_1, p_1, q_2) = \dot{p}_1 = -\frac{\partial H^{(2)}(q_1, p_1, q_2, p_2)}{\partial q_1} \bigg|_{H^{(2)}(q_1, p_1, q_2, p_2) = 0} = \dot{p}_1;$$

$$B_3(x_1, x_2, x_3) = B_3(q_1, p_1, q_2) = \dot{q}_2 = \frac{\partial H^{(2)}(q_1, p_1, q_2, p_2)}{\partial p_2} \bigg|_{H^{(2)}(q_1, p_1, q_2, p_2) = 0} = \dot{q}_2.$$  

Here we make use of (2.21), which identifies $x_1, x_2, x_3$ with $q_1, p_1, q_2$. To be more explicit as to what we mean by evaluating the partial derivatives on the energy surface, we can rewrite (2.24a) by using equation (2.23):

$$B_1(q_1, p_1, q_2) = \frac{\partial f}{\partial p_1} + \frac{\partial g}{\partial p_1},$$

such that $f(q_1, p_1, q_2) = g(q_1, p_1, q_2)$, and similarly for $B_2$ and $B_3$. This is the general construction we have been seeking. The crucial feature of this construction is that if the Hamiltonian $H^{(2)}$ possesses a second integral of motion (such as $p_2$ in § IIb), then the phase flow will be restricted to the intersections of the 3D level surfaces of this second integral with the 3D energy.
surface. In other words, the phase flow will lie on a family of 2D surfaces. This corresponds, via (2.24), to the magnetic field lines lying on 2D magnetic surfaces. If no second integral of \( H^{(2)} \) exists, the phase flow will ergodically fill the energy hypersurface and, correspondingly, we will not have magnetic surfaces.

In closing this section, we shall slightly digress from our main line of argument in order to explicitly construct \( H^{(2)} \) for a magnetic field obeying the weak symmetry condition using, not the \( H^{(2)} = 0 \) hypersurface, but (as in § IIb) the \( p_2 = 0 \) hypersurface as the locus for evaluating the partial derivatives of \( H^{(2)} \). In this case, it is particularly easy to see the connection between our formulae and the more familiar ones for generating \( B \) from a vector potential \( A(x_1, x_2, x_3) \).

Recall that \( B \) is derived from a vector potential \( A \) via

\[ B = \frac{\partial A_3}{\partial x_2}, \quad \frac{\partial A_3}{\partial x_2}, \quad \frac{\partial A_3}{\partial x_2}, \quad \frac{\partial A_3}{\partial x_2} \]

Given \( B \), this does not uniquely determine \( A \); there are an infinite number of possible \( A \)'s satisfying (2.26), all related by a gauge transformation. In order to determine \( A \), we must impose some restrictions on its form; that is, we must choose a gauge. For our purposes, it is convenient to choose a gauge in which

\[ A_1 = 0. \]

In this gauge, (2.26) reduces to

\[ B(x_1, x_2, x_3) = \frac{\partial A_3(x_1, x_2, x_3)}{\partial x_2} \]

where the fact that \( A_3 \) must be independent of \( x_3 \) follows from our symmetry constraint \( \partial_3 B_3 = 0 \). Thus \( B \) is completely specified by the two functions

\[ A(x_1, x_2, x_3) = A_3(x_1, x_2), \quad g(x_1, x_2) = A_2(x_1, x_2). \]

Now let us construct \( H^{(2)} \). As in § IIb, we choose

\[ H^{(2)} = a(q_1, p_1, q_2) + p_2 b(q_1, p_1) = a(x_1, x_2, x_3) + p_2 b(x_1, x_2). \]

Then, from Hamilton’s equations,

\[
\begin{align*}
B_1 &= \frac{\partial H^{(2)}}{\partial p_1} \bigg|_{p_2=0} = \frac{\partial a}{\partial x_2}, \\
B_2 &= -\frac{\partial H^{(2)}}{\partial q_1} \bigg|_{p_2=0} = -\frac{\partial a}{\partial x_1}, \\
B_3 &= \frac{\partial H^{(2)}}{\partial p_2} \bigg|_{p_2=0} = b(x_1, x_2).
\end{align*}
\]

Comparing (2.28) and (2.31), we see that

\[
\begin{align*}
a(x_1, x_2, x_3) &= A(x_1, x_2, x_3) + \phi_1(x_3), \\
h_1 h_2 b(x_1, x_2) &= \frac{\partial g(x_1, x_2)}{\partial x_1},
\end{align*}
\]

where \( \phi_1 \) is an arbitrary function of \( x_3 \). Now our gauge choice \( A_1 = 0 \) has not entirely specified the gauge. In particular, we are still free to make two further gauge transformations, which depend on \( x_2 \) alone, and \( x_3 \) alone, respectively. We can thus completely specify the gauge by requiring

\[ \phi_1(x_3) = 0. \]

and, upon integrating (2.32b), to find \( g(x_1, x_2) = \int h_1 h_2 b(\xi_1, x_2) d\xi_1 + \phi_2 \), with \( \xi \) an integration variable, and \( \phi_2(x_2) \) an arbitrary function of \( x_2 \).

\[ \phi_2(x_2) = 0. \]

Thus

\[ g(x_1, x_2) = \int_0^{x_1} d\xi_1 b(\xi_1, x_2) h_1 h_2, \quad A(x_1, x_2, x_3) = a(x_1, x_2, x_3). \]

Equations (2.32)–(2.34) thus allow us to transform \( A(r) \) to find \( H^{(2)} \), and vice versa, and hence we see that for magnetic fields obeying the weak symmetry condition, the 2D Hamiltonian and the vector potential are two completely equivalent ways of characterizing the magnetic field.
We now consider the problem which originally motivated the present study: can magnetostatic equilibria having some symmetry (expressible in the form $\partial_i B = 0$ for some $i$) be subjected to perturbations which lead to states lacking any symmetry, but still satisfying the equilibrium equations (2.2)?

To answer this question, we consider the equivalent question in the dynamical domain; that is, we consider a nonintegrable perturbation $H_1^{(2)}$ of the equilibrium system defined by an integrable Hamiltonian $H^{(2)}$ corresponding to (as discussed above) the given magnetohydrostatic equilibrium, with $H_1^{(2)} \ll H^{(2)}$. The perturbed Hamiltonian then has the form

$$H^{(2)} = H_1^{(2)} + H_2^{(2)}.$$  

In the unperturbed system, the nested set of integral surfaces are 2D level surfaces (defined by, for example, the integrals corresponding to the constraints of constant energy and $p_z = \text{constant}$); in this nested topology, one can choose some parameter [say, $z = A(x_1, x_2)$] labeling distinct surfaces. Now, as discussed in Appendix B, the KAM theorem (Moser 1973; Arnold 1978) shows that a finite measure of these integral surfaces of $H^{(2)}$ is destroyed by the application of the nonintegrable perturbation; that is, for a range of values of $z$ lying in an interval of nonvanishing measure, “most” orbits associated with former integral surfaces labeled by this range of $z$ values no longer remain on a 2D surface: in terms of the discussion of §§ IIb and IIc, the corresponding phase space flow $v$ in the hypersurface $H^{(2)} = \text{constant}$ is no longer confined on two-dimensional surfaces (a fact noted by Hénon 1966 for the special case of force-free fields). Thus, the destruction of a finite range of integral surfaces implies that there now exist finite intervals in the range of $z$ for which there do not exist corresponding 2D integral surfaces.

What does this imply about the corresponding MHD case? Translating from integral surfaces to magnetic flux surfaces, we recall that the plasma pressure is a flux surface parameter which is, like the flux function $A$ (corresponding to $z$ above), constant on flux surfaces, and that a nonintegrable perturbation corresponds to a symmetry-breaking field/plasma perturbation (viz., one that violates the constraint $\partial_i B = 0$). Such perturbations thus result in a nonvanishing measure of surfaces of constant pressure to be disrupted; that is to say, it is no longer possible to define a scalar field (corresponding to some new pressure field) everywhere such that pressure lines lie on level (2D) surfaces of that scalar field. It follows that in the perturbed system, one can no longer assume that, for some appropriate $P(r)$, $VP$ is balanced everywhere by the Lorenz force, so that static equilibrium, in the sense described by equations (2.2) no longer exists; the system may still be in equilibrium, but then must of necessity involve flows, otherwise nonequilibrium prevails, and the system is forced to evolve in time. Hence, symmetry-breaking perturbations of magnetostatic equilibria necessarily result in the violation of the hydrostatic equations of motion. This result corresponds precisely to the conclusions of Yu (1973) and Parker (1972, 1979), with the restriction to infinitesimal perturbations removed, and the result generalized to arbitrary, symmetry-breaking, perturbations.

IV. ASTROPHYSICAL APPLICATIONS

In the above, we have extended a fundamental property of Hamiltonian systems to symmetric hydromagnetic systems by exploiting the analogy between dynamical systems and magnetic fields in highly conducting plasmas, showing that such equilibria, if perturbed, continue to satisfy the magnetostatic equilibrium equations only for exceptional forms of the applied perturbations. Because astrophysical magnetic field configurations are not expected to possess the special symmetry properties apparently required for equilibrium, the question which naturally arises is: Is there any cause-and-effect relationship between the above mentioned property of equations (2.2) and the observed “activity” in astrophysical systems? If so, then the intimate connection between magnetic fields and “activity” may be quite fundamental (Parker 1979). In the following, we argue that the assumptions made in order to derive the result discussed in § III are not overly restrictive, so that our conclusions can be applied to astrophysical problems as occur, for example in the solar atmosphere.

Consider first the assumption that the magnetostatic equilibrium which is subject to perturbation possesses some symmetry. It is a matter of record that available nonsingular magnetostatic equilibria all show some symmetry; with the exception of the mirror-symmetric field discussed by Lortz (1970), these symmetries may also be described in terms of an invariance property of the field geometry under translation along some appropriate (generalized) coordinate. Grad (1967) was first to conjecture that the absence of asymmetric magnetostatic equilibria is not a consequence of our mathematical limitations, but instead that only “highly symmetric” solutions to the equations of magnetostatic equilibrium should be expected. Parker (1972, 1979) subsequently provided the first quantitative evidence that magnetostatic equilibria lacking topological invariance may not exist (see also Low 1975, 1980; Edenstrasser 1980). Furthermore, it can be shown that if nonsymmetric magnetostatic equilibria exist, then these are not smoothly accessible from neighboring symmetric states (Rosner and Knobloch 1982). Of course, these arguments do not exclude the existence of asymmetric magnetostatic equilibria, but our results show that modeling of astrophysical systems via symmetric magnetostatic equilibria misses a crucial piece of physics—namely, that such equilibria are topologically unstable.

Now consider the assumption that all boundary stresses vanish (i.e., that $B \cdot \hat{n} \to 0$ at the boundaries, where $\hat{n}$ is the local normal to the boundary surface). When studying the (MHD) stability of magnetic configurations in the solar corona (for example), it is of course important to note that the time scale of motions at the photospheric boundary is far larger than that associated with coronal MHD instabilities; in that case, it is a fair assumption to regard the photosphere as a rigid boundary, so that re-entrant magnetic field lines are “line-tied” (Solov’ev 1975). These concerns are not relevant to our study. From the point of view of the topological structure of magnetic flux surfaces (and the structural stability of this topology), the Sun does not have any “rigid” boundaries; indeed, this would seem to be the case throughout astrophysics, with the possible exception of singular systems such as planetary magnetospheres and neutron stars.
Finally, consider the apparently restrictive assumption made in writing the set of MHD equations (2.1) in the absence of gravity or other constraining forces acting on the system; because of the virial theorem (Chandrasekhar and Fermi 1953; Parker 1958), it would then seem that we are severely limited in the construction of MHD equilibria (Field 1982); i.e., the magnetic field cannot contain its own stresses in the absence of gravitation or external forces acting on some confining boundary. However, the geometrical structure of the MHD equilibria that we investigate is largely controlled by the interplay of the nonconservative Lorentz force and all other conservative forces in the momentum balance equation (2.1b); the pressure gradient force can in this case be viewed as simply a typical representative of all such conservative forces, and hence the addition of forces such as gravitation will not change the qualitative behavior of the solutions of (2.1). For example, if \( \rho \) is constant on equipotential surfaces (defined by \( \phi = \) constant), then the gravitational force per unit volume can be written as the gradient of a scalar field and hence the gravitational force can be subsumed in the gas pressure gradient (i.e., \( \rho \phi + P \rightarrow P' \)).

The question of finding the conditions under which a magnetic field can be constrained to lie on level surfaces of some scalar function is of course highly relevant to laboratory studies of plasma confinement (e.g., Field et al. 1966; Cohen and Killeen 1983). That is, fields describable by a dense set of flux surfaces ensure particle confinement to lowest order in the ratio of Larmor radius to the scale length, and absence of magnetic “islands” reduces the transport of low-energy particles (cf. Cary 1982; Freidberg 1982). A related interesting astrophysical application of the possible destruction of stationary isobaric surfaces by perturbations is to the cross-field heat diffusion problem in tenuous, hot plasmas such as occur in the solar corona. Conventional solar theory views the coronal, magnetically confined plasma as residing on nested flux tubes, so that plasma heating mechanism which posses very localized energy release (viz., Rosner et al. 1978; Galeev et al. 1981; Hinata 1980; Benford 1982) are faced with a cross-field heat dispersal problem. Since perturbations of ambient coronal magnetic fields are unlikely to share any symmetry properties with the preexisting flux, the present results suggest that interior isobaric surfaces are not static; if the corresponding flux surfaces are “destroyed,” our result suggests that field lines within coronal loops are very likely stochastic. Because in this case neighboring stochastic field lines separate exponentially, and because parallel heat transport is uninhibited, we infer that field line stochasticity would cause mixing of such field lines throughout the loop volume, and hence would also cause enhanced “cross-field” heat dispersal, quite analogous to the situation in laboratory plasmas (cf. Rechester and Rosenbluth 1978). This consequence of our results on perturbations of magnetohydrostatic equilibria thus provides theoretical support for Colgate’s (1978) phenomenal model of lateral heat transport in flaring coronal “loops,” which posits field line stochasticity (but on rather different grounds from those argued here).

In summary, we have addressed the question of the topological stability of magnetostatic equilibria by asking for the necessary constraints on perturbations such that the perturbed system still satisfies the magnetohydrostatic equations. Our method of attack focused on the fact that in magnetostatic systems, isobaric surfaces generally coincide with magnetic flux surfaces (subject to the constraint that the pressure gradient vanish only at isolated points); and we have sought the constraints on perturbations which may either lead to destruction of these isobaric surfaces, or to noncoincidence of isobaric and flux surfaces. Our results show that arbitrary symmetry-breaking perturbations of such symmetric magnetostatic equilibria inevitably lead to a violation of hydrostatic equilibrium. Our method of proof depends upon an extension of previous work by Kerst (1962) and others, who connected magnetostatic equilibria with the dynamics of a 1D Hamiltonian system, to the dynamics of 2D time-independent Hamiltonian systems (see also recent work of Boozer and White 1982 and Littlejohn and Cary 1982), and it uses the results of the KAM theorem of dynamical theory (cf. Moser 1973; Arnold 1978) for the behavior of integral surfaces in the dynamical phase space under nonintegrable perturbations to derive the corresponding behavior of isobaric surfaces.

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APPENDIX A

SPECIFIC EXAMPLES OF MAGNETOSTATIC EQUILIBRIA

In order to place the discussion of § IIa on more concrete grounds, we exemplify the correspondence of magnetostatic equilibrium equations and Hamilton’s equations for one-dimensional systems by considering three specific cases of interest. In the first two cases, the full solution to equation (2.2) contains a finite number of singularities; these must be removed when constructing the volume \( V \) discussed above (most expeditiously, by simply excluding the immediate volume surrounding the unwanted singularities).

\[ a) \text{ Magnetic Field around an O-Type Neutral Point} \]

We begin by considering the magnetic field due to a straight line current:

\[ J = (c/4\pi)(0, 0, 2a), \]

where \( a = B_0/r_0, B_0 \) and \( r_0 \) are constants. The magnetic field is then given simply by

\[ B = (-ay, ax, b), \]
with \( b \) an arbitrary constant. By writing \( t = z/b, \ q = x, \) and \( p = y \), the corresponding Hamiltonian is

\[
H = \frac{1}{2}a(p^2 + q^2),
\]  

(A3)

Upon substituting equation (A3) into equations (2.4), one immediately obtains the familiar equations of motion for a classical harmonic oscillator; thus, the curves generated by the intersection of the plane \( z = 0 \) and the magnetic flux surfaces due to a uniform current density correspond to the phase portrait of a classical 1D harmonic oscillator, a result one would expect by simple inspection of the field geometry and the coordinate definitions.

**b) Magnetic Field around an X-Type Neutral Point**

We next consider the magnetic field geometry in the vicinity of an X-type neutral point (cf., Sweet 1969). In this case, the magnetic field is described by the equations

\[
B = (ay, ax, b),
\]

(A4)

where again \( a = B_0/r_0 \), and \( b \) is an arbitrary constant; the required transformation is, as before, \( t = z/b, \ q = x, \) and \( p = y \).

The Hamiltonian now reads instead

\[
H = \frac{1}{2}a(p^2 - q^2),
\]

(A5)

and the Hamilton’s equations corresponding to equation (2.2) are, again, equations (2.4). The curves in the surface of section defined by the magnetic field line geometry and the plane \( z = 0 \) in this case correspond to the phase portrait of an unstable one-dimensional Hamiltonian system.

**c) Magnetic Field of a Screw Pinch**

Finally, consider a magnetic field geometry with both O and X type neutral points. In particular, consider the simpler solution of equation (2.3b), which is translationally symmetric in cylindrical coordinates \((r, \phi, z)\) and which is generated by expanding the two surface functions \( P(A) \) and \( B_z(A) \) in a Taylor series,

\[
P = P_0 + \frac{\lambda B_0}{4\pi} A + \cdots,
\]

(A6)

\[
B_z = B_0 - \lambda A + \cdots,
\]

(A7)

where \( P_0, B_0, \) and \( \lambda \) are constants. It can be easily verified that the magnetic field which satisfies the equilibrium equation (2.2) is given by the expressions

\[
B_z = B_0 + \sum_m J_m(x)(a_m \cos m\phi + b_m \sin m\phi),
\]

(A8)

\[
B_r = \sum_m \frac{m J_m(x)}{x}(a_m \sin m\phi - b_m \cos m\phi),
\]

(A9)

\[
B_\phi = \sum_m J_m'(x)(a_m \cos m\phi + b_m \sin m\phi),
\]

(A10)

where \( x = \lambda r \) and \( J_m(x) \) are the familiar Bessel functions. By defining \( q \equiv r \) and \( p \equiv \phi \), and if \( t(z) \) is given by the expression \( t(z) = \int dz/rB_z \), then the corresponding Hamiltonian is

\[
H = -\sum_m \frac{J_m(x)}{\lambda} (a_m \cos m\phi + b_m \sin m\phi),
\]

(A11)

and the equations for a field line correspond to Hamilton’s canonical equations (2.4).

**APPENDIX B**

**HAMILTONIAN DYNAMICS AND THE KAM THEOREM**

In this Appendix we briefly review the background of the Kolmogorov-Arnold-Moser (KAM) theorem (cf. Arnold and Avez 1968; Arnold 1978) relevant to the problem considered in the present paper. The results of dynamical theory given here are not new (cf. Contopoulos 1963, 1982; Ford 1974; Whiteman 1977), but we seek to place the discussion of dynamical systems in the context of the magnetic flux problem.

Suppose we are given an integrable Hamiltonian \( H_0 \), with \( H_0 \) having no explicit time dependence. Hamilton-Jacobi theory then allows us to find, using a generating function \( S \) (referred to as Hamilton’s characteristic function by Goldstein 1965), cyclic coordinates whose conjugates are constants of the motion; we choose these to be the action-angle variables \((J, \theta)\) such that \( H_0 = H_0(J) \) alone (Goldstein 1965). If we now consider a nonintegrable perturbation \( \epsilon H_1(J, \theta) \), with \( \epsilon \ll 1 \), such that

\[
H(J, \theta) = H_0(J) + \epsilon H_1(J, \theta),
\]

(B1)
we can then ask whether there always exist (possibly new) action-angle variables \((J, \theta')\) such that

\[ H(J, \theta) = H'(J), \]  

(B2)
i.e., such that \(H'\) is a function of the action variable alone. The existence of these (possibly new) action-angle variables would guarantee that the perturbed Hamiltonian was still integrable (or, translating into the MHD domain, that the corresponding perturbed magnetic state still satisfies the magnetostatic equations). Now, if such new action-angle variables exist, then there must exist a generating function \(S(J, \theta)\), with the property that \(J = V_\theta S\) (cf. Landau and Lifshitz 1960); \(S\) must then satisfy the relation

\[ H[V_\theta S(J, \theta), \theta] = H'(J). \]  

(B3)

One way to construct this generating function is to expand \(S\) in a perturbation expansion, say in powers of some parameter \(\epsilon\),

\[ S = J \cdot \theta + \epsilon S_1(J, \theta) + \cdots; \]  

(B4)
then, by substituting equation (B4) into equation (B3), and using the definition of \(H\) given in (B1) above, we obtain

\[ H_0(J + \epsilon V_\theta S_1 + \cdots) + \epsilon H_1(J + \cdots, \theta) = H'(J). \]  

(B5)
The lowest-order nontrivial result is obtained by equating coefficients of the terms linear in \(\epsilon\); thus, retaining terms to order \(\epsilon\), we have

\[ H_0(J) + \epsilon [V_\theta H_0(J) \cdot V_\theta S_1 + H_1(J, \theta)] = H'(J). \]  

(B6)

Now, since \(H_0\) is cyclic in \(\theta\), it follows that

\[ V_\theta H_0(J) = \omega_0(J) = [\omega_{01}(J), \omega_{02}(J), \ldots] \]  

(B7)
is the frequency vector of the unperturbed motions; furthermore,

\[ H_1(J, \theta) = \sum_{m=0} \frac{H_{1m}(J) e^{im \cdot \theta}}{m}; \]  

(B8)
because \(H_1\) is a function of \(p\) and \(q\), and these are periodic in \(\theta\); therefore \(S_1\) is periodic in \(\theta\), with an arbitrary constant term we set to zero. Thus

\[ S_1(J, \theta) = \sum_{m \neq 0} S_{1m}(J) e^{im \cdot \theta}. \]  

(B9)
Equating Fourier coefficients, we obtain

\[ m \neq 0: \quad S_{1m}(J) = \frac{iH_{1m}(J)}{m \cdot \omega_0(J)} + \cdots, \]  

(B10a)
\[ m = 0: \quad H'(J) = H_0(J) + \epsilon H_{10}(J) + \cdots. \]  

(B10b)
The generating function \(S\) is thus given by

\[ S(J, \theta) = \theta \cdot J + \epsilon i \sum_{m \neq 0} \frac{H_{1m}(J) e^{im \cdot \theta}}{m \cdot \omega_0(J)} + \cdots. \]  

(B11)
Cursory inspection of equation (B11) immediately shows a great difficulty: if there exist motions such that the frequencies \(\omega_0\) are commensurable, then there exist terms in the sum with \(m \cdot \omega_0 = 0\),

\[ (B12) \]
so that the perturbation series diverges. In fact, even if the frequencies \(\omega_0\) are incommensurate, one can always find values for \(m\) such that \(m \cdot \omega_0\) is arbitrarily small. Hence (B11) can diverge even in the incommensurate case. This is the well-known problem of "small divisors" in classical mechanics (cf. Arnold and Avez 1968); note in particular that the evident singular behavior does not reflect the behavior of the dynamical system under study, but rather a failure of the method of solution (i.e., the method of solving for the required generating function). The above discussion, which simply summarizes a classical result of dynamical theory, thus shows explicitly that a straightforward perturbation expansion of the corresponding MHD problem cannot answer the crucial question regarding the "structural" or topological stability of hydrostatic stationary MHD solutions (Rosner and Knobloch 1982).

The way out of this dilemma rests on the KAM theorem. The essence of the method underlying the KAM theorem is to replace the series expansion of (B11) by an iteration scheme (analogous to Newton's method) to calculate the form of the perturbed tori (Arnold and Avez 1968; Moser 1973). That is, rather than expanding about the unperturbed Hamiltonian (as in eqs. [B10] and [B11]) in \(\epsilon\) so as to directly calculate the generating function \(S(J, \theta)\) of the perturbed tori, one instead calculates a sequence of generating functions by iteration. This procedure accelerates convergence for \(\epsilon \ll 1\) for almost all unperturbed tori; for present purposes, the crucial point is that it can be shown (cf. Moser 1966, 1973) that for a set of
unperturbed tori of nonvanishing measure, convergence is not obtained. The key point here is that, unlike the case discussed above in connection with the straightforward perturbation leading to equation (B11), the lack of convergence for these exceptional tori is not a defect of the approximation scheme, but is instead intrinsic to these particular sets of unperturbed tori. Thus, one has shown that these particular integral surfaces are destroyed by the symmetry-breaking perturbation; this is the basic result required in § III.

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