WAVE PROPAGATION IN INTENSE FLUX TUBES

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Abstract. The nature of non-adiabatic wave propagation in a slender magnetic flux tube is explored. The results of the theory are compared with the observations of Giovanelli et al. (1978), and found to be in general agreement. Those observations, of tubes in the photosphere-chromosphere, show outwardly propagating waves, with periods of 300 s, which take some 19 s to propagate from one level of line formation to another level higher in the atmosphere. In sharp contrast to this, is the time of 7 s for a similar disturbance outside the tube to propagate between the same two levels of line formation, estimated to be some 600 km apart in the field-free atmosphere.

It is argued that the sharply contrasting propagation times for the tube and its environment is principally due to the elasticity of the tube and its subsequent propensity for propagation. A non-adiabatic disturbance may be essentially propagating within the tube but essentially non-propagating outside, with considerably slower phase speeds thus arising inside the tube.

The theory suggests that the observed disturbances are non-adiabatic, acoustic-gravity waves channelled along a magnetic flux tube and modulated by external pressure variations.

1. Introduction

Intense magnetic flux tubes, located in the boundaries of supergranules, provide a direct means of communication between the photosphere and the chromosphere. Waves excited in the tube are ducted along the tube, changing their propagation characteristics in response to the geometry of the tube and the stratified atmosphere within it.

Flux tubes may support a variety of waves. In particular, a vertical slender tube supports the linear propagation of a longitudinal wave (Defouw, 1976; Roberts and Webb, 1978), in which motions are predominantly along the tube and are compressive, and a transversal wave (Spruit, 1981), in which motions are predominantly transverse and noncompressive. Both modes may be described by the Klein-Gordon equation (Roberts, 1981). From such an equation it follows that an impulsively generated disturbance gives rise to a wave-front which propagates along the tube, trailing a wake oscillating at the cut-off frequency of the atmosphere for the mode generated (see Rae and Roberts (1982) for a detailed discussion of the wake of the longitudinal wave).

To date, the only direct observational study of waves in intense flux tubes is that carried out by Giovanelli et al. (1978). They were able to obtain simultaneous time-sequence observations at two heights, both within and external to the tube, and concluded that observed oscillations within an intense tube were outwardly propagating. The amplitude of the disturbance within the tube increased from about 0.27 km s$^{-1}$ in the line Fe 5166 Å to about 0.49 km s$^{-1}$ in the line Mg 5183 Å, an increase by a factor of 1.8 in a distance of some 600 km (estimated, however, from heights of line formation in a field-free atmosphere). This may be compared with an amplitude of 0.29 km s$^{-1}$ in Fe 5166 Å and 0.41 km s$^{-1}$ in Mg 5183 Å for disturbances outside the tube, giving an
increase by a factor of 1.4 over the 500 or 600 km range of propagation. There was no
evidence of shocks. The disturbance outside the tube took an estimated 7 s to cover the
observed height range, but the transit time for the disturbance inside the tube was 19 s.
The dominant period for disturbances in the photosphere-low chromosphere in 5 min
(see also Giovanelli and Brown, 1977); higher in the chromosphere, it is 3 min.
Giovanelli et al. (1978) concluded that their data was insufficient to decide on the precise
nature of the interaction of the tube with its environment, be it one of common excitation
or strong interaction.

The observations of Giovanelli et al. (1978) raise an immediate question: why is the
disturbance in the tube observed to move so much slower than that in the field-free
environment? It is important to try to answer such a question if theories of flux tube
propagation are to be extensively developed; it should be remembered that flux tube
oscillations may eventually provide a diagnostic tool for probing magnetic regions.

We consider here what factors might contribute to giving a significantly slower
propagation speed for disturbances within a tube as compared with those outside. Our
discussion will be confined to the behaviour of an isolated slender flux tube in a field-free
environment, even though such a model is strictly appropriate only for the low
photospheric layers of the atmosphere. In the chromosphere, where expanding tubes are
beginning to merge with their neighbours, the notion of an isolated slender tube becomes
suspect and, correspondingly, a description of the allowable wave modes will be more
complicated. In this respect, the recently described magnetic canopy structure (Gio-
vanelli, 1980), with a merging level that is perhaps as low as the temperature minimum,
would require a somewhat different theoretical description, incorporating both the
isolated tubes in the low photosphere and the merged canopy fields in the chromosphere
and above*. This is a more complicated undertaking than we consider here, being
instead content to examine the waves of an isolated flux tube and their behaviour in
comparison with the modes of a field-free atmosphere.

The role of an isolated tube in wave propagation may be simply described by noting,
firstly, that the tube provides a wave-guide for disturbances, guiding them into the higher
layers of the atmosphere in an anisotropic fashion (in contrast to the isotropic nature of
sound waves). Secondly, the magnetic field imparts an elasticity to the medium, which
tends to increase the effective compressibility of the gas within the tube. This dual role
of a flux tube results in slower wave-speeds and slower amplitude growths (with height).
In an isothermal flux tube in temperature balance with its surroundings, a propagating
tube wave (longitudinal or transversal) e-folds in four scale-heights, compared with two
scale-heights for an acoustic-gravity wave in a field-free gas.

Not all frequencies propagate, however; the tube possesses cut-off frequencies, which
depend upon the nature of the particular wave that is generated. To clarify this feature
it is useful to introduce the dispersion relations associated with the tube modes. Now,
since both the longitudinal and transversal waves in a vertical, slender, flux tube obey
the Klein-Gordon equation, which is also the governing equation for the vertical

* Some investigations along these lines are in progress (H. Spruit, 1982, private communication).
propagation of acoustic-gravity waves in the field-free environment of the tube, the
dispersion relations of the three waves may be conveniently discussed in a single form.
For a mode of frequency $\omega$ and longitudinal wavenumber $k$, propagating adiabatically
in an isothermal atmosphere, the dispersion relation appropriate to the Klein-Gordon
equation is

$$\omega^2 = k^2 c^2 + \Omega^2,$$

(1)

where $c$ is the propagation speed of the wave (longitudinal, transversal, or acoustic-
gravity) and $\Omega$ its cut-off frequency. Only frequencies above $\Omega$, wave periods below $2\pi/\Omega$, result in propagation. We write $\Omega = \omega_v$ for the longitudinal wave, and $\Omega = \omega_a$ for the acoustic-gravity wave.

For an intense flux tube under photospheric conditions the cut-off of the longitudinal
wave is numerically close to the value of the acoustic cut-off, both having cut-off periods
of about 200 s. Thus, a sustained oscillator at the dominant photospheric period of 300 s
(and so above acoustic cut-off) will not generate a propagating wave, but instead will
support an oscillation that declines exponentially with height. The transversal wave, on
the other hand, has a markedly different cut-off period, of in fact about 670 s, and so
is likely to propagate in the photosphere (Spruit, 1981).

The above conclusion, that a longitudinal wave is unlikely to propagate at photospheric-chromospheric levels, would seem to be in contradiction to the observations of Giovanelli et al. (1978). However, the resolution of this difficulty is not hard
to find; for it is well-known that conditions in the photosphere and chromosphere favour
non-adiabatic wave propagation. When dissipative effects are included in the description
of the propagation of a wave, we no longer find a sharp cut-off between frequencies that
propagate and those that are evanescent. Instead, as Souffrin (1966) has shown for the
propagation of acoustic-gravity waves under the assumption that non-adiabatic pro-
cesses may be described by Newton’s law of cooling, it is convenient to classify
disturbances as either mainly propagating or mainly evanescent, accordingly as the
frequency $\omega$ is greater than or less than $\omega_{v_1}$ for a tube, $\omega_{a_1}$ for a field-free medium (see Webb and Roberts, 1980). In the limit of almost adiabatic propagation, corresponding
to wave periods very much less than the damping time-scale, the frequencies $\omega_{v_1}$ and
$\omega_{a_1}$ reduce to their adiabatic counterparts, $\omega_v$ and $\omega_a$. For disturbances of frequency
above cut-off, then, we expect (as concluded above) phase-speeds that are roughly
comparable (being, in fact, slightly slower inside the tube than outside). But for
disturbances of frequency below cut-off, as, for example, those of 5-min period, radiative
effects are important in determining the phase-speed, and the elasticity of the tube leads
to a greater propensity for propagation in this mainly evanescent zone. The result is
substantially slower phase-speeds inside the tube than those outside.

This difference in phase-speeds is especially noticeable under almost isothermal
conditions, corresponding to wave periods very much greater than damping time-scales.
Such is the case in the first one or two scale-heights above the solar photosphere at
$\tau_{5000} = 1$, where damping time-scales of several seconds (compared with typical periods
of 300 s) arise. In this region $\omega_{v_1}$ is much smaller than $\omega_{a_1}$ and so, in general, there will
be a range of frequencies that are mainly propagating inside the tube but mainly
evanescent outside the tube. A mainly propagating disturbance has a slower phase-
speed than a mainly evanescent disturbance; in the adiabatic limit, a mainly evanescent
disturbance propagates with infinite phase-speed, whereas a mainly propagating wave
has a finite phase-speed somewhere above the tube speed.

These considerations, then, suggest that the observations of Giovanelli et al. (1978)
may relate to longitudinal tube waves. In the remainder of this paper we explore in detail
the theoretical nature of tube waves as described by the slender flux tube equations. To
begin, though, it is convenient to discuss vertical propagation in a nonmagnetic
atmosphere (Section 2), turning afterwards (in Sections 3 to 5) to the slender flux tube.

2. Radiative Relaxation of Acoustic-Gravity Waves

We are interested in linear disturbances about a basic state of hydrostatic equilibrium,
in the absence of magnetic fields. The gas pressure in the basic state is denoted by \( p_e(z) \),
its density by \( \rho_e(z) \); and the two are related by

\[
p_e' = -\rho_e g ,
\]

where the dash refers to differentiation with respect to the z-axis (pointing vertically
upwards). We will suppose the atmosphere to be *isothermal*.

Linear perturbations \( p, \rho, \) and \( v \) in pressure, density, and velocity are taken to be
governed by (see, for example, Bray and Loughhead, 1974)

\[
\frac{\partial p^e}{\partial t} + \text{div} \rho_e v^e = 0 , \quad \rho_e \frac{\partial v^e}{\partial t} = -\nabla p^e - \rho^e g \hat{z} ,
\]

\[
\frac{\partial p^e}{\partial t} + p_e' v^e = c_e^2 \left( \frac{\partial p^e}{\partial t} + \rho_e' v^e \right) - \frac{p_e}{\tau_{Re}} \left( \frac{p_e}{\rho_e} - \frac{\rho_e^e}{\rho_e} \right) ,
\]

where \( \tau_{Re} \) is the radiative decay time of the atmosphere, \( c_e = (\gamma p_e/\rho_e)^{1/2} \) is the sound
speed, and \( v^e \) the z-component of velocity \( v^e \). The relaxation time \( \tau_{Re} \) varies rapidly
from the photosphere to the chromosphere. However, in the following we shall neglect any
gradients in \( \tau_{Re} \), and allow for its variation with height in a ‘local approximation’ only
(see, for example, Souffrin, 1972). The adiabatic index \( \gamma \) is taken to be constant (= 5/3).

The solution of (2) and (3) is particularly simple if propagation is *adiabatic* \((\tau_{Re}^{-1} = 0)\)
and one-dimensional \((v^e = v^e \hat{z})\). For then (2) and (3) yield (see Lamb, 1932, Art. 309)

\[
\frac{\partial^2 v^e}{\partial t^2} = c_e^2 \frac{\partial^2 v^e}{\partial z^2} - \gamma g \frac{\partial v^e}{\partial z} .
\]

(4)

For an isothermal atmosphere \((c_e \text{ and } \rho_e' / \rho_e \text{ constant})\), the transformation

\[
v^e(z, t) = \rho_e^{-1/2}(z) Q^e(z, t)
\]

(5)
results in
\[
\frac{\partial^2 Q^e}{\partial t^2} - c_e^2 \frac{\partial^2 Q^e}{\partial z^2} + \omega_a^2 Q^e = 0,
\]
(6)
where \(\omega_a = \gamma g / 2 c_e = c_e / 2 A_e\) is the acoustic cut-off frequency of the atmosphere. Equation (6) is of the Klein-Gordon type (see, for example, Morse and Feshbach, 1953); it arises also in the tube problem (see Section 3). For a disturbance of the form
\[Q^e \sim \exp [i(\omega t - k^e z)],\]
Equation (6) yields the dispersion relation
\[(k^e)^2 = (\omega^2 - \omega_a^2) / c_e^2,\]
(7)
showing that propagation \((k^e)^2 > 0\) occurs only for frequencies \(\omega\) above \(\omega_a\); frequencies below \(\omega_a\) result in evanescent disturbances. (See Leibacher (1977) for a clear discussion of this feature.)

Returning to (2) and (3) under nonadiabatic conditions, and restricting attention to one-dimensional propagation along the \(z\)-axis, we find, on writing \(v^e(z, t) = \psi^e(z) e^{i\omega t}\), that \(\psi^e(z)\) satisfies
\[
\frac{d^2 \psi^e}{dz^2} + \left( \frac{\rho_e^*}{\rho_e} \right) \frac{d \psi^e}{dz} + \frac{1}{c_e^2} \left( \frac{\gamma + i \omega \gamma \tau_{Re}}{1 + i \omega \gamma \tau_{Re}} \right) \psi^e = 0.
\]
(8)
A transformation of the form (5) then casts (8) into the canonical form
\[
\frac{d^2 \hat{Q}^e}{dz^2} + \left( \frac{\omega^2 - \omega_a^2}{c_e^2} \right) + \frac{(\gamma - 1)}{1 + i \omega \gamma \tau_{Re}} \left( \frac{\omega^2}{c_e^2} \right) \hat{Q}^e(z) = 0.
\]
(9)
Equation (8) governs the vertical propagation of acoustic-gravity waves in an isothermal atmosphere, under the assumption that radiative relaxation of those waves may be described by Newton’s law of cooling (see Spiegel, 1957). Equation (8), and its generalization to include propagation at an angle to the vertical, have been discussed previously by a number of authors (see, for example, Souffrin, 1972).

With
\[\hat{Q}^e \sim e^{-ik^e z},\]
Equation (8) yields the dispersion relation
\[(k^e)^2 = \frac{\omega^2 - \omega_a^2}{c_e^2} + \frac{(\gamma - 1)}{1 + i \omega \gamma \tau_{Re}} \left( \frac{\omega^2}{c_e^2} \right),\]
(10)
showing that, in general, \(k^e\) is complex with real and imaginary parts, \(k^e_\ell\) and \(k^e_\im\).

Under adiabatic conditions \((\tau_{Re})^{-1} = 0\), (10) reduces to (7) showing that only those frequencies \(\omega\) above the acoustic cut-off \(\omega_a\) propagate. Under non-adiabatic conditions, no such clear division between propagating and non-propagating disturbances exists.
However, following Souffrin (1966), we may classify disturbances as *mainly propagating* or *mainly damped* accordingly as \((k^{e})^2\) is greater or less than \((k^r)^2\). Thus, the equation \(\text{Re}[(k^e)^2] = 0\), that is
\[
\omega^2(\omega^2 - \omega_R^2) + \frac{1}{\gamma^2 \tau_{\text{Re}}} (\gamma \omega^2 - \omega_R^2) = 0 ,
\]
provides a convenient division between the two classes of disturbances.

Equations (10) and (11) provide us with the means of discussing the propagation of an acoustic-gravity wave, and comparing our results with the observations. However, it is convenient to defer discussion of their solution until we have investigated the behaviour of longitudinal tube waves, to which we now turn. In this way an easy comparison between the two modes may be made.

### 3. Propagation in a Slender Flux Tube

#### 3.1. The General Equations

We consider now the propagation of small amplitude longitudinal waves in a slender flux tube, allowing for the effects of radiative relaxation and for pressure coupling of the motions within the tube of those outside. We may formulate the theory most conveniently by first considering a general *elastic* tube, specifying to a magnetic flux tube later.

The equilibrium state of the tube is taken to be one of cross-sectional area \(A_0(z)\), pressure \(p_0(z)\), and density \(\rho_0(z)\) related by hydrostatic equilibrium:
\[
p_0 = -\rho_0 g 
\]

The linear form of the equation of continuity for longitudinal motions \(v(z, t)\), dominating over any transverse motions, in an elastic tube of cross-sectional area \(A(z, t)\) is
\[
\frac{\partial}{\partial t} (\rho A_0 + \rho_0 A) + \frac{\partial}{\partial z} (\rho_0 A_0 v) = 0 ,
\]
where \(\rho(z, t)\) is the perturbed density of the gas. The momentum equation has the linear form
\[
\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} - \rho_0 g ,
\]
for pressure perturbations \(p(z, t)\). The energy equation, allowing disturbances to relax under the effect of radiative cooling, assumed describable by Newton’s law, is taken to be
\[
\frac{\partial p}{\partial t} + p_0' v = c_0^2 \left( \frac{\partial p}{\partial t} + \rho_0' v \right) - \rho_0 c_0^2 \left( \frac{p}{\gamma \tau_R} p_0 - \frac{\rho}{\rho_0} \right) ,
\]
where $\tau_R$ is the radiative relaxation time for the atmosphere within the tube, and $c_0 = (\gamma p_0/\rho_0)^{1/2}$ is the sound speed.

To illustrate the form of (13)–(15) in the general case of an elastic tube, consider first the case of \textit{adiabatic} disturbances ($\tau_R^{-1} = 0$). Additionally, suppose that changes in the cross-section $A(z, t)$ arise as a result of changes in pressure $p(z, t)$, so that we may define a speed $c$ through the relation (see Lighthill, 1978, Chapter 2)

$$
\frac{1}{c^2} = \frac{1}{c_0^2} + \rho_0 A_0 \left( \frac{\partial A}{\partial p} \right)_{p - p_0}.
$$

(16)

Then, Equations (13)–(15) may be reduced to a single wave-like equation for $v$ and this, in turn, may be transformed by writing

$$
v(z, t) = (\rho_0 A_0 c^2)^{-1/2} Q(z, t)
$$

(17)

to yield (after considerable algebra) the equation

$$
\frac{\partial^2 Q}{\partial t^2} - c^2 \frac{\partial^2 Q}{\partial z^2} + \omega_v^2 Q = 0,
$$

(18)

where

$$
\omega_v^2 = N_0^2 + c^2 \left\{ \frac{1}{2} \left( \frac{\rho_0'}{\rho_0} + \frac{A_0'}{A_0} + \frac{c_0^2'}{c^2} \right) + \frac{1}{4} \left( \frac{\rho_0'}{\rho_0} + \frac{A_0'}{A_0} + \frac{c_0^2'}{c^2} \right)^2 + \left( \frac{g}{c_0^2} - \frac{A_0'}{A_0} \right) \frac{g}{c_0^2} + \left( \frac{g}{c_0^2} - \frac{A_0'}{A_0} \right) \left( \frac{\rho_0'}{\rho_0} + \frac{c_0^2'}{c^2} + \frac{g}{c_0^2} \right) \right\},
$$

(19)

and $N_0$, defined by

$$
N_0^2 = -g \left( \frac{\rho_0'}{\rho_0} + \frac{g}{c_0^2} \right),
$$

is the Brünt-Väisälä frequency.

Equation (18), which is of the Klein-Gordon type, describes the adiabatic propagation of longitudinal acoustic-gravity waves in a non-isothermal atmosphere contained within an elastic tube.

There are a number of special cases of the above of interest. For example, in a \textit{rigid} tube (16) yields $c = c_0$, the speed of sound. If, further, the gas is \textit{isothermal} and the tube straight, Equation (19) gives

$$
\omega_v^2 = N_0^2 + c_0^2 \left\{ \frac{1}{4} \left( \frac{\rho_0'}{\rho_0} \right)^2 + \frac{g}{c_0^2} \left( \frac{\rho_0'}{\rho_0} + \frac{g}{c_0^2} \right) \right\} =
$$

$$
= \frac{c_0^2}{4A_0^2},
$$
where $A_0$ is the scale-height of the atmosphere within the tube. Thus $\omega_c = \omega_a$, the cut-off frequency for vertically propagating acoustic-gravity waves, and Equation (18) reduces to (6) discussed earlier.

3.2. Adiabatic propagation in an isothermal tube

A second illustration of (18) may be given for a gas in a magnetic flux tube. In this case, to relate changes in cross-sectional area of the tube to changes in gas pressure, we take the transverse component of the momentum equation to be

$$p + \frac{B_0}{\mu} B = \pi_e,$$

where $\pi_e(z, t)$ is the external perturbed pressure field on the boundary of the tube and $B(z, t)$ is the linear perturbation in the longitudinal induction field. The field $B$ is related to changes in cross-sectional area by flux conservation, so that

$$\frac{1}{B_0} \frac{\partial B}{\partial t} = -\frac{1}{A_0} \frac{\partial A}{\partial t}.$$

The boundary term $\pi_e(z, t)$ complicates the discussion of waves in a tube since it incorporates of number of dissimilar effects arising from the presence of an external atmosphere. To simplify our discussion, we will first suppose (in keeping with Defouw (1976), Roberts and Webb (1978), Parker (1979), and Spruit (1981)) that $\pi_e$ may be neglected. Some aspects of including $\pi_e$ will be discussed in Section 4.

With $\pi_e$ neglected, (20) and (21) allow us to relate $A$ and $p$, and thus, through (16), to calculate $c$. We find that $c = c_T$, where the longitudinal tube speed $c_T$ is given by

$$c_T^2 = c_0^2 v_A^2/(c_0^2 + v_A^2)$$

for Alfvén speed $v_A = B_0/\mu \rho_0^{1/2}$. Thus, the characteristic wave speed for a longitudinal disturbance in a magnetic flux tube is reduced from the sound speed $c_0$ to the tube speed $c_T$, a result due to the increase in effective compressibility of the gas within the field. (Of course, if the magnetic pressure greatly exceeds the gas pressure in the tube, then $v_A \gg c_0$ and so $c_T = c_0$.)

The frequency $\omega_a$, given in general by (19), simplifies considerably in an isothermal atmosphere. With the isothermal tube in temperature balance with its surroundings, so that the scale-height $A_0 = p_0/\rho_0 g$ is constant and equal to the scale-height in the external atmosphere, the equilibrium state of a flux tube is simply

$$p_0(z) = p_0(0) e^{-z/A_0}, \quad \rho_0(z) = \rho_0(0) e^{-z/A_0},$$

$$B_0(z) = B_0(0) e^{-z/2A_0}, \quad A_0(z) = A_0(0) e^{z/2A_0},$$

for arbitrary reference level $z = 0$, chosen to be at optical depth unity in the solar atmosphere. Equation (23) shows that the sound, Alfvén and tube speeds are all
constant. Under these circumstances, (19) yields

\[ \omega^2 = N_0^2 + c_T^2 \left( \frac{1}{4} \left( \frac{\rho_0}{\rho_0} + \frac{A_0'}{A_0} \right)^2 + \left( \frac{g}{c_T^2} - \frac{A_0}{A_0} \right) \left( \frac{\rho_0}{\rho_0} + \frac{g}{c_T^2} \right) \right) = \]

\[ = N_0^2 + \left( \frac{3}{4} - \frac{1}{\gamma} \right) \frac{c_T^2}{A_0^2}. \]

(24)

Thus, longitudinal waves in an isothermal slender magnetic flux tube grow in amplitude like (see (17)) \((\rho_0 A_0)^{-1/2}\), and so e-fold in four scale-heights (for the basic state (23)) in preserving constancy of energy flux \(\rho_0 A_0 |v|^2\). (In contrast, as is well known, a vertically propagating acoustic-gravity wave grows like \(\rho_0^{-1/2}\), and so e-folds in two scale-heights in preserving constancy of energy density \(1/2 \rho_0 |v|^2\).)

For the isothermal atmosphere, \(\omega_c\) is constant and so normal modes of (18) may be found by writing

\[ Q \sim \exp[i(\omega t - kz)], \]

(25)

and then the frequency \(\omega\) and wavenumber \(k\) are related by the dispersion relation

\[ \omega^2 = k^2 c_T^2 + \omega_c^2, \]

(26)

the general form of which has been discussed in the introduction.

As a numerical illustration of (26), we may set \(c_0 = v_A\), \(\gamma = \frac{5}{3}\) and \(A_0 = 125\) km, all typical of flux tube conditions in the region from the photosphere to the temperature minimum. Then \(\omega_c\) is about 0.03 s\(^{-1}\) (with corresponding period of 210 s), and so a wave of period 300 s (as observed by Giovanelli et al., 1978) is non-propagating under adiabatic conditions. However, as discussed earlier, radiative relaxation allows propagation at all frequencies. We turn, then, to considering the effect of radiative relaxation on the longitudinal tube wave.

3.3. NONADIABATIC PROPAGATION

If conditions are no longer close to adiabatic, then we must consider Equations (13) to (15) and (20) to (23) with \(\tau_R\) finite. For a tube with an isothermal atmosphere, equal to the temperature outside the tube, these equations yield a single ordinary differential equation for the spatial dependence \(\hat{v}(z)\) of the perturbation velocity \(v = \hat{v}(z) e^{i\omega t}\). As in the adiabatic case, it is convenient to introduce \(\hat{Q}(z)\) by writing

\[ \hat{v}(z) \sim (\rho_0 A_0)^{-1/2} \hat{Q}(z) \sim e^{z/A_0} \hat{Q}(z). \]

(27)

Then, retaining the assumption that \(\pi_e\) is negligible, we find that \(\hat{Q}(z)\) satisfies

\[ \hat{Q}'' + (a + ib)\hat{Q} = 0, \]

(28)

where

\[ a = \frac{\omega^2 - \omega_c^2}{c_T^2} + \frac{1}{A_0^2(1 + \omega^2 \gamma^2 c_T^2)} \left( \frac{\gamma - 1}{\gamma} \right) \left( \frac{\omega^2 A_0}{g} + \frac{c_T^2}{\gamma v_A^2} + \frac{1}{2} \right), \]
\[
b = - \frac{1}{A_0^2} \left( \frac{\gamma - 1}{\gamma} \right) \frac{\gamma \omega \tau_R}{1 + \gamma^2 \omega^2 \tau_R^2} \left( \frac{\omega^2 A_0}{g} + \frac{c_s^2}{\gamma v_A^2} + \frac{1}{2} \right),
\]

with \( \omega_c \) defined in (24).

With \( \hat{Q}(z) \sim e^{-ikz} \), for complex wavenumber \( k = k_r + ik_i \), Equation (28) yields the dispersion relation (Webb and Roberts, 1980)

\[
k^2 = a + ib.
\]

(29)

Just as in the case of a field-free atmosphere (Section 2), we may broadly divide disturbances into \textit{mainly propagating} or \textit{mainly evanescent} modes by setting \( a = 0 \) in (29), with the result that

\[
\omega^2 (\omega^2 - \omega_c^2) + \frac{1}{\gamma^2 \tau_R^2} \left( \omega^2 + (\gamma - 1)\omega^2 \frac{c_T^2}{c_s^2} - \frac{c_T^2}{16A_0^2} \right) = 0.
\]

(30)

Equations (29) and (30) should be compared with their field-free counterparts, (10) and (11). The differences arise principally as a result of the tube's geometry and field strength.

The positive solution \( \omega^2 = \omega_{aT}^2 \) of (30) provides the broad division of disturbances (into mainly propagating or mainly evanescent) that we seek. The form of \( \omega_{aT} \) is shown in Figure 1, together with the field-free conterpart \( \omega_{aT} \) (the positive solution of (11)). The adiabatic (\( \tau_R, \tau_{Re} \to \infty \)) cut-offs, \( \omega_c \) and \( \omega_a \), are also shown for comparison.

![Fig. 1. The broad-band division between mainly propagating and mainly evanescent waves for the tube, \( \omega_{aT} \) (-----) and for vertically propagating acoustic-gravity waves in the field-free environment, \( \omega_{aT} \) (---). The adiabatic values \( \omega_c \) and \( \omega_a \) (\( \omega_c \approx \omega_a \)) are also shown (--- ...) and merge into \( \omega_c \) and \( \omega_a \) at about \( z = 400 \) km above \( z = 0 \) (\( \tau_{5000} = 1 \)). We have taken \( \gamma = \frac{\beta}{\alpha} \) and \( c_0 = c_s = v_A \).](image-url)
In illustrating our results we have used local values of the scale-height (taken from HSRA) and the radiative decay time $\tau_{Re}$ (given in Ulmschneider, 1971; see also Bray and Loughhead, 1974). The adiabatic index is $\gamma = \frac{5}{3}$, and $z = 0$ refers to $\tau_{5000} = 1$ in a field-free atmosphere. For the radiative decay time in the tube we have followed Webb and Roberts (1980) and set $\tau_R = (\rho_e / \rho_0)^{1/2} \tau_{Re}$.

The generalized cut-offs $\omega_{\nu \tau}$ and $\omega_{a \tau}$, illustrated in Figure 1 for the case $c_0 = v_A = c_e$, should not be interpreted as sharp boundaries between propagation and nonpropagation, as is the case with their adiabatic values $\omega_{\nu}$ and $\omega_{a}$, but as broad-banded divisions between mainly propagating and mainly evanescent frequencies. It is clear from the figure that $\omega_{\nu \tau}$ and $\omega_{a \tau}$ differ only slightly from their adiabatic values in the height range $z = 400$ km to $1000$ km*. However, below this level conditions are almost isothermal and $\omega_{\nu \tau}$ and $\omega_{a \tau}$ depart strongly from their adiabatic values, and also strongly from one another. It is this sharp departure of $\omega_{\nu \tau}$ and $\omega_{a \tau}$ in the lower atmosphere that there gives rise to markedly different propagation speeds for a tube wave and an acoustic-gravity wave.

![Graph](image)

Fig. 2. The damping per scale-height, $A_0 |k_i|$, of a tube wave (——) as a function of height $z$. We have taken $c_0 = v_A$ and $\omega = 0.02$ s$^{-1}$. Also shown (--------) is the damping per scale-height, $A_e |k_e^f|$, for an acoustic-gravity wave (with $A_e = A_0$).

The dispersion relation (29) and its field-free counterpart (10) are illustrated in Figures 2 and 3, where we have sketched the behaviour with height of $k_i$, $k_e^f$, and the phase-speeds $\omega/k_i$ and $\omega/k_e^f$, all for $\omega = 0.02$ s$^{-1}$. Again, the results for the tube and those for the environment show a marked contrast.

Finally, in Figure 4 we have sketched the transit times for a tube wave and an acoustic-gravity wave in propagating from $z = 0$ to a height $z$, the transit time in the tube

* It should be noted that the notion of an isolated flux tube is dubious at this high level.
Fig. 3. The phase-speeds, $\omega/k_z$ and $\omega/k_\zeta$, of a tube wave (-----) and an acoustic-gravity wave (------).
We have taken $\omega = 0.02$ s and $c_0 = v_A = c_c$.

Fig. 4. The transit time for a disturbance to propagate to a height $z$ from $z = 0$, illustrated for $\omega = 0.02$ s$^{-1}$ and $c_0 = v_A$. -----: tube wave; ------: acoustic-gravity wave.
being the time a tube wave takes to traverse the distance $z$ at a speed $\omega/k_r$ (or $\omega/k_c^e$ for an acoustic-gravity wave). Also, in Figure 5, we sketch the (relative) velocity amplitudes $v(z)/v(0)$ for the tube and $v^e(z)/v^e(0)$ for the field-free medium.

How do these results compare with the observations? The observations, it will be recalled, indicate a transit time of 19 s for a disturbance within a flux tube to travel from the level where Fe 5166 Å is formed to that where Mg 5183 Å arises. In a non-magnetic atmosphere it is estimated (see Giovanelli et al., 1978) that such lines are formed at heights of 270 and 835 km, respectively, above $\tau_{5000} = 1$. The heights of formation in a flux tube are, unfortunately, uncertain. However, for roughly similar formation heights, it is apparent from Figure 4 that transit times of 19 s are consistent with a lower level of formation of Fe 5166 at about $z = 200$ km*; the transit times are not sensitive to precise levels of formation higher in the atmosphere, since tube disturbances are slowest in the lower atmosphere. Outside the tube, an observed time of 7 s is seen to be in general accord with the curve for acoustic-gravity waves given in Figure 4 (though, again, a lower formation height of Fe 5166 than the level 270 km given by Giovanelli et al. (1978) is suggested*).

Turning to the amplitude factors, we see that for the external wave Figure 5 gives a relative growth of 1.6, compared with 1.4 estimated from the observations. Thus, good agreement exists for the acoustic-gravity wave. However, for the tube wave the

* We have made no attempt to take into account temperature differences between the tube and its environment, so these differences in height should be treated cautiously.
observations give a relative growth of 1.8, whereas the theory predicts a decline in height. In this one respect, then, the theory and observations are in conflict.

There is, however, an effect, omitted in the above theory, which may be responsible for this discrepancy: the interaction of a tube wave with waves in the external medium, an effect neglected by assuming $\pi_e = 0$. We therefore turn to a consideration of the term $\pi_e$.

4. The Effect of External Disturbances

Confining attention to an isothermal tube, but retaining a non-zero $\pi_e$ in (20), we may show (after some detailed algebra) that $Q$ and $\pi_e$ are related by

$$
\dot{Q}'' + (a + ib)\dot{Q} = \frac{i\omega e^{-(z/4A_0)}}{\rho_0(z)v_A^2} \left\{ \pi_e + \frac{\gamma g}{c_0^2} \left( 1 + i\omega \tau_R \right) \pi_e \right\},
$$

with $v$ and $Q$ related as before (see (27)).

A complete discussion of (31) would have to be accompanied by a full investigation of the equations in the environment that govern $\pi_e$, together with the velocity coupling between the tube and the surroundings. Such a discussion would be involved and is postponed for a future investigation. However, there are a number of observations that can be made. If we ignore stratification then the coupling between the tube and its environment can be fully described. In particular, the propagation of the longitudinal tube wave is rendered dispersive by the inertia of the tube's surroundings. This dispersive effect can be balanced by nonlinearities, with a soliton resulting (Roberts and Mangeney, 1982). It is clear too that the tube can act as an acoustic radiator, with sound waves being generated on the oscillating boundary of the tube and then propagating out to infinity (see Spruit, 1982; Roberts and Webb, 1979). Under these circumstances, tube waves lose energy to their surroundings but the damping rate is small (of order $k^2A_0$) for a thin tube (Spruit, 1982).

Motions generated in the environment may also modulate those in the tube. We may illustrate such an effect in the following simple way. Suppose that the motions in the environment are in the form of acoustic-gravity waves, propagating vertically according to the description in Section 2. (We are ignoring the slight influence of the tube on those motions.) Then, with $\pi_e$ thus specified, we may regard (31) as a differential equation with a forcing term. The solution of (31) will thus be made up of two contributions, the pure tube mode (as calculated in Section 3) and an additional term due to the forcing. For an outgoing wave, those terms combine (on using (27) and (31)) to give

$$
v(z, t) \sim \exp \left[ \left( \frac{1}{4A_0} + k_l \right) z \right] e^{i(\omega t-k_lz)} + \alpha \exp \left[ \left( \frac{1}{2A_0} + k_l^e \right) z \right] e^{i(\omega t-k_l^ez)},
$$

where the (complex) constant $\alpha$ depends upon the precise details of the interaction (not evaluated here).

The point we wish to make is that the motions in the environment (the second term
in (32)) will modify the amplitude of the tube mode. In fact, we see that

\[ |v(z, t)| \leq \exp \left( \frac{1}{4A_0} + k_z \right) z \right) + |x| \exp \left( \frac{1}{2A_0} + k_x \right) z \right), \]

(33)

and so, even though the first term declines with height (as found earlier), the velocity amplitude \(|v|\) may actually increase with height, depending upon the value of \(|x|\). We have illustrated this behaviour in Figure 5 where we have sketched the right-hand side of (33), normalized to unity at \(z = 0\), for \(|x| = 1\). It is apparent that the influence of an external wave, modulating the behaviour of the 'pure' tube wave, may bring about the observed amplitude increase with height.

5. Concluding Remarks

The observations of Giovanelli et al. (1978) have established the existence of outwardly propagating waves in intense magnetic flux tubes. These observations, however, present a puzzle in that they indicate a propagation time over a height range of some 600 km of about 19 s inside the tube compared with only 7 s outside the tube. Why does such a large difference arise?

It is clear that physical conditions inside a flux tube are different from those in the tube’s environment, and such differences, in density and temperature, presumably lead to differences in line formation heights. But, as Giovanelli et al. (1978) point out, it is unlikely that such conditions are so extreme as to result in a 12 s time difference. Admitting the existence of differences in formation heights, we argue instead that the main component of the 12 s time difference arises from the character of acoustic-gravity waves in a magnetic flux tube as compared with their field-free counterparts.

Assuming for simplicity equal and isothermal atmospheres within and without a slender tube, we have discussed how, under adiabatic conditions, the geometry and elasticity of a magnetic flux tube give rise to a cut-off \(\omega_v\), analogous to the cut-off \(\omega_a\) for vertically propagating acoustic-gravity waves in a field-free medium:

\[ \omega_v = \left( \frac{9}{16} - \frac{1}{2\gamma} + \left( \frac{\gamma - 1}{\gamma^2} \right) \left( \frac{c_0^2}{v_A^2} \right) \right)^{1/2} \frac{c_T}{A_0}, \quad \omega_a = \frac{c_0}{2A_0}. \]

The presence of these cut-offs implies that only frequencies above the out-off propagate under adiabatic conditions; frequencies below cut-off are evanescent. Now in an intense tube, with \(c_0 = v_A\) and \(\gamma = \frac{5}{3}\), \(\omega_v\) and \(\omega_a\) are almost equal and for a scale-height of 125 km have the value 0.03 s\(^{-1}\), corresponding to a period of 210 s. Only those waves — be they acoustic-gravity modes in a field-free medium of tube waves — of period below 210 s propagate from the photosphere to the temperature minimum. (Non-isothermality changes \(\omega_v\), somewhat, and results in a minimum critical period of some 195 s (Roberts and Webb, 1978).) Thus, a wave of period of 300 s cannot propagate (under adiabatic conditions) from the photosphere to the temperature minimum, while one of 180 s can.
Now radiative relaxation of temperature perturbations is known to be important in the photosphere and low chromosphere and, when included in the above description, leads to the conclusion that all frequencies propagate. However, one may still classify disturbances as mainly propagating or mainly evanescent (Webb and Roberts, 1980), and this gives rise to the critical frequencies \( \omega_{\nu \tau} \) and \( \omega_{\alpha \tau} \), generalizing the adiabatic frequencies \( \omega_c \) and \( \omega_a \) to include radiative effects. Such a broad-band division, found from (11) for acoustic-gravity waves and (30) for tube waves, is illustrated in Figure 1. It is immediately apparent from this figure that in the region of the temperature minimum, where conditions are almost adiabatic, \( \omega_{\nu \tau} \) and \( \omega_{\alpha \tau} \) approach their adiabatic values which, as we have remarked earlier, are almost equal. However, in the other extreme, of almost isothermal conditions, arising at the photosphere, the values of \( \omega_{\nu \tau} \) and \( \omega_{\alpha \tau} \) are quite distinct. Indeed, under almost isothermal conditions we have

\[
\omega_{\nu \tau} \simeq \bar{c}_T/4A_0, \quad \omega_{\alpha \tau} \simeq \bar{c}_0/2A_0,
\]

where \( \bar{c}_T \) and \( \bar{c}_0 \) denote isothermal (sound speed replaced by its isothermal value) values of the tube speed and sound speed. For example, with \( A_0 = 150 \text{ km} \) we find \( \omega_{\alpha \tau} = 0.021 \text{ s}^{-1} \) (with corresponding period of 294 s). In a tube with Alfvén and adiabatic sound speeds equal (\( v_A^2 = c_0^2 = \gamma g A_0 \)), \( \bar{c}_T = 5 \text{ km s}^{-1} \) and so \( \omega_{\nu \tau} = 0.008 \text{ s}^{-1} \) (with corresponding period 784 s), a difference of almost a factor of three.

In general terms, then, a wave of frequency \( \omega \) may be mainly non-propagating outside the tube but mainly propagating inside the tube. A mainly non-propagating disturbance has large phase-speed (an evanescent mode has ‘infinite’ phase-speed), and so we expect a disturbance to propagate slower (and so take longer to travel up the atmosphere) inside the tube than in the field-free medium.

In qualitative terms this is in general agreement with the observations of Giovanelli et al. (1978). A more quantitative comparison may be made using Figures 4 and 5, which show both the external transit time of 7 s and the tube transit time of 19 s are consistent with the calculations presented here (though a slightly lower formation height of Fe 5166 than the level 270 km given by Giovanelli et al. (1978) is suggested). The observed amplitude growth would seem to require an interaction of a tube wave with an external (acoustic-gravity) wave for agreement with our calculations to be possible.

A more detailed comparison of theory and observations than we have so far presented would, at this stage, be over-ambitious: there are simply too many uncertainties in both the observations and the theory. But the general level of accordance that we have indicated argues well for such a programme at some future date when further observations become available.

From the present studies, however, it seems reasonable to conclude that the outwardly propagating disturbances observed by Giovanelli et al. (1978) are the tube modes described in this paper. These modes are non-adiabatic, acoustic-gravity waves propagating along a slender magnetic flux tube. The tube both guides and, through its elasticity, slows down the waves, which are further modulated by acoustic-gravity waves propagating in the field-free environment of the tube.
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