SIMPLE MODELS FOR MAGNETIC FLUX TUBES

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Abstract. Known potential field solutions can be used to model the structure of magnetic fields in the solar photosphere. Several two-dimensional and axisymmetric solutions are compared. In the most satisfactory model the vertical component of the field is prescribed on a horizontal plane so as to be uniform within a finite disc and zero outside it. The resulting flux distribution provides a good description of small scale intergranular magnetic fields and of the observed field structure in a pore, but is inadequate for sunspots.

1. Introduction

Magnetic fields in the solar atmosphere are spatially intermittent: flux is confined to isolated tubes with fields that are locally intense. These tubes emerge as sunspots or pores or as tiny unipolar features, which have only recently been resolved (Ramsey et al., 1977; Wilson and Simon, 1983; Title, 1982; Tarbell, 1983). Within these features, the magnetic energy density approaches the thermal energy density of the external gas. Such strong fields must be approximately force-free and models with idealized geometry should therefore have potential fields. The simplest models have axial symmetry, with fields maintained by currents flowing at the surface of the flux tube, where the magnetic pressure balances the pressure of the external field-free gas (Parker, 1979; Spruit, 1981). Appropriate solutions can readily be obtained in the thin flux tube approximation (Meyer et al., 1977), and Schmidt and Wegmann (1983) have recently developed a general procedure for calculating fields with an arbitrary external pressure distribution. We shall, however, follow an alternative approach, and use known potential field solutions to represent the magnetic fields in pores and smaller magnetic features (Newkirk and Harvey, 1968; Simon and Weiss, 1970; Spruit, 1976; Meyer et al., 1977).

In this paper we extend the work of Simon and Weiss (1970, referred to as Paper I) to provide more realistic models of the fields in photospheric flux tubes, and compare the results with observations. These models fall into two classes. In the first, the field lines emerging from a flux tube expand to fill a half-space and behave like a monopole field at infinity. We discuss such models in the next section and investigate the rate at

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which the flux spreads out with height above the photosphere. The second class of models have fields that are contained within isolated flux tubes, which spread out with increasing height. In Section 3 we show that such models, with the trial solution introduced by Spruit (1976), provide an excellent representation of the field in a pore, though they fail to describe a sunspot. Finally, we indicate how these field models might be elaborated and improved.

2. Model Solutions

Consider a magnetic field \( B \) whose vertical component is prescribed on the horizontal surface \( z = 0 \). If the field in the half-space \( z > 0 \) is everywhere current-free, so that curl \( B = 0 \), we may introduce a potential \( \Phi \) such that \( B = -\nabla \Phi \) and

\[
\Phi(\mathbf{r}) = \frac{1}{2\pi} \iint \frac{B_z(r_0)}{|\mathbf{r} - \mathbf{r}_0|} \, dx_0 \, dy_0 ,
\]

(1)

where \( \mathbf{r}_0 = (x_0, y_0, 0) \) (Schmidt, 1964). We wish to investigate the field structures above flux concentrations associated, for example, with granular or supergranular convection. Thus we model behaviour at the periphery of a convection cell by considering isolated sheets of flux and at the corners by considering isolated tubes.

For two-dimensional models we take cartesian co-ordinates \((x, y, z)\) such that \( \partial B/\partial y = 0 \). Then

\[
\mathbf{B} = (B_x, 0, B_z) = -\left( \frac{\partial \Phi}{\partial x}, 0, \frac{\partial \Phi}{\partial z} \right) = \left( -\frac{\partial \psi}{\partial x}, 0, \frac{\partial \psi}{\partial z} \right),
\]

(2)

where \( \psi \) is the magnetic flux function. At a great distance from the origin the field resembles that of a monopole, with

\[
\Phi = -\frac{1}{2} \ln (x^2 + z^2) , \quad \psi = \arctan(x/z)
\]

(3)

and

\[
B_z(x, z) = z/(x^2 + z^2).
\]

(4)

Figure 1(a) shows the vertical component, \( B_z \), of the magnetic field as a function of \( x \) for different values of \( z \). The field is normalized so that \( B_z(0, 1) = 1 \); at the centre of the flux sheet, \( B_z \) is inversely proportional to the height.

In order to model discrete concentrations of flux, we first consider fields with a gaussian profile, so that

\[
B_z(x, 0) = e^{-(1/2)x^2}.
\]

(5)

Then

\[
\psi(x, z) = \left( \frac{2}{\pi} \right)^{1/2} \int_0^\infty k^{-1} e^{-(1/2)k^2} \sin kx \, e^{-(1/2)k^2} \, dk
\]

(6)
Fig. 1. Vertical magnetic field distributions in two-dimensional cartesian coordinates \((x, z)\) and in axisymmetric cylindrical coordinates \((r, z)\) for potential field models: monopole (a and b), gaussian (c and d), and step function (e and f).
and

\[
B_z(x, z) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty e^{-(1/2)k^2} \cos kx e^{-kz} \, dk.
\]  

Profiles of \(B_z\), given by (7), are shown in Figure 1(c); they are initially flatter than those for a monopole field but the two sets become indistinguishable as \(z\) increases. A more realistic field distribution is obtained by taking a step function rather than a gaussian, so that

\[
B_z(x, 0) = \begin{cases} 
1 & (|x| < 1) \\
0 & (|x| > 1) \end{cases}.
\]  

Then

\[
\psi(x, z) = \frac{2}{\pi} \int_0^\infty k^{-2} \sin k \sin kx \, e^{-kz} \, dk
\]

\[
= \frac{1}{\pi} \left[ \tan^{-1}\left(\frac{2xz}{z^2 - x^2 + 1}\right) + x \tan^{-1}\left(\frac{2z}{z^2 + x^2 - 1}\right) + \frac{x}{2} \ln\left(\frac{z^2 + (1-x)^2}{z^2 + (1+x)^2}\right) \right]
\]  

and

\[
B_z(x, z) = \frac{2}{\pi} \int_0^\infty k^{-1} \sin k \cos kx \, e^{-kz} \, dk
\]

\[
= \frac{1}{\pi} \tan^{-1}\left[2z/(x^2 + z^2 - 1)\right]
\]  

(Gradshteyn and Ryzhik, 1965). The profiles shown in Figure 1(e) are initially flatter than those in Figure 1(c), and might be appropriate for describing magnetic fields in the chromospheric network.

For axisymmetric models we take cylindrical polar co-ordinates \((r, \phi, z)\) with \(\partial B_r/\partial \phi = \partial B_z/\partial \phi = 0\) and

\[
\mathbf{B} = (B_r, 0, B_z) = -\left(\frac{\partial \Phi}{\partial r}, 0, \frac{\partial \Phi}{\partial z}\right) = \frac{1}{r} \left( -\frac{\partial \psi}{\partial z}, 0, \frac{\partial \psi}{\partial r} \right).
\]  

For a monopole field,

\[
\Phi = (r^2 + z^2)^{-1/2}, \quad \psi = -z(r^2 + z^2)^{-1/2}
\]
and

$$B_z(r, z) = z(r^2 + z^2)^{-3/2},$$  \hspace{1cm} (13)

so that $B_z$ falls off inversely as $z^2$ on the axis. With a gaussian field profile, we have

$$B_z(r, 0) = e^{-(1/2)r^2},$$  \hspace{1cm} (14)

so that

$$\psi(r, z) = \int_0^\infty e^{-(1/2)k^2} r J_1(kr) e^{-kz} \, dk$$  \hspace{1cm} (15)

and

$$B_z(r, z) = \int_0^\infty k e^{-(1/2)k^2} J_0(kr) e^{-kz} \, dk,$$  \hspace{1cm} (16)

where $J_n(kr)$ is a Bessel function of order $n$. Finally, for an axisymmetric step function with

$$B_z(r, 0) = \begin{cases} 1 & (r < 1) \\ 0 & (r > 1) \end{cases},$$  \hspace{1cm} (17)

we obtain

$$\psi(r, z) = \int_0^\infty k^{-1} J_1(k) r J_1(kr) e^{-kz} \, dk$$  \hspace{1cm} (18)

and

$$B_z(r, z) = \int_0^\infty J_1(k) J_0(kr) e^{-kz} \, dk.$$  \hspace{1cm} (19)

It can be shown that, on the axis, (19) reduces to

$$B_z(0, z) = 1 - z(1 + z^2)^{-1/2}$$  \hspace{1cm} (20)

(Gradshteyn and Ryzhik, 1965). This field is identical to that used by Spruit (1976, Equation (12)). Profiles of $B_z$ derived from (13), (16), and (19) are shown in Figures 1(b), (d), and (f).

Comparing the results illustrated in Figure 1, we see that the axisymmetric fields fall off more rapidly than the twodimensional fields, since flux can spread out over two dimensions. The step-function field profiles remain flatter than the others, and provide a better approximation to observed fields above pores and isolated structures. At any level $z$, we may define the width of the flux concentration in terms of the radius $\bar{R}(z)$ such that $B_z(\bar{R}, z) = \frac{1}{2}B_z(0, z)$. From Figure 1(f) it can be seen that $\bar{R}$ increases fairly slowly with $z$; thus $\bar{R} \approx 1$ for $z \leq 0.5$, rising to $\bar{R}(2) \approx 1.5$ and $\bar{R}(4) \approx 3$. Hence the fields above isolated flux tubes do not spread out very rapidly with height and, as they
approach the field of a monopole, \( \bar{R}/z \rightarrow (2^{2/3} - 1)^{1/2} \approx 0.77 \), from (13). (For two-dimensional fields, \( \bar{R}/z \rightarrow 1 \), from (4).)

In order to relate these dimensionless results more directly to observable quantities, we confine our attention to the axisymmetric step-function of Figure 1(f) and suppose that at the photosphere the vertical field within the flux tube is uniform, with a strength \( B_0 = 2500 \) G. Fields are measured using the Zeeman effect in lines (e.g. \( \lambda 5250 \)) formed several hundred km higher in the atmosphere. Figure 2 shows the behaviour of the field at a height \( z = 250 \) km for different values of the total flux \( F \), measured in terawebers (1 TWb = \( 10^{20} \) Mx). \( B_z \) falls off with radius but \( B_r \) increases, so that the total field \( B = (B_r^2 + B_z^2)^{1/2} \) remains almost constant out to the radius \( R_0 = (F/\pi B_0)^{1/2} \) of the flux concentration. Indeed, it can be seen that as \( F \) increases, the radius at which \( B \) has its maximum moves from \( r = 0 \) to \( r \approx R_0 \). This behaviour contrasts markedly with that of the simple Bessel function model illustrated in Figure 3 of Paper I, where \( B \) falls rapidly with \( r \).

**Fig. 2.** Magnetic field profiles computed for a height 250 km above an axisymmetric step function of radius \( R_0 \) and field strength \( B_0 = 2500 \) G, for four values of the total flux \( F \): (a) 0.5 TWb, (b) 1.0 TWb, (c) 2.0 TWb, (d) 4.0 TWb. The four curves in each figure show the vertical \( (B_z) \), radial \( (B_r) \), and total \( (B) \) magnetic field, and the percent of the total flux contained within any radius; the arrows indicate the direction and magnitude of \( B \) at various radii.

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These results confirm that the geometrical spreading of flux tubes with height is not very rapid. Hence different techniques for detecting magnetic fields (Wilson, 1981) ought to give similar estimates for the diameters of small magnetic features. On the other hand, adjacent flux tubes will merge with each other if the spacing between them is comparable with their radii, producing fewer, larger features. This may explain the relatively large features found by Simon and Zirker (1974).

3. Magnetic Fields in Pores

In order to represent the magnetic fields in small photospheric features or in pores it is convenient to consider an isolated flux tube bounded by the surface $\psi = F/2\pi$, where $F$ is the total flux within the tube. Within the tube, the field is assumed to be given by some known potential field distribution, with parameters adjusted to match conditions at or near the photosphere. In Paper I we used a separable Bessel function solution to represent the field in a pore; subsequently, Spruit (1976) introduced the solution (18)

Fig. 3. Field lines for Bessel and step-function models with pressure balancing at the photosphere (represented by horizontal line at $z = 0$) and at $z = -100$ km, for four values of the flux from 0.01 TWb up to 2.56 TWb, the limit for the Bessel model. Each set of three lines (0.1F, 0.5F, 1.0F) represents field lines on flux tubes containing 10%, 50%, and 100% of the total flux. The step function is located 400 km below the photosphere.
to describe the structure of small photospheric flux tubes. We show here that the latter solution provides a much better approximation to the observed field profile in a pore (Stesenko, 1967).

We suppose that there is a uniform vertical field, of magnitude \(B_0\), over a disc of radius \(R^*\) at a height \(z = z_2 < z_0\) (where \(z = z_0\) at the photosphere) so that

\[
B_z(r, z) = B_0 R^* \int_0^\infty J_1(kR^*)J_0(kr) e^{-k(z-z_2)} \, dk ,
\]

\[
\psi(r, z) = B_0 R^* r \int_0^\infty k^{-1} J_1(kR^*)J_1(kr) e^{-k(z-z_2)} \, dk .
\]

(21)

The radius of the flux tube is given by \(R(z)\), where \(\psi(R, z) = F/2\pi\) and \(R_2 \equiv R(z_2) = (F/\pi B_0)^{1/2} < R^*\). We require that the magnetic pressure within the flux tube should balance the external pressure at two levels, \(z = z_0, z_1\); for given values of \(z_0, z_1, z_2\) the model is then uniquely determined.

Following the procedure of Paper I, we set \(z_0 = 0\) (photosphere), \(z_1 = -100\) km and represent the external pressure by the expression given in Equation (4) of Paper I, which is sufficiently accurate for our purposes. Values of \(B_0\) and \(R^*\) were obtained numerically, by an iterative method. Figure 3 shows a comparison between the Bessel function model of Paper I and yields derived from (21), with \(z_2 = -400\) km. In the Bessel function model the field becomes horizontal at some finite radius and the field lines therefore spread out more rapidly with height, eventually bending over and re-entering the surface. For small fluxes (e.g. \(F = 0.01\) T WB) the two models are indistinguishable for \(z \leq 250\) km, but as \(F\) increases the difference becomes more apparent. For \(F \approx 2.56\) T WB, the Bessel function model gives a field such that \(B_z(R, 0) = 0\), i.e. the field at the edge of the flux tube becomes horizontal at the photosphere. This is the greatest flux that can be represented by the model of Paper I. However, the step function of (21) can be used to describe any desired flux, since the field lines continue to rise through the entire half-space.

The best available measurements of the variation of the magnetic field across a pore are still those of Stesenko (1967). He observed a large pore with a flux \(F \approx 1.96\) T WB, a central field of 2246 G and a radius \(R(0) = 2030\) km. In Figures 4(a), (b), and (c) we show a comparison between his observations, the Bessel function model of Paper I and the model defined in (21), for \(z_2 = -500, -400, \) and \(-150\) km, respectively. Field strengths are given at the photospheric level (\(z = 0\)). The new model has a flatter profile than the old and for \(z_2 = -400\) km the fit is extremely good.

Spruit (1976) employed the field given by (21) to produce an extensive series of models that successfully described small photospheric flux tubes and we confirm that this model can also be used quite successfully for pores. There is, however, a limit to its validity. Several observers (Beckers and Schröter, 1969; Wittmann, 1974; Gurman and House, 1981) have measured field profiles across sunspots. In Figure 4(d) we show a comparison...
Fig. 4. In (a), (b), and (c) we compare the success of our earlier Bessel function and the new step function in modelling Steshenko's observations of a small pore. The only parameter varied in $z_2$, the depth where the step occurs. The curve labeled 'Steshenko fit' is a least-squares fit to his data. In (d) we show our attempt to model the field of a sunspot observed by Beckers and Schröter. By choosing the right step function depth (1512 km) we can match the central photospheric field strength (2550 G), but the radial dependence of the field is not well represented.

between the field measured by Beckers and Schröter (1969) in a small sunspot ($F \approx 94$ TWb) and that computed from our model. As can be seen, the model yields too flat a profile, with a field that does not fall off rapidly enough in the penumbra. We have investigated other trial solutions, with a ring of reversed polarity around a central spot so as to produce fields that return below the photosphere, but have not found a potential field model that matches the observed profile in sunspots.

This discrepancy is not surprising. All models give photospheric fields $\mathbf{B}(R, z_0)$ that become more steeply inclined to the vertical as $F$ is increased. (The separable Bessel function solution of Paper I is an extreme case, for field lines must re-enter at some radius.) When $F$ is sufficiently large the field structure is affected by the gas pressure within the flux tube. Schmidt and Wegmann (private communication) have constructed potential field models with an internal pressure $p_\gamma(z)$ that increases downwards, so that the pressure jump is modified, and they obtain fields that fit the sunspot observations quite well.
4. Conclusion

We have shown that Equation (21) can be used to provide a family of models that adequately describe the structure of magnetic fields in small flux tubes and pores. These models also allow us to investigate the rate at which the field spreads out with height in the photosphere and chromosphere. When \( z \) is sufficiently large, adjacent flux tubes will merge together, yielding more complicated structures (Newkirk and Harvey, 1968; Kopp and Kuperus, 1968; Gabriel, 1976; Anzer and Galloway, 1983). This procedure, of adopting known potential field solutions, suffices for many purposes, but in order to investigate the stability of flux tubes it is necessary to ensure that pressure balancing applies continuously at the boundary (Meyer et al., 1977). The technique developed by Schmidt and Wegmann (1983) makes it possible to calculate improved models, with pressure balancing at all levels.

As we saw in the last section, our potential field models break down for fluxes greater than a few TWb. For large fluxes, the almost horizontal field leads to the formation of a penumbra, where the gas pressure, and tension along the magnetic field, cannot necessarily be neglected. Most sunspot models rely on similarity assumptions (e.g. Low, 1980; Osherovich, 1982). To go further requires a better understanding of compressible magnetoconvection than is currently available.

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References

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