POLARIZATION IN SPECTRAL LINES

I: A Unifying Theoretical Approach

E. LANDI DEGL'INNOCENTI
Astrophysical Observatory of Arcetri, Firenze

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Il suffit de passer le pont,
C'est tout de suite l'aventure!
Laisse-moi tenir ton jupon,
J'emmène' visiter la nature!

Georges Brassens

Abstract. A unifying theoretical approach is presented to derive from the general principles of Quantum Electrodynamics both the radiative transfer equations for polarized radiation and the statistical equilibrium equations for an atomic system interacting with a polarized radiation field. The radiation field is described by means of Stokes parameters while the atomic system is described in terms of its density-matrix operator. The non-diagonal terms of the density matrix are fully accounted for so that this formalism can be suitably employed to describe a wide variety of physical phenomena like resonance scattering, the Hanle effect and the Zeeman effect, either in optically thin or optically thick atmospheres, together with all the possible intermediate situations.

The general formulae derived in the first sections of the paper are subsequently particularized introducing the dipole approximation in the relevant matrix elements describing the interaction between the atomic system and the radiation field. The final equations assume a quite compact expression by the introduction of suitable spherical tensors connected with the components of the polarization unit vectors associated with each direction of the radiation field. The general expressions and the main properties of these tensors are discussed in the Appendix.

1. Introduction

Increasing attention has been devoted in recent years to the study of polarized radiation originating from solar active regions, like sunspots, plages, and prominences, as well as from the quiet solar photosphere and the solar corona. As a result of a vigorous technological effort, high quality instruments especially conceived for polarimetric observations (as magnetographs, polarimeters and Stokes polarimeters) are now in operation or have been operating in several observatories as well as on board of a few scientific satellites. In typical cases, these instruments provide the possibility of obtaining wavelength profiles of the solar radiation in two or more directions of polarization with a spectral resolution of the order of a few tens of mÅ. Unfortunately, no recent review is available on the subject, so that the interested reader is here referred to a series of individual papers where the relevant information can be found concerning the main characteristics and performances of the single instruments (Beckers, 1968; Hagyard and Cumings, 1975; Smithson, 1975; Ratier, 1975; Wiehr, 1974; Nishi and

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The interpretation of the data obtained from such instruments requires a good knowledge of the physics of the generation and transfer of polarized radiation through the solar atmosphere. Polarization can be originated in solar spectral lines by several physical mechanisms. The best known among them is the Zeeman effect, which induces circular and linear polarization across the profile of a magnetic sensitive line when a magnetic field is present in the line forming layers. The Zeeman effect is the basic principle for the operation of standard magnetographs and it is generally employed as a diagnostic tool to infer the intensity and direction of the magnetic field at the photospheric level in solar active regions (Stenflo, 1978a).

Other mechanisms able to generate polarization include resonance scattering and impact polarization. Resonance scattering occurs in the higher layers of the solar atmosphere and in the corona and is induced by anisotropic illumination of the emitting atoms. The anisotropy of the incident radiation field, which is generally due to a combination of factors such as geometrical effects, limb darkening, and inhomogeneities of the solar atmosphere, is able to produce differences in populations and interferences between the atomic magnetic sublevels (atomic level polarization) and can then bring, as a net result, to the emission of polarized radiation. The degree of atomic polarization, and consequently, the degree of polarization of the emitted radiation, is controlled by different factors. Apart from the magnetic field, an important role is played by depolarizing collisions with surrounding particles (electrons, ions, and neutral atoms) and by the interlocking of the atomic energy levels through various mechanisms of radiative and collisional excitation and deexcitation. Observations of resonance polarization can then provide original information about the physical situation of the line-forming layers (upper photosphere and chromosphere for visible lines and chromosphere-corona transition region and corona for UV lines). Important insight into the mechanism of line formation can also be gained through such observations.

When the scattering process takes place in the presence of a magnetic field, there is a further effect, called the Hanle effect, which affects the polarization of the resonance radiation through a combined mechanism of depolarization and rotation of the plane of linear polarization. This results from the fact that, when the Larmor frequency associated with the magnetic field is of the order of magnitude or greater than the inverse lifetime of the upper level, the interferences between magnetic sublevels, which are excited by the incident radiation field, are much reduced and the scattering polarization is considerably modified from the zero-magnetic field situation. The Hanle effect has been recently employed as a diagnostic tool to derive the amplitude and direction of the weak magnetic fields found in solar prominences (Bommier et al., 1981; Landi Degl’Innocenti, 1982).

Finally, impact polarization can result from the process of collisional excitation of atoms by anisotropic particle streams. The anisotropy of the colliding particles is able to induce a given amount of atomic level polarization, which then results in the emission of polarized radiation. Impact polarization will not be considered in this paper; if
observed, it could however bring to important information about the electron streams supposed to be present near flaring regions.

Once polarized radiation has originated in a given point of the solar atmosphere through one, or a combination of more, of the effects outlined above, its polarization may be considerably modified by several mechanisms before it can radiate into space and reach the observer. These mechanisms may be due either to the presence of a magnetic field (inverse Zeeman effect, magneto-optical effects) or to the existence of atomic polarization in the lower levels involved in the radiative excitation. The proper understanding of this phenomena is of crucial importance if a safe interpretation of the observations has to be given in those cases where the optical thickness of the emitting region is not negligible.

The present situation of the theoretical literature on these arguments is rather unsatisfactory. Several interesting papers have indeed been published, dealing with one (or in some cases more than one) aspect of the general problem of the generation and transfer of polarized radiation. However, as the formalisms employed and the kind of theoretical approaches followed in the single papers differ from author to author, this makes it impossible to establish interesting connections and interrelations between different aspects of the same problem. Moreover, the limitations and assumptions which are at the basis of each single paper prevent the results from being extended to broader physical situations. As a result, there are several intermediate cases for which the theory still needs to be established.

Since this paper is not intended to be a review one, we limit ourselves to quote here a reduced list of papers which have contributed substantial results in the field. We will see in the following sections of this paper and in subsequent papers of this series, how most of these results can be recovered employing a unique and self-contained theoretical approach in the framework of Quantum Electrodynamics.

The theory of line formation in a magnetic field, for a normal Zeeman triplet, supposed to be formed in LTE and neglecting magneto-optical effects, was first established by Unno (1956). Magneto-optical effects were then introduced in a phenomenological way by Rachkovsky (1962). The first attempt to establish the LTE theory on a purely quantum-mechanical basis was performed by Landi Degl'Innocenti and Landi Degl'Innocenti (1972). Their theory was subsequently generalized by Šidlichovský (1974) and House and Steinitz (1975) to account for departures from LTE in the populations of the Zeeman sublevels in the approximation where quantum interferences between different Zeeman sublevels are neglected ('strong field' approximation). A general transfer equation for polarized radiation in a magnetic field has also been presented by Dolginov and Pavlov (1974).

The theory for resonance polarization in spectral lines has been reviewed many years ago by Mitchell and Zemansky (1934). Since then, important contributions have been given by Hamilton (1947), Chandrasekhar (1950), Lamb and Ter Haar (1971) and Stenflo (1976, 1980). The influence of a magnetic field on the properties of the emitted radiation in a scattering process depends on its amplitude and direction. If the Zeeman splitting is so large that there is no overlapping between magnetic sublevels ('strong field'
approximation), the populations of each sublevel can be treated independently. Several papers have been devoted to the problem of establishing the statistical equilibrium equations for the magnetic sublevels in a polarized radiation field (Charvin, 1965; Šidlichovský, 1974; Sahal-Brechot, 1974; Landi Degl'Innocenti et al., 1976; House, 1977). Alternatively, when the magnetic field produces a Zeeman splitting which is of the order of magnitude of the natural broadening of the energy levels, the single sublevels cannot be described independently and the density-matrix formalism has to be employed to have an appropriate description of the interferences, or coherences, between different sublevels. This physical situation, which is usually referred to as the Hanle effect, has been theoretically investigated by Lamb (1970), Lamb and Ter Haar (1971), House (1970a, b, 1971), Stenflo (1978b), Bommier and Sahal-Brechot (1978), Bommier (1980), and Landi Degl'Innocenti (1982).

In this paper we will follow a pure quantum-mechanical approach in the framework of non-relativistic Quantum Electrodynamics. This approach is similar to the one previously followed by Landi Degl'Innocenti and Landi Degl'Innocenti (1974) to derive a transfer equation for polarized radiation and by Landi Degl'Innocenti et al. (1976) to derive the statistical equilibrium equations for an atomic assembly embedded in a polarized radiation field. In this paper we will however release the 'strong field' approximation and will obtain a more general set of equations; in this sense the present paper can be considered an obvious generalization of the two papers quoted above.

A final introductory remark is necessary concerning the main limitations contained in the theory here presented. The first limitation concerns the absorption and emission profiles which result in being Dirac's delta distributions instead of Lorentzian profiles. This is due to the fact that the perturbative expansion of Quantum Electrodynamics which is employed in the derivation is extended up to the second order and not to higher orders. Typical problems of line radiative transfer, like the redistribution in frequency problem, are beyond the purposes of this paper which is primarily devoted to the polarization aspects of the radiation field. However, these problems can, in principle, be addressed with the same kind of formalism and will be the subject of further investigations. Another limitation concerns the collisional rates which enter the equations of statistical equilibrium. These rates are disregarded in the theoretical derivation as the study of collisions is a problem in itself and would require a detailed treatment which is again beyond the purposes of this paper. The collisional rates can however be added to the statistical equilibrium equations in a purely phenomenological way (Mihalas, 1970).

2. Formulation

We consider the interaction between an atomic assembly of \( N \) atoms per unit volume and the radiation field. The total hamiltonian of the coupled system can be written, in the Schroedinger picture, in the form

\[
H = H_0 + V_S,
\]

where \( H_0 \) is the imperturbed hamiltonian which is the sum of the energies of the free
radiation field and of the atomic system:

\[ H_0 = H_R + H_A , \]  

(2)

while \( V_S \) is the interaction hamiltonian expressed in the Schroedinger picture.

In the formalism of second quantization (Heitler, 1954) \( H_R \) can be written as a bilinear combination of creation and destruction operators:

\[ H_R = \sum_{v, \Omega, \lambda} h v a^\dagger(v, \Omega, \lambda) a(v, \Omega, \lambda) , \]  

(3)

where \( a(v, \Omega, \lambda) \) is the destruction operator of a photon having frequency \( v \), direction along the unit vector \( \Omega \), and polarization characterized by the index \( \lambda \), and \( a^\dagger(v, \Omega, \lambda) \) is the corresponding creation operator. In the same formalism, the interaction hamiltonian can be expressed in the form

\[ V_S = \sum_{v, \Omega, \lambda} \{ Q(v, \Omega, \lambda) a(v, \Omega, \lambda) + Q^\dagger(v, \Omega, \lambda) a^\dagger(v, \Omega, \lambda) \} , \]  

(4)

where the operator \( Q(v, \Omega, \lambda) \) is explicitly given by

\[
Q(v, \Omega, \lambda) = \frac{e_0}{m} \left( \frac{\hbar}{2 \pi v \nu} \right)^{1/2} \mathbf{p} \cdot \mathbf{e}_\lambda(\Omega) \exp(i \mathbf{k} \cdot \mathbf{r}) = \\
= d_v \mathbf{p} \cdot \mathbf{e}_\lambda(\Omega) \exp(i \mathbf{k} \cdot \mathbf{r}) ,
\]

(5)

where again, \( e_0, m, \) and \( \hbar \) are the usual atomic constants, \( \nu \) is the volume of the normalization box, \( \mathbf{p} \) and \( \mathbf{r} \) are the momentum and position operators of the optical electrons of the atom \( \mathbf{k} = 2 \pi v \Omega / c \) is the photon’s momentum, \( \mathbf{e}_\lambda(\Omega) \) \((\lambda = 1, 2)\) are the polarization unit vectors relative to the direction \( \Omega \) and, finally, \( d_v = e_0 \hbar^{1/2} / [m(2 \pi v \nu)^{1/2}] \). Being \( \mathbf{p} = \mathbf{p}^\dagger \) and \( \mathbf{r} = \mathbf{r}^\dagger \), we have:

\[ Q^\dagger(v, \Omega, \lambda) = d_v \mathbf{p} \cdot \mathbf{e}_\lambda^\ast(\Omega) \exp(-i \mathbf{k} \cdot \mathbf{r}) . \]  

(6)

The interaction hamiltonian can then be expressed in the interaction picture through the transformation

\[ V_I(t) = \exp(2 \pi i H_0 t / \hbar) V_S \exp(-2 \pi i H_0 t / \hbar) \]  

(7)

which can be written in the form

\[ V_I(t) = B(t) + B^\dagger(t) , \]  

(8)

where

\[
B(t) = \sum_{v, \Omega, \lambda} \{ \exp(2 \pi i H_A t / \hbar) Q(v, \Omega, \lambda) \exp(-2 \pi i H_A t / \hbar) \} \times \\
\times \{ \exp(2 \pi i H_R t / \hbar) a(v, \Omega, \lambda) \exp(-2 \pi i H_R t / \hbar) \} .
\]

(9)

The second bracket can be easily evaluated, taking into account Equation (3), to obtain

\[ a(v, \Omega, \lambda) \exp(-2 \pi i vt) . \]  

(10)
To express in a more suitable form the first bracket we introduce a complete set \( \{ |n\rangle \} \) of eigenvectors of the atomic hamiltonian \( H_A \):

\[
H_A |n\rangle = E_n |n\rangle ,
\]
where \( E_n \) is the energy eigenvalue corresponding to the eigenvector \( |n\rangle \); the set \( |n\rangle \) obeys the completeness relation:

\[
\sum_n |n\rangle \langle n| = 1 .
\]

Introducing Equation (12) at the left and at the right of the operator \( Q(v, \Omega, \lambda) \) we obtain:

\[
\{ \exp (2\pi i H_A t/\hbar) Q(v, \Omega, \lambda) \exp (-2\pi i H_A t/\hbar) \} = \sum_{nm} d_n q(v, \Omega, \lambda)_{nm} |n\rangle \langle m| \exp (2\pi iv_{nm} t) ,
\]
where we have introduced the notations:

\[
v_{nm} = (E_n - E_m)/\hbar
\]
and

\[
q(v, \Omega, \lambda)_{nm} = \langle n| p \cdot e_\lambda(\Omega) \exp (i k \cdot r) |m\rangle .
\]

From Equations (10) and (13) we have:

\[
B(t) = \sum_{v, \Omega, \lambda} \sum_{nm} d_n q(v, \Omega, \lambda)_{nm} |n\rangle \langle m| a(v, \Omega, \lambda) \exp [2\pi i (v_{nm} - v)t]
\]
and, analogously

\[
B^\dagger(t) = \sum_{v, \Omega, \lambda} \sum_{nm} d_n q^\dagger(v, \Omega, \lambda)_{nm} |n\rangle \langle m| a^\dagger(v, \Omega, \lambda) \exp [2\pi i (v_{nm} + v)t] ,
\]
where

\[
q^\dagger(v, \Omega, \lambda)_{nm} = \langle n| p \cdot e_\lambda^\ast(\Omega) \exp (-i k \cdot r) |m\rangle = [q(v, \Omega, \lambda)_{mn}]^\ast .
\]

In some particular case, it can be easier to calculate the matrix elements of the \( Q \) operator on a basis different from the energy eigenvectors one. If \( \{ |A\rangle \} \) is such a basis, which we suppose orthonormal and complete, it is connected to the basis \( \{ |n\rangle \} \) by a unitary transformation of the form

\[
|n\rangle = \sum_A U_{A,n} |A\rangle .
\]

The following properties can be easily proved:

\[
\sum_A U_{A,m}^\ast U_{A,n} = \delta_{mn} ,
\]
\[
\sum_n U_{A,m}^\ast U_{B,n} = \delta_{AB} ,
\]
\[
\sum_n U_{B,n}^\ast U_{A,n} E_n = \langle A| H_A |B\rangle ,
\]
so that, for the matrix elements, one can write

$$q(v, \Omega, \lambda)_{nm} = \sum_{AB} U_{A,n}^* U_{B,m} \langle A \mid p \cdot e_{\lambda}^{(i)}(\Omega) \exp(i \mathbf{k} \cdot \mathbf{r}) \mid B \rangle,$$  \hspace{1cm} (23)

$$q^+(v, \Omega, \lambda)_{nm} = \sum_{AB} U_{B,n}^* U_{A,m} \langle A \mid p \cdot e_{\lambda}^{(i)}(\Omega) \exp(i \mathbf{k} \cdot \mathbf{r}) \mid B \rangle^*.$$

After having expressed the interaction hamiltonian in a suitable form, we are in the position of deducing the time evolution of the physical quantities we are interested in. We describe the whole system by means of a density operator \( \rho_f(t) \) expressed in the interaction picture. A given observable, \( O(t) \), varies with time according to the following equation:

$$O(t) = \text{Tr} \left\{ \hat{O}_f(t) \rho_f(t) \right\},$$  \hspace{1cm} (25)

where \( \hat{O}_f(t) \) is the quantum operator which corresponds to the observable \( O \) expressed in the interaction picture, and where the symbol \( \text{Tr} \) means the trace operation. For the time derivative of \( O(t) \) one has

$$\frac{dO(t)}{dt} = \text{Tr} \left\{ \frac{d\hat{O}_f(t)}{dt} \rho_f(t) \right\} + \text{Tr} \left\{ \hat{O}_f(t) \frac{d\rho_f(t)}{dt} \right\}.$$  \hspace{1cm} (26)

Taking into account the Schroedinger equation for the density operator,

$$\frac{d\rho_f(t)}{dt} = -\frac{2\pi i}{\hbar} \left[ V_f(t), \rho_f(t) \right],$$  \hspace{1cm} (27)

where the symbol \( [A, B] \) means the commutator \( AB - BA \), we obtain the following expansion which relates the time derivative of the density operator with the same operator evaluated at previous times:

$$\frac{d\rho_f(t)}{dt} = -\frac{2\pi i}{\hbar} \left[ V_f(t), \rho_f(0) \right] - \frac{4\pi^2}{\hbar^2} \int_0^t \left[ V_f(t'), \left[ V_f(t), \rho_f(t') \right] \right] dt',$$  \hspace{1cm} (28)

where \( t = 0 \) is the time corresponding to the beginning of the interaction. Substituting Equation (28) into Equation (26) and taking into account the cyclic property of the trace, we obtain:

$$\frac{dO(t)}{dt} = \text{Tr} \left\{ \frac{d\hat{O}_f(t)}{dt} \rho_f(t) \right\} - \frac{2\pi i}{\hbar} \text{Tr} \left\{ \left[ \hat{O}_f(t), V_f(t) \right] \rho_f(0) \right\} -$$

$$- \frac{4\pi^2}{\hbar^2} \text{Tr} \left\{ \int_0^t \left[ \left[ \hat{O}_f(t), V_f(t) \right], V_f(t') \right] \rho_f(t') dt' \right\}. $$  \hspace{1cm} (29)

This is an exact equation which will be used in the following to deduce both the radiative transfer equations for polarized radiation and the statistical equilibrium...
equations for the atomic system interacting with the polarized radiation field. In the derivation we will however introduce some restrictive assumptions on the density operator that will allow us to obtain a self-contained set of equations for the time evolution of the polarization tensor of the radiation field and of the density-matrix elements of the atomic system. The most relevant assumption is the one of supposing that the $\rho_I$ operator for the whole system is the direct product of the single $\rho$-operators referring to the two interacting systems, or, with self-evident notations:

$$\rho_I(t') = \rho_I^{(R)}(t') \otimes \rho_I^{(A)}(t') \quad (0 \leq t' \leq t) . \quad (30)$$

If $R$ is an operator which acts only on the radiation field variables, and $A$ is an operator which acts only on the atomic variables, Equation (30) allows the evaluation of the trace of the product of two such operators in the form:

$$\text{Tr} \{RA \rho_I\} = \text{Tr} \{R \rho_I^{(R)}\} \text{Tr} \{A \rho_I^{(A)}\} . \quad (31)$$

The physical meaning of the approximation described by Equation (30) can be expressed by the statement that the radiation field and the atomic system are supposed to be uncorrelated at the time when the interaction begins. This uncorrelation is kept up to the time $t$ and this is equivalent to consider a second-order perturbative expansion to the solution of the Schroedinger equation, when Equation (30) is substituted into Equation (29). The approximation described by Equation (30) is a reasonable one to describe the interaction between light and matter in a stellar atmosphere; the same approximation could not be applied to the description of phenomena taking place in a different environment, like for instance in a laser, where strong correlations between the radiation field and the atomic system are known to exist.

A second approximation concerns the properties of the radiation field and can be expressed through the set of conditions:

$$\text{Tr} \{a(v, \Omega, \lambda) \rho_I^{(R)}\} = \text{Tr} \{\dot{a}^+(v, \Omega, \lambda) \rho_I^{(R)}\} = 0 , \quad (32a)$$

$$\text{Tr} \{a(v, \Omega, \lambda) a(v', \Omega', \lambda') \rho_I^{(R)}\} = \text{Tr} \{\dot{a}^+(v, \Omega, \lambda) \dot{a}^+(v', \Omega', \lambda') \rho_I^{(R)}\} = 0 , \quad (32b)$$

$$\text{Tr} \{\dot{a}^+(v, \Omega, \lambda) a(v', \Omega', \lambda') \rho_I^{(R)}\} = 0 \quad \text{unless} \quad v = v' , \Omega = \Omega' . \quad (33)$$

Equations (32) mean that the expectation value of any observable which is described by a linear or bilinear operator in either the creation or the destruction operators of the radiation field is zero. This holds, in particular, for the electric and magnetic fields associated with each mode of the radiation field, so that we obtain the analogue of the classical result that the time average of such observables is zero. Equations (32) are, however, somewhat restrictive in the sense that they don't allow us to consider those physical phenomena which involve coherent states of the radiation field (see, e.g., Glauber, 1964). Nevertheless, these approximations are quite reliable to describe the radiation field typical of a stellar atmosphere, where optical-coherence phenomena are supposed to play a totally insignificant role. Finally, the approximation described by Equation (33) means that any correlation between different modes of the radiation field
is zero, except for correlations between polarization modes. Again, this approximation appears to be fairly reliable for the description of the radiation field in a stellar atmosphere.

By means of Equations (30), (32), and (33) it is possible to simplify considerably the time evolution equation for those observables whose associated quantum operator does not contain any creation or destruction operator of the radiation field, or, alternatively, is bilinear in $a^\dagger$ and $a$. For such observables Equation (29) assumes the simpler form:

$$
\frac{dO(t)}{dt} = \text{Tr} \left\{ \frac{d\hat{O}_f(t)}{dt} \rho_f(t) \right\} - \frac{4\pi^2}{\hbar^2} \text{Tr} \left\{ \left[ \hat{O}_f(t), B(t) \right], B^\dagger(t') \right\} \rho_f(t') dt' \right\} - \frac{4\pi^2}{\hbar^2} \text{Tr} \left\{ \left[ \hat{O}_f(t), B^\dagger(t') \right], B(t') \right\} \rho_f(t') dt' \right\} \tag{34}
$$

which is the fundamental equation which will be used in the following.

To deduce the radiative transfer equations for polarized radiation, we consider the time evolution of the polarization tensor $I_{\gamma\delta}(v, \Omega)$ relative to a generic direction $\Omega$ and a frequency $v$. The tensor is here defined as in Landau and Lifshitz (1971), and is linearly related to the Stokes parameters, which will be introduced in the following, in a way which depends on the choice of the two mutually perpendicular unit vectors, $e_\gamma(\Omega)$ and $e_\delta(\Omega)$, relative to the direction $\Omega$. For the time being this choice is irrelevant and will not be introduced until necessary. The properties of the polarization unit vectors are summarized by the orthogonality relations

$$
e_\gamma \cdot \Omega = 0 \quad (\gamma = 1, 2), \tag{35}$$

$$
e_\gamma \cdot e_\delta^\ast = \delta_{\gamma\delta} \quad (\gamma, \delta = 1, 2). \tag{36}$$

With the observable $I_{\gamma\delta}(v, \Omega)$, following Landi Degl'Innocenti and Landi Degl'Innocenti (1974), we associate the quantum operator $\hat{I}_{\gamma\delta}(v, \Omega)$ defined by (both in the Schroedinger and the interaction pictures)

$$
\hat{I}_{\gamma\delta}(v, \Omega) = b_v a^\dagger(v, \Omega, \gamma) a(v, \Omega, \delta), \tag{37}
$$

where $b_v$ is a constant which is chosen in such a way that the polarization tensor results in having the dimensions of an intensity (erg cm$^{-2}$ s$^{-1}$ Hz$^{-1}$):

$$
b_v = \hbar v^3/c^2. \tag{38}
$$

The physical observable is obtained from the quantum operator by the usual rule:

$$
I_{\gamma\delta}(v, \Omega; t) = \text{Tr} \left\{ \hat{I}_{\gamma\delta}(v, \Omega) \rho_f^{(R)}(t) \right\}. \tag{39}
$$
Analogously, the statistical equilibrium equations are obtained considering the time evolution of the quantities $\rho_{nm}$ which are the matrix elements of the density operator of the atomic system expressed in the Schrödinger picture; introducing the projection operator $|m\rangle \langle n|$, we have

$$\rho_{nm}(t) = \text{Tr} \{ |m\rangle \langle n| \rho_S^{i\lambda}(t) \}$$

(40)

or, transforming this expression in the interaction picture and noting that

$$\exp(2\pi i H_0 t/\hbar) |m\rangle \langle n| \exp(-2\pi i H_0 t/\hbar) = |m\rangle \langle n| \exp(2\pi i v_{mn} t),$$

(41)

we also have

$$\rho_{nm}(t) = \text{Tr} \{ |m\rangle \langle n| \exp(2\pi i v_{mn} t) \rho_f^{i\lambda}(t) \}.$$  

(42)

### 3. Radiative Transfer Equations

Substituting in Equation (34) $\hat{\mathcal{O}}$ with $\hat{I}_{\gamma\delta}(v_0, \Omega_0)$ we obtain the time evolution of the polarization tensor at the frequency $v_0$, propagating in direction $\Omega_0$:

$$\frac{d}{dt} I_{\gamma\delta}(v_0, \Omega_0) = -\frac{4\pi^2}{\hbar^2} \int_0^t \text{Tr} \{ [[\hat{I}_{\gamma\delta}(v_0, \Omega_0), B(t)], B^\dagger(t')] \} \rho_f(t') dt' + c.c. (\gamma \Leftrightarrow \delta),$$

(43)

where the symbol c.c. ($\gamma \Leftrightarrow \delta$) means the quantity which is obtained by taking the complex conjugate of the previous term and interchanging the indices $\gamma$ and $\delta$. This result is easily obtained by the conjugation property of the polarization tensor operator:

$$\hat{I}_{\gamma\delta}^*(v_0, \Omega_0) = \hat{I}_{\delta\gamma}(v_0, \Omega_0)$$

(44)

used in conjunction with the hermiticity property of the density operator ($\rho_f^* = \rho_f$).

Evaluating the first commutator and taking into account the usual commutation rules for the creation and destruction operators, we obtain:

$$[\hat{I}_{\gamma\delta}(v_0, \Omega_0), B(t)] = -b_{v_0} d_{v_0} \sum_{nm} q(v_0, \Omega_0, \gamma)_{nm} |n\rangle \langle m| \times$$

$$\times a(v_0, \Omega_0, \delta) \exp[2\pi i(v_{nm} - v_0)t].$$

(45)

We then evaluate the second commutator to get:

$$[[\hat{I}_{\gamma\delta}(v_0, \Omega_0), B(t)], B^\dagger(t')] = -b_{v_0} d_{v_0} \sum_{v, \Omega, \lambda} \sum_{nm} \sum_{rs} d_v q(v_0, \Omega_0, \gamma)_{nm} \times$$

$$\times q^\dagger(v, \Omega, \lambda)_{rs} [ |n\rangle \langle m| a(v_0, \Omega_0, \delta), |r\rangle \langle s| a^\dagger(v, \Omega, \lambda)] \times$$

$$\times \exp \{2\pi i[(v_{nm} - v_0)t + (v_{rs} + v)t'] \}.$$  

(46)

The commutator appearing in Equation (46) is of the form $[AX, BY]$ with $[A, X] = [A, Y] = [B, X] = [B, Y] = 0$, and can be written in the form

$$[AX, BY] = AB(YX + [X, Y]) - BAYX$$

(47)
so that we obtain:

\[
\begin{align*}
\bra{n} \bra{m} a(v_0, \Omega_0, \delta) \ket{r} \bra{s} a^\dagger(v, \Omega, \lambda) \ket{\delta} &= \\
= \delta_{mr} \bra{n} \bra{s} \{ a^\dagger(v, \Omega, \lambda) a(v_0, \Omega_0, \delta) + \delta_{v_0} \delta_{\Omega\Omega_0} \delta_{\lambda\delta} \} - \\
- \delta_{sn} \bra{r} \bra{m} a^\dagger(v, \Omega, \lambda) a(v_0, \Omega_0, \delta). 
\end{align*}
\] (48)

Substituting this result into Equations (46) and (43), taking into account the hypotheses on the density operator described by Equations (32) and (33), observing that

\[
\text{Tr} \{ \ket{a} \bra{b} \exp(2\pi i v_{ab} t') \rho_{f}^{4}(t') \} = \rho_{ba},
\] (49)

and that

\[
\begin{align*}
\lim_{t \to \infty} \int_{0}^{t} \exp[2\pi i (v_{ab} - v)(t - t')] \, dt' &= \frac{1}{2} \delta(v_{ab} - v) + \frac{i}{2\pi} P \frac{1}{v_{ab} - v} = \\
&= \frac{1}{2} \Phi(v_{ab} - v),
\end{align*}
\] (50)

where \(\delta(x)\) is the Dirac's delta and \(P\) means the principal part in the distribution theory sense, we finally obtain:* 

\[
\begin{align*}
\frac{d}{dt} I_{\gamma} (v_0, \Omega_0) &= \frac{2\pi^2}{h^2} d_\Omega^2 \sum_{\lambda} \sum_{nmr} \{ g(v_0, \Omega_0, \gamma)_{nr} g^\dagger(v_0, \Omega_0, \lambda)_{rm} \rho_{mn} \times \\
&\quad \times [I_{\lambda\delta}(v_0, \Omega_0) + b_{\omega} \delta_{\lambda\delta}] \Phi(v_{nr} - v_0) - g(v_0, \Omega_0, \gamma)_{nm} g^\dagger(v_0, \Omega_0, \lambda)_{nr} \rho_{mn} \times \\
&\quad \times I_{\lambda\delta}(v_0, \Omega_0) \Phi(v_{rm} - v_0) \} + \text{c.c.} (\gamma \Leftrightarrow \delta).
\end{align*}
\] (51)

Equation (51) is a very general radiative transfer equation for polarized radiation, expressed in terms of the polarization tensor instead of the usual Stokes parameters. The transformation in terms of Stokes parameters will be done in Section 5. We want to remark here that the single terms which appear in Equation (51) represent, respectively, stimulated and spontaneous emission and absorption. The \(\Phi\) profiles which appear in Equation (51) contain a real and an imaginary part:

\[
\Phi(x) = \phi(x) + i\psi(x).
\] (52)

Their expression in terms of distributions (Equation (50)) results from the fact that we have followed a second-order perturbation expansion. In practical applications \(\phi\) and \(\psi\) can be substituted, respectively, by Voigt and Faraday—Voigt profiles to take properly

* The approximation of taking the limit \(t \to \infty\) is justified by the fact that the characteristic time over which the exponential in Equation (50) varies is much shorter than the corresponding time for the variation of the density matrix operator \(\rho_f\). For the same reason it is not necessary to specify a time argument in the quantities \(\rho_{mn}\) and \(I_{\lambda\delta}\) appearing in the right-hand side of Equation (51).
into account the line broadening effects which result from natural and collisional damping and from thermal motions (see Landi Degl’Innocenti and Landi Degl’Innocenti, 1972, and references therein). The imaginary part, \( \psi \), of the absorption profile is responsible for the anomalous dispersion terms which play an important role in the physics of radiative transfer and which are found to result, in our theoretical approach, in a quite natural and consequential way. Equation (51) has been derived for a single atom interacting with the radiation field. As a consequence, the time derivative of the polarization tensor contains a factor \( \mathcal{N}^{-1} \) which results from the term \( d_o^2 \). For an assembly of uncorrelated atoms, the factor \( \mathcal{N}^{-1} \) has to be substituted by \( \mathcal{N} \), the number density of atoms of the same species. Equation (51) can be transformed into a more compact form; introducing the quantities

\[
\begin{align*}
    f_{\alpha \beta} &= 4\pi^2 d_o^2 b_o e^{-1} h^{-2} \sum_{n} q(v_o, \Omega_o, \alpha) q^*(v_o, \Omega_o, \beta) \rho_{mn} \Phi(v_{nr} - v_o), \\
    g_{\alpha \beta} &= 2\pi^2 d_o^2 e^{-1} h^{-2} \sum_{n} q(v_o, \Omega_o, \alpha) q^*(v_o, \Omega_o, \beta) \rho_{mn} \Phi(v_{rm} - v_o), \\
    h_{\alpha \beta} &= \frac{1}{2} f_{\alpha \beta}/b_o,
\end{align*}
\]

and taking into account that, in stationary situations, the operator \( d/dt \) can be substituted by \( c d/ds \) (where \( s \) is the coordinate measured along the ray path; see Landi Degl’Innocenti and Landi Degl’Innocenti, 1974), we obtain:

\[
\frac{d}{ds} I_{\gamma \delta}(v_o, \Omega_o) = \frac{1}{2} (f_{\gamma \delta} + f^*_{\gamma \delta}) - \sum_{\lambda} [g_{\gamma \lambda} I_{\lambda \delta}(v_o, \Omega_o) + g^*_{\delta \lambda} I_{\gamma \lambda}(v_o, \Omega_o)] + \\
+ \sum_{\lambda} [h_{\gamma \lambda} I_{\lambda \delta}(v_o, \Omega_o) + h^*_{\delta \lambda} I_{\gamma \lambda}(v_o, \Omega_o)],
\]

where the terms \( f, g, \) and \( h \) represent spontaneous emission, absorption and stimulated emission, respectively.

4. Statistical Equilibrium Equations

The statistical equilibrium equations can be obtained by the same procedure used to derive the radiative transfer equations. In this case, however, the operator to be substituted for \( \hat{O}_f(t) \) in Equation (34) is the projection operator appearing in Equation (42):

\[
|m\rangle \langle n| \exp(2\pi i v_{mn} t)
\]

which, being time dependent, brings an extra contribution due to the first term appearing in the right-hand side of Equation (34). Substituting this operator in Equation (34), we obtain:

\[
\frac{d}{dt} \rho_{nm} + 2\pi i v_{nn} \rho_{nm} = -\frac{4\pi^2}{\hbar^2} \int_0^t \text{Tr} \{[[m\rangle \langle n|, B(t)], B^+(t')] \} \times \\
\times \exp(2\pi i v_{nn} t) \rho_f(t') \} \, dt' + c.c. (n \not= m),
\]

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where the symbol c.c. \((n \equiv m)\) means the quantity which is obtained by taking the complex conjugate of the previous term and interchanging the indices \(n\) and \(m\).

Evaluating the inner commutator, we get:

\[
\lbrack m \lbrack n, B(t) \rbrack \rbrack = \sum_{\nu, \Omega', \lambda} \sum_{rs} d_{\nu} q(v, \Omega, \lambda)_{rs} \lbrack m \lbrack n, r \rbrack \rbrack \times \times a(v, \Omega, \lambda) \exp[2\pi i(v_{rs} - v)t] \]  

(56)

and, taking into account that

\[
\lbrack m \lbrack n, r \rbrack \rbrack \rbrack = \lbrack m \lbrack n, \delta_{nr} \rbrack \rbrack \rbrack = \lbrack m \lbrack n, \delta_{ms} \rbrack \rbrack \rbrack,
\]

(57)
we obtain for the outer commutator:

\[
[[m \lbrack n, B(t) \rbrack \rbrack, B^{\dagger}(t')] \exp(2\pi i v_{nm} t) = \sum_{\nu, \Omega, \lambda} \sum_{\nu', \Omega', \lambda'} \sum_{rsuv} d_{\nu} d_{\nu'} \times \times q(v, \Omega, \lambda)_{rs} q^\dagger(v', \Omega', \lambda')_{uv} \lbrack (m \lbrack n, \delta_{nr} \rbrack \rbrack \rbrack \rbrack \rbrack \times \times a(v, \Omega, \lambda), \lbrack u \lbrack v \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrack \rbrak...
evolution of the populations of the single levels (diagonal terms of the form $\rho_{nn}$) and of the interferences or coherences between different levels (non diagonal terms). Neglecting polarization effects, the equations for the diagonal terms reduce to the well known equations of statistical equilibrium (see for instance Mihalas, 1970). Due to the presence of the homogeneous term $2\pi i v_{nm} \rho_{nm}$, the coherences between non-degenerate levels ($v_{nm} \neq 0$) decrease in importance as the separation between the energy levels becomes increasingly larger. Similar equations for the diagonal terms have been deduced by Landi Degl’Innocenti et al. (1976).

5. The Stokes Parameters and the Dipole Approximation

The formulae deduced up to now are expressed in terms of the polarization tensor $I_{\alpha \beta}(v, \Omega)$. In order to introduce the Stokes parameters we need a clear and unambiguous definition of these quantities. For each direction $\Omega$ we choose a real unit vector $e_\alpha(\Omega)$ belonging to the plane perpendicular to $\Omega$. The direction of the vector $e_\alpha(\Omega)$ will be referred to, in the following, as the 'reference direction' relative to $\Omega$. We then consider a second unit vector, $e_\beta(\Omega)$, perpendicular to both $e_\alpha(\Omega)$ and $\Omega$ and such that $e_\alpha(\Omega)$, $e_\beta(\Omega)$, and $\Omega$ form, in this order, a right-handed coordinate system. Clearly, $e_\beta(\Omega)$ is fixed once that the reference direction has been assigned. The Stokes parameters are defined, at each frequency $v$, according to Table I. These operational definitions are the same as those given by Shucliff (1962).

<table>
<thead>
<tr>
<th>Stokes parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>Intensity of the radiation beam.</td>
</tr>
<tr>
<td>$Q$</td>
<td>Intensity of linearly polarized radiation along $e_\alpha$ minus intensity of linearly polarized radiation along $e_\beta$.</td>
</tr>
<tr>
<td>$U$</td>
<td>Intensity of linearly polarized radiation along the unit vector $(e_\alpha + e_\beta)/\sqrt{2}$ minus intensity of linearly polarized radiation along $(-e_\alpha + e_\beta)/\sqrt{2}$.</td>
</tr>
<tr>
<td>$V$</td>
<td>Intensity of right circularly polarized radiation minus intensity of left circularly polarized radiation. (Right circularly polarized radiation means that the electric vector associated with the radiation beam is seen rotating clockwise by an observer looking toward the source).</td>
</tr>
</tbody>
</table>

While the unit vectors $e_\alpha(\Omega)$ and $e_\beta(\Omega)$ are particularly suitable to give a proper definition of the Stokes parameters, for theoretical calculations it is more convenient to introduce two complex unit vectors, $e_{\pm 1}(\Omega)$, which are linear combinations of the previous ones being defined by the expressions:

$$
e_{+1}(\Omega) = \left[ -e_\alpha(\Omega) + ie_\beta(\Omega) \right]/\sqrt{2},$$
$$
e_{-1}(\Omega) = \left[ e_\alpha(\Omega) + ie_\beta(\Omega) \right]/\sqrt{2},$$

(61)
with
\[ e^*_{\pm 1}(\Omega) = -e_{\mp 1}(\Omega). \] (62)

These unit vectors have the advantage of having simpler spherical components than \( e_a \) and \( e_b \). From now on, we suppose that the unit vectors \( e_{\pm}(\Omega) \) introduced in the expression for \( V_S \) (Equations (4), (5)) and appearing in the matrix elements \( q \), are just the complex unit vectors \( e_{\pm 1}(\Omega) \). The relationships between the Stokes parameters and the components of the polarization tensors are then the following:
\[
I = I_{++} + I_{--}, \quad U = -i(I_{+-} + iI_{-+}), \\
Q = -I_{+-} - I_{-+}, \quad V = I_{++} - I_{--},
\] (63)

where we have shortened the notations writing \( I_{++} \) instead of \( I_{+1+1} \), and so on.

Equations (63) can be inverted to give:
\[
I_{++} = \frac{1}{2}(I + V), \quad I_{+-} = \frac{1}{2}(-Q + iU), \\
I_{--} = \frac{1}{2}(I - V), \quad I_{-+} = \frac{1}{2}(-Q - iU).
\] (64)

By means of Equations (63) and (64), we obtain, by substitution into Equation (54), the following transfer equation for the Stokes parameters propagating at the frequency \( \nu_0 \) in the direction \( \Omega_0 \):
\[
\frac{d}{ds} \begin{pmatrix}
I \\
Q \\
U \\
V
\end{pmatrix} =
\begin{pmatrix}
\varepsilon_I \\
\varepsilon_Q \\
\varepsilon_U \\
\varepsilon_V
\end{pmatrix} - 
\begin{pmatrix}
\eta_I & \eta_Q & \eta_U & \eta_V \\
\eta_Q & \rho_V - \rho_U & \rho_I & \eta_V \\
\eta_U - \rho_V & \rho_I & \eta_I & \rho_Q \\
\eta_V & \rho_U - \rho_Q & \eta_I & \eta_I
\end{pmatrix}
\begin{pmatrix}
I \\
Q \\
U \\
V
\end{pmatrix} + 
\begin{pmatrix}
\eta_I & \eta_Q & \eta_U & \eta_V \\
\eta_Q & \eta_I & \rho_V - \rho_U & \rho_U \\
\eta_U - \rho_V & \eta_I & \eta_I & \rho_Q \\
\eta_V & \rho_U - \rho_Q & \rho_I & \eta_I
\end{pmatrix}
\begin{pmatrix}
I \\
Q \\
U \\
V
\end{pmatrix},
\] (65)

where, in terms of the quantities defined by Equations (53), we have:
\[
\varepsilon_I = \text{Re}(f_{++} + f_{--}), \quad \eta_I = \text{Re}(g_{++} + g_{--}), \\
\varepsilon_Q = -\text{Re}(f_{+-} + f_{-+}), \quad \eta_Q = -\text{Re}(g_{+-} + g_{-+}), \quad \rho_Q = -\text{Im}(g_{++} + g_{--}), \\
\varepsilon_U = \text{Im}(f_{+-} - f_{-+}), \quad \eta_U = \text{Im}(g_{+-} - g_{-+}), \quad \rho_U = -\text{Re}(g_{++} - g_{--}), \\
\varepsilon_V = \text{Re}(f_{+-} - f_{-+}), \quad \eta_V = \text{Re}(g_{+-} - g_{-+}), \quad \rho_V = \text{Im}(g_{++} - g_{--}),
\] (66)

with analogous expressions for \( \eta_I^*, \ldots, \eta_V^*, \rho_Q^*, \ldots, \rho_V^* \), the \( g_{\alpha \beta} \) having to be substituted by the corresponding \( h_{\alpha \beta} \).

The radiative transfer equation now established is a very general one, as it takes properly into account the existence of coherences between different energy levels. In this respect it can be considered a generalization of the similar equation presented in Landi Degl'Innocenti and Landi Degl'Innocenti (1974), where the non diagonal terms \( \rho_{nm} \) with \( n \neq m \) had been deliberately neglected. The symmetry structure of the absorption and stimulated emission matrices is the usual one which generally appears in radiative transfer equations for polarized radiation. It can be shown that these symmetry properties are due to very general physical principles and do not depend on the
particular material system interacting with the radiation field (Landi Degl'Innocenti and Landi Degl'Innocenti, 1981). Finally, we want to remark that the equation derived can be applied to any type of atomic transition, which makes them suitable to describe radiative transfer processes in forbidden lines too. The more common case of electric-dipole transitions will be considered in the final part of this section.

We now return to the statistical equilibrium equations and note that Equation (60) involves general expressions of the form:

$$\mathcal{A}_{abcdef} = \sum_{v, \Omega} \sum_{\lambda, \lambda'} 2 \pi d v h^{-2} q(v, \Omega, \lambda)_{ab} q^*(v, \Omega, \lambda')_{cd} \delta_{\lambda \lambda'} \Phi(v_f - v), \quad (67)$$

$$\mathcal{B}_{abcdef} = \sum_{v, \Omega} \sum_{\lambda, \lambda'} 2 \pi d v b^{-1} h^{-2} q(v, \Omega, \lambda)_{ab} q^*(v, \Omega, \lambda')_{cd} I_{\lambda' \lambda}(v, \Omega) \Phi(v_f - v).$$

These expressions involve an integral of the radiation field over the solid angle. This integral depends on the angular dependence of the matrix elements $q$ and $q^*$, which is different for different types of atomic transitions (electric dipole, magnetic dipole, electric quadrupole, and so on). In the following we will introduce the dipole approximation, neglecting the factor $\exp(\pm i \mathbf{k} \cdot \mathbf{r})$ in Equations (15) and (18). By means of usual transformations we have:

$$q(v, \Omega, \lambda)_{ab} = 2 \pi i m v_{ab} \langle a | \mathbf{r} \cdot \mathbf{e}_\lambda(\Omega) | b \rangle,$$  \quad (68)

where $m$ is the electron mass and $\mathbf{r}$ the position operator of the optical electrons. Introducing the irreducible spherical components (defined as in Brink and Satchler, 1968) of the vectors $\mathbf{r}$ and $\mathbf{e}_\lambda(\Omega)$ we can write:

$$q(v, \Omega, \lambda)_{ab} = 2 \pi i m v_{ab} \sum_q (-1)^q (r_{qab} (e_\lambda(\Omega))_{-q},$$  \quad (69)

and, analogously:

$$q^*(v, \Omega, \lambda')_{cd} = [q(v, \Omega, \lambda')_{cd}]^* = -2 \pi i m v_{dc} \sum_q (-1)^q (r_{qdc}^*(e_{\lambda'}(\Omega))_{-q}^*,$$  \quad (70)

where the meaning of the last symbols introduced is the following:

$$(r_{q})_{dc}^* = [(r_{q})_{dc}]^*,$$

$$(e_{\lambda'}(\Omega))_{-q}^* = [(e_{\lambda'}(\Omega))_{-q}]^*,$$

or, in other words, the complex conjugate has to be taken after the evaluation of the pertinent spherical component. This remark is non-trivial because, if $v$ is a complex vector, we have, in general $(v_q)^* \neq (v^*)_q$. Substituting Equations (68) and (69) in Equations (67) and transforming the sum over the radiation field modes in an integral over the solid angle and the frequencies through the expression

$$\sum_{v, \Omega} \to 4 \pi c^{-3} \mathcal{V} \int_0^\infty d\Omega \frac{d\Omega}{4 \pi} \int_0^\infty d\nu \nu^2,$$  \quad (71)
we obtain:

\[
\mathcal{A}_{abcdef} = 16\pi^4 \frac{e_0^2}{\hbar c^3} v_{ab} v_{dc} \sum_{qq'} (-1)^{q+q'} (r_{q})_{ab}(r_{q'})_{dc} \int_0^\infty \frac{d\Omega}{4\pi} \int_0^\infty \frac{dv}{v^2} v \times \\
\times \sum_{\lambda,\lambda'} (e_\lambda(\Omega))_{-q}(e_{\lambda'}(\Omega))^{*}_{-q'} \delta_{\lambda,\lambda'} \Phi(v_{ef} - v),
\]

\[
\mathcal{B}_{abcdef} = 16\pi^4 \frac{e_0^2}{\hbar^2 c} v_{ab} v_{dc} \sum_{qq'} (-1)^{q+q'} (r_{q})_{ab}(r_{q'})_{dc} \int_0^\infty \frac{d\Omega}{4\pi} \int_0^\infty \frac{dv}{v^2} v \times \\
\times \sum_{\lambda,\lambda'} (e_\lambda(\Omega))_{-q}(e_{\lambda'}(\Omega))^{*}_{-q'} I_{\lambda,\lambda'}(v, \Omega) \Phi(v_{ef} - v). 
\]

(72)

It is now useful to introduce some tensorial quantities which are directly connected to the tensorial components of the polarization unit vectors and to the Stokes parameters. The main properties of these tensors are summarized in the Appendix. Defining

\[
\delta_{qq'}(\alpha, \beta, \Omega) = (e_\alpha(\Omega))_{q} (e_\beta(\Omega))^{*}_{q'},
\]

\[
\mathcal{J}_{qq'}(v, \Omega) = \sum_{\alpha\beta} \delta_{qq'}(\alpha, \beta, \Omega) I_{\beta\alpha}(v, \Omega),
\]

\[
J_{qq'}(v) = \int_0^\infty \frac{d\Omega}{4\pi} \mathcal{J}_{qq'}(v, \Omega),
\]

\[
\bar{J}_{qq'}(v_0) = \int_0^\infty \frac{v^2_0}{v^2} J_{qq'}(v) \phi(v_0 - v) \, dv,
\]

\[
\bar{F}_{qq'}(v_0) = \int_0^\infty \frac{v^2_0}{v^2} J_{qq'}(v) \psi(v_0 - v) \, dv,
\]

(73)

Equations (72) can be written in the form:

\[
\mathcal{A}_{abcdef} = \frac{32}{3} \pi^4 \frac{e_0^2}{\hbar c^3} v_{ab} v_{dc} \sum_{q} (r_{q})_{ab}(r_{q})_{dc} \int_0^\infty dv \, v \Phi(v_{ef} - v),
\]

(74)

\[
\mathcal{B}_{abcdef} = 16\pi^4 \frac{e_0^2}{\hbar^2 c} v_{ab} v_{dc} v_{ef}^{-2} \sum_{qq'} (-1)^{q+q'} (r_{q})_{ab}(r_{q'})_{dc} \times \\
\times [\bar{J}_{-q-q'}(v_{ef}) + i\bar{F}_{-q-q'}(v_{ef})].
\]

(75)

The integral appearing in Equation (74) is made of a real part and an imaginary one. The real part is easily evaluated and gives

\[
v_{ef} \Theta(v_{ef}),
\]

where \(\Theta(x)\) is the step-function, being 1 for positive arguments and 0 otherwise. The imaginary part brings to a diverging integral which is connected to renormalization problems (Lamb-shift). This part will be simply neglected in our development and we
simply write:

\[ A_{abcdef} = \frac{32}{3} \pi^4 \frac{e^2}{\hbar c^3} v_{ab} v_{dc} v_{ef} \Theta(v_{ef}) \sum_q (r_q)_{ab} (r_q)_{dc}^*. \]  

(76)

In terms of the quantities \( A \) and \( B \) given by Equations (75) and (76) the statistical equilibrium equations can be finally expressed in the more compact form:

\[ \frac{d}{dt} \rho_{nm} + 2\pi i v_{nm} \rho_{nm} = -R^{(1)}_{nm} - R^{(2)}_{nm} + R^{(3)}_{nm} + R^{(4)}_{nm} + R^{(5)}_{nm} - R^{(6)}_{nm}, \]  

(77)

where the single rates \( R^{(i)}_{nm} \), which are graphically represented in Figure 1, are explicitly given by:

\[
\begin{align*}
R^{(1)}_{nm} &= \sum_{rs} A_{nssrs} \rho_{rm} + \text{c.c. (} n \geq m, \\
R^{(2)}_{nm} &= \sum_{rs} B_{nssrs} \rho_{rm} + \text{c.c. (} n \geq m, \\
R^{(3)}_{nm} &= \sum_{rs} B_{nssrn} \rho_{sr} + \text{c.c. (} n \geq m, \\
R^{(4)}_{nm} &= \sum_{rs} A_{rnrsn} \rho_{sr} + \text{c.c. (} n \geq m, \\
R^{(5)}_{nm} &= \sum_{rs} B_{rnrsn} \rho_{sr} + \text{c.c. (} n \geq m, \\
R^{(6)}_{nm} &= \sum_{rs} B_{rmsrn} \rho_{ns} + \text{c.c. (} n \geq m, \\
\end{align*}
\]

(78)

Fig. 1. The time evolution of the matrix element \( \rho_{nm} \) depends on six radiative rates \( R^{(i)}_{nm} \), each associated with a positive or negative sign in Equation (77). Solid lines refer to spontaneous emission while wiggled lines refer to absorption (arrows pointing upward) or stimulated emission (arrows pointing downward).
with $\mathcal{A}$ and $\mathcal{B}$ given by Equations (75) and (76) in the dipole approximation and by Equations (67) in general.

Equation (77) contains only the radiative rates. The collisional rates can be added in a pure phenomenological way and the equations can be solved, together with the normalization condition

$$
\sum_n \rho_{nn} = 1, \quad (79)
$$

to obtain the time dependence of the density matrix. In stationary situations $[(d/dt)\rho_{nm} = 0]$, the problem is reduced to the solution of a system of $N$ equations in $N$ unknowns, where $N$ is the total number of matrix elements $\rho_{nm}$ to be considered.

The dipole approximation has been introduced so far only in the equations of statistical equilibrium. When the same approximation is introduced in the radiative transfer equation, it is possible to give a more detailed expression for the absorption and stimulated emission matrices and for the emission vector appearing in Equation (65). Defining the formal vectors

$$
\epsilon_i = (\epsilon_{\ell}, \epsilon_{\gamma}, \epsilon_{U}, \epsilon_{\nu}) \quad (i = 0, 1, 2, 3),
$$

$$
\eta_i = (\eta_{\ell}, \eta_{\gamma}, \eta_{U}, \eta_{\nu}) \quad (i = 0, 1, 2, 3),
$$

$$
\rho_i = (\rho_{\ell}, \rho_{\gamma}, \rho_{U}, \rho_{\nu}) \quad (i = 1, 2, 3),
$$

$$
\eta_i^t = (\eta_i^t, \eta_i^t, \eta_i^t, \eta_i^t) \quad (i = 0, 1, 2, 3),
$$

$$
\rho_i^t = (\rho_i^t, \rho_i^t, \rho_i^t, \rho_i^t) \quad (i = 1, 2, 3), \quad (80)
$$

we have, for the radiation propagating at the frequency $\nu$ in the direction $\Omega$:

$$
\epsilon_i(\nu, \Omega) = \sum_{nmr} \sum_{q \neq q'} (-1)^{q + q'} \frac{16 \pi^2 e_0^2}{c^3} v_{nmr} \nu \nu^2 N \times
$$

$$
\times \text{Re} \{ (r_{q,nr}(r_q)_{nmr}^* \rho_{nm} \Phi(\nu_{nr} - \nu) T_{-q-q'}(i, \Omega) \},
$$

$$
\eta_i(\nu, \Omega) = \sum_{nmr} \sum_{q \neq q'} (-1)^{q + q'} \frac{8 \pi^2 e_0^2}{c h \nu} v_{nmr} \nu N \times
$$

$$
\times \text{Re} \{ (r_{q,nr}(r_q)_{nr}^* \rho_{nm} \Phi(\nu_{nr} - \nu) T_{-q-q'}(i, \Omega) \},
$$

$$
\rho_i(\nu, \Omega) = \eta_i(\nu, \Omega) \{ \text{Re} \rightarrow \text{Im} \} \quad (i = 1, 2, 3),
$$

$$
\eta_i^t(\nu, \Omega) = c^2 \frac{2h \nu^3}{\varepsilon_i(\nu, \Omega)},
$$

$$
\rho_i^t(\nu, \Omega) = \eta_i^t(\nu, \Omega) \{ \text{Re} \rightarrow \text{Im} \} \quad (i = 1, 2, 3), \quad (81)
$$

where the tensor $T_{-q-q'}(i, \Omega)$ is defined in Appendix 1, and, with the symbol $A = B \{ \text{Re} \rightarrow \text{Im} \}$, we mean that the expression for $A$ is obtained from the expression for $B$ substituting the operator Re with the operator Im.
Finally, we want to remark that, if the dipole matrix elements \((r_q)_{ab}\) have to be calculated on the basis \(\{|A\rangle\}\) defined by Equation (19), instead of the basis \(\{|n\rangle\}\) of the energy eigenvectors, formulae (75), (76), and (81) have to be suitably modified taking into account that

\[
(r_q)_{ab} = \langle a | r_q | b \rangle = \sum_{AB} U_{A,a}^* U_{B,b} \langle A | r_q | B \rangle = \sum_{AB} U_{A,a}^* U_{B,b} (r_q)_{AB}.
\]

(82)

6. Conclusions

In this paper we have presented a unifying theoretical approach in the formalism of Quantum Electrodynamics to derive, from the same general principles, both the equations of radiative transfer for polarized radiation and the equations of statistical equilibrium for an atomic system embedded in a polarized radiation field. The most relevant improvement of the present derivation, with respect to previous ones, is the fact that the non-diagonal terms of the density matrix of the atomic system (the so called coherences) are fully accounted for, both in the radiative transfer equations and in the equations of statistical equilibrium. The formulae derived in this paper are quite general and are directly expressed in terms of Stokes parameters, which makes them particularly suitable for the interpretation of polarimetric observations.

Most of the mechanisms which are responsible for the appearance of polarization in spectral lines, under a wide variety of astrophysical situations, can be easily described through the formalism presented in this paper. These include the Zeeman effect (normal or anomalous, direct or inverse), the Hanle effect, resonance polarization, magneto-optical effects, and so on. The particularization of the present formalism to the various physical problems will be the subject of further papers in this series.

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Appendix 1: Spherical Tensors for Polarimetry

In this appendix we will introduce several spherical tensors which are particularly suitable to handle in a simple way the geometrical relationships which are often associated with polarimetric observations and their interpretation. A quantity which usually appears, when expressing the dipole operator through spherical components, is
Fig. 2. For each direction $\Omega$, the choice of the reference direction, $e_a$, is specified by the angle $\gamma$.

the tensor:

$$
\delta_{qq'}(\alpha, \beta, \Omega) = (e_{\alpha}(\Omega))_q (e_{\beta}(\Omega))_{q'} \quad (\alpha, \beta = \pm 1; q, q' = 0, \pm 1). \tag{A1}
$$

This is a reducible spherical tensor, being a diadic product of two irreducible tensors of rank 1. Its explicit expression depends on the coordinate system employed and on the choice, for each direction $\Omega$, of the polarization unit vectors $e_a$ and $e_b$ (and of the associated linear combinations $e_{\pm 1}$) which are needed to define the Stokes parameters. Such a choice is summarized in Figure 2. In the right-handed coordinate system $K$, each direction $\Omega$ is characterized by the polar angles $\theta$ and $\chi$ and the relative ‘reference direction’, $e_{\alpha}(\Omega)$, is chosen, in the plane perpendicular to $\Omega$, in such a way to form an angle $\gamma$ with the plane containing $\Omega$ and the $z$-axis (see Figure 2 for the sign conventions). $\gamma$ is, in general, an arbitrary function of $\Omega$ which is left unspecified for greater generality. The expression of the tensor $\delta_{qq'}(\alpha, \beta, \Omega)$ is particularly simple when the direction $\Omega$ is coincident with the $z$-axis and the corresponding $\gamma$ angle is null. In this case, remembering the definitions of the unit vectors $e_{\pm 1}(\Omega)$ (Equations (61)), and the connection between spherical and rectangular components of a vector (Brink and Satchler, 1968), it can be easily proved that

$$
\delta_{qq'}(\alpha, \beta, \Omega = \hat{z}) = \delta_{aq} \delta_{bq'}. \tag{A2}
$$
From this expression one can show, by means of suitable rotations of the coordinate system that, in general,

$$
\mathcal{E}_{qq'}(\alpha, \beta, \Omega) = D_{qq'}^1(R) D_{\beta q}^{1*}(R),
$$

(A3)

where $R$ is the rotation which brings the system $(e_a(\Omega), e_b(\Omega), \Omega)$ into the system $K$, and where the rotation matrices are defined according to Brink and Satchler (1968). As $R$ is characterized by the Euler angles $(-\gamma, -\theta, -\chi)$, one can write explicitly:

$$
\mathcal{E}_{qq'}(\alpha, \beta, \Omega) = \exp\{i[(q-q')\chi + (\alpha-\beta)\gamma]\} d_{q\alpha}^1(\theta) d_{q'\beta}^1(\theta).
$$

(A4)

From Equations (A1) and (A3) it is possible to prove that

$$
\sum_{\alpha = \pm 1} \delta_{qq'}(\alpha, \alpha, \Omega) = \delta_{qq'} - D_{0q}^1(-\gamma, -\theta, -\chi) D_{0q}^{1*}(-\gamma, -\theta, -\chi)
$$

(A5)

and, consequently, that

$$
\sum_{\alpha, \beta} \delta_{\alpha\beta} \int \frac{d\Omega}{4\pi} \mathcal{E}_{qq'}(\alpha, \beta, \Omega) = \frac{2}{3} \delta_{qq'}.
$$

(A6)

Moreover, one can prove directly from the definition that

$$
\mathcal{E}_{qq'}^{*}(\alpha, \beta, \Omega) = \mathcal{E}_{q'q}(\beta, \alpha, \Omega)
$$

(A7)

and that

$$
\sum_q \mathcal{E}_{qq}(\alpha, \beta, \Omega) = \delta_{\alpha\beta}.
$$

(A8)

The tensor $\mathcal{I}_{qq'}(v, \Omega)$ is obtained from $\mathcal{E}_{qq'}$ through a saturation with the polarization tensor:

$$
\mathcal{I}_{qq'}(v, \Omega) = \sum_{\alpha\beta} \mathcal{E}_{qq'}(\alpha, \beta, \Omega) I_{\beta\alpha}(v, \Omega)
$$

(A9)

which, by means of Equations (64), can be expressed through the Stokes parameters in the form:

$$
\mathcal{I}_{qq'}(v, \Omega) = \sum_{i=0}^{3} \mathcal{T}_{qq'}(i, \Omega) S_i(v, \Omega),
$$

(A10)

where $S_i(v, \Omega)$ is the formal vector whose components are the Stokes parameters at frequency $v$ and relative to the direction $\Omega$:

$$
S_i(v, \Omega) = [I(v, \Omega), Q(v, \Omega), U(v, \Omega), V(v, \Omega)] \quad (i = 0, 1, 2, 3)
$$

(A11)

and where the tensors $\mathcal{T}_{qq'}(i, \Omega)$ are defined according to the expressions:

$$
\begin{align*}
\mathcal{T}_{qq'}(0, \Omega) &= \frac{1}{2} [ \mathcal{E}_{qq'}(+, +, \Omega) + \mathcal{E}_{qq'}(-, -, \Omega)], \\
\mathcal{T}_{qq'}(1, \Omega) &= -\frac{1}{2} [ \mathcal{E}_{qq'}(+, -, \Omega) + \mathcal{E}_{qq'}(-, +, \Omega)], \\
\mathcal{T}_{qq'}(2, \Omega) &= -\frac{i}{2} [ \mathcal{E}_{qq'}(+, -, \Omega) - \mathcal{E}_{qq'}(-, +, \Omega)], \\
\mathcal{T}_{qq'}(3, \Omega) &= \frac{1}{2} [ \mathcal{E}_{qq'}(+, +, \Omega) - \mathcal{E}_{qq'}(-, -, \Omega)].
\end{align*}
$$

(A12)
| \( i = 0 \) | \( 1 \) | \(-2\sqrt{2}\) | \(-2\sqrt{2}\) | \(-2\) | \(-2\) |
| \( j = 2 \) | \(-4\sqrt{2}\) | \(-4\sqrt{2}\) | \(-4\) | \(-4\) |

Explicit expressions of the tensor \( F_{ij} (\Omega, \Theta) \). The indexing of the rows and columns runs from \(-1\) to \(1\).
The tensor here defined satisfies the conjugation property:

\[ T_{qq}(i, \Omega) = T_{q'q}(i, \Omega) \]  

so that, from Equation (A10) one obtains:

\[ T_{qq}^*(v, \Omega) = T_{q'q}(v, \Omega). \]  

The explicit expressions of the tensors \( T_{qq}(i, \Omega) \) as functions of \( \theta, \chi, \) and \( \gamma \) is given in Table II. From these expressions one obtains for \( T_{qq}^* \):

\[ T_{-1-1}(v, \Omega) = \frac{1}{4} (1 + \cos^2 \theta) I(v, \Omega) - \frac{1}{4} \sin^2 \theta [\cos 2\gamma Q(v, \Omega) - \sin 2\gamma U(v, \Omega)] - \frac{1}{2} \cos \theta V(v, \Omega), \]

\[ T_{00}(v, \Omega) = \frac{1}{2} \sin^2 \theta I(v, \Omega) + \frac{1}{2} \sin^2 \theta [\cos 2\gamma Q(v, \Omega) - \sin 2\gamma U(v, \Omega)], \]

\[ T_{11}(v, \Omega) = \frac{1}{4} (1 + \cos^2 \theta) I(v, \Omega) - \frac{1}{4} \sin^2 \theta [\cos 2\gamma Q(v, \Omega) - \sin 2\gamma U(v, \Omega)] + \frac{1}{2} \cos \theta V(v, \Omega), \]

\[ T_{01}(v, \Omega) = \frac{1}{2} \sqrt{2} \exp(i\chi) \{ -\sin \theta \cos \theta I(v, \Omega) - \sin \theta \cos \theta [\cos 2\gamma Q(v, \Omega) - \sin 2\gamma U(v, \Omega)] - i \sin \theta [\cos 2\gamma U(v, \Omega) + \sin 2\gamma Q(v, \Omega)] + \sin \theta V(v, \Omega) \}, \]

\[ T_{10}(v, \Omega) = \frac{1}{2} \sqrt{2} \exp(i\chi) \{ \sin \theta \cos \theta I(v, \Omega) + \sin \theta \cos \theta [\cos 2\gamma Q(v, \Omega) - \sin 2\gamma U(v, \Omega)] + i \sin \theta [\cos 2\gamma U(v, \Omega) + \sin 2\gamma Q(v, \Omega)] + \sin \theta V(v, \Omega) \}, \]

\[ T_{11}(v, \Omega) = \frac{1}{4} \exp(2i\chi) \{ \sin^2 \theta I(v, \Omega) - (1 + \cos^2 \theta) [\cos 2\gamma Q(v, \Omega) - \sin 2\gamma U(v, \Omega) + \sin 2\gamma Q(v, \Omega)] \}, \]

(A15)

the other components being easily recovered by the conjugation property (A14). It is important to remark that the linear polarization Stokes parameters, \( Q \) and \( U \), enter Equations (A15) only through the combinations: \([\cos 2\gamma Q - \sin 2\gamma U]\) and \([\cos 2\gamma U + \sin 2\gamma Q]\), which are invariant with respect to the arbitrary choice of the angle \( \gamma \) which defines the ‘reference direction’ and, consequently, the Stokes parameters \( Q \) and \( U \). If the most natural choice is performed (\( \gamma = 0 \)) the two combinations simply reduce to \( Q \) and \( U \), respectively.

An important property relates the trace of \( T_{qq}^* \) to the total intensity; taking into account Equations (A8) we have:

\[ \sum_q T_{qq} = \sum_{\alpha \beta} \sum_q S_{qq}(\alpha, \beta, \Omega) I_{\alpha\beta}(v, \Omega) = \sum_\alpha I_{\alpha\alpha}(v, \Omega) = I(v, \Omega). \]  

(A16)
From the tensor $\mathcal{J}_{qq'}(v, \Omega)$ one can define the other tensor $J_{qq'}(v)$ by integrating over the solid angle:

$$J_{qq'}(v) = \int \frac{d\Omega}{4\pi} \mathcal{J}_{qq'}(v, \Omega) = \int \frac{d\Omega}{4\pi} \sum_{i=0}^{3} \mathcal{T}_{qq'}(i, \Omega) S_i(v, \Omega)$$  \hspace{1cm} (A17)

with

$$J_{qq'}^{*}(v) = J_{q'q}(v).$$ \hspace{1cm} (A18)

This tensor depends on the polarized radiation field over the whole solid angle. Its expression can be obtained, in general, by an integration of Equations (A15). However, if the radiation field is cylindrically symmetric around the z-axis it is easy to prove that the non-diagonal terms are identically zero; in addition, if the radiation field is non-polarized, we simply have:

$$J_{-1-1}(v) = J_{11}(v) = \frac{1}{8} \int_{-1}^{1} (1 + \mu^2) I(v, \mu) \, d\mu,$$

$$J_{00}(v) = \frac{1}{4} \int_{-1}^{1} (1 - \mu^2) I(v, \mu) \, d\mu,$$ \hspace{1cm} (A19)

where $\mu = \cos \theta$.

From Equation (A16) it is possible to deduce an important property for the trace of $J_{qq'}^{*}(v)$:

$$\sum_{q} J_{qq}(v) = \int \frac{d\Omega}{4\pi} I(v, \Omega) = J(v)$$ \hspace{1cm} (A20)

which is the usual definition of the mean intensity of the radiation field over the solid angle (Mihalas, 1970).

Finally, from the tensor $J_{qq'}(v)$, the associated mean values over the absorption and anomalous dispersion profiles can be obtained:

$$\bar{J}_{qq'}(v_0) = \int_{0}^{\infty} \frac{v_0^2}{v^2} J_{qq'}(v) \phi(v_0 - v) \, dv,$$

$$\bar{F}_{qq'}(v_0) = \int_{0}^{\infty} \frac{v_0^2}{v^2} J_{qq'}(v) \psi(v_0 - v) \, dv.$$ \hspace{1cm} (A21)

When the radiation field has no relevant structure around the frequency $v_0$, the profiles can be simply substituted with the expressions

$$\phi(v_0 - v) = \delta(v_0 - v),$$

$$\psi(v_0 - v) = \frac{1}{\pi} \frac{P}{v_0 - v}$$ \hspace{1cm} (A22)
so that one gets:

$$\tilde{J}_{qq'}(v_0) = J_{qq'}(v_0),$$

$$\tilde{F}_{qq'}(v_0) = \frac{v_0^2}{\pi} \int_0^\infty \frac{d\nu}{\nu^2} \left( \frac{1}{v_0 - \nu} \right),$$

(A23)

where the notation $\tilde{\int}$ has been used to mean the principal part of the integral. The main properties of the tensor $\tilde{F}_{qq'}(v_0)$ are discussed in Appendix 2.

All the tensors introduced are reducible tensors. Under a rotation of the coordinate system, the new components, $T_{qq'}$, are obtained from the old ones, $T_{pp'}$, by means of the transformation

$$T_{qq'} = \sum_{pp'} \mathcal{D}_{pq}^1 (\alpha \beta \gamma) \mathcal{D}_{p'q}^{1*} (\alpha \beta \gamma) T_{pp'},$$

(A24)

where $(\alpha \beta \gamma)$ are the Euler angles of the rotation which brings the old system into the new system. Irreducible spherical components can be obtained by means of suitable linear combinations of the $T_{qq'}$:

$$T_{qq'}^K = \sum_{q'} (-1)^{1 + q} [3(2K + 1)]^{1/2} \begin{pmatrix} 1 & 1 & K \\ q & -q' & -Q \end{pmatrix} T_{qq'},$$

(A25)

with the inverse:

$$T_{qq'} = \sum_{KQ} (-1)^{1 + q} [(2K + 1)/3]^{1/2} \begin{pmatrix} 1 & 1 & K \\ q & -q' & -Q \end{pmatrix} T_{q'q'}^K.$$

(A26)

By means of this formulae it is possible to define the irreducible tensors $\mathcal{E}_{Q}^K(\alpha, \beta, \Omega)$, $\mathcal{F}_{Q}^K(i, \Omega)$, $\mathcal{J}_{Q}^K(v, \Omega)$, $\mathcal{P}_{Q}(v, \Omega)$, $\mathcal{L}_{Q}(v_0)$, and $\mathcal{F}_{Q}^K(v_0)$ from the corresponding reducible tensors $\mathcal{E}_{qq'}(\alpha, \beta, \Omega), \ldots$, and so on. These tensors will be needed in the following papers of this series.

Appendix 2: Some Properties of the Tensor $\tilde{F}_{qq'}$

The tensor $\tilde{F}_{qq'}(v)$ is defined by the expression

$$\tilde{F}_{qq'}(v) = \frac{v^2}{\pi} \int_0^\infty \frac{d\nu'}{\nu'^2(v - \nu')} J_{qq'}(v'),$$

(B1)

It enters, together with $\tilde{J}_{qq'}$, the statistical equilibrium equations for the matrix elements $\rho_{nm}$ (see Equations (75)) and it is responsible for the appearance of a particular kind of effect which consists in a modification of the coherences between different energy-levels due to a non-resonant interaction between the atomic system and the radiation field. Some discussion on the importance of this effect on the polarization properties.
of the He I D₃ lines from prominences can be found in Landi Degl’Innocenti (1982). To have an estimate of the importance of this tensor we can calculate its trace supposing that the radiation field is isotropic and non-polarized and has a black-body spectrum characteristic of the temperature $T$:

$$\bar{F}(v) = \sum_q \bar{F}_{qq} = \frac{
u^2}{\pi} \int_0^\infty P(v', T) \frac{1}{v'^2(v - v')} \, dv',$$  \hfill (B2)

where $P(v', T)$ is the Planck function. Introducing the dimensionless quantities $x = h\nu'/kT$ and $z = h\nu/kT$, we can write:

$$\bar{F}(v) = \frac{2k^3 T^3}{h^2 c^2} \mathcal{F}(z),$$  \hfill (B3)

where

$$\mathcal{F}(z) = \frac{z^2}{\pi} \int_0^\infty \frac{x}{z - x} (e^x - 1)^{-1} \, dx.$$  \hfill (B4)

Introducing the new variable $y = x/z$, one obtains:

$$\mathcal{F}(z) = \frac{z^3}{\pi} \int_0^\infty \frac{y}{1 - y} (e^{zy} - 1)^{-1} \, dy,$$  \hfill (B5)

and noting that

$$(e^{zy} - 1)^{-1} = \sum_{n=1}^\infty e^{-ny},$$

the integral appearing in $\mathcal{F}(z)$ results:

$$\sum_{n=1}^\infty \left\{ - \int_0^\infty e^{-ny} \, dy + \int_0^\infty \frac{1}{1 - y} e^{-ny} \, dy \right\}. \hfill (B6)$$

The first integral can be easily evaluated; for the second one, by the substitution $nz(y - 1) = w$, we have:

$$\int_0^\infty \frac{1}{1 - y} e^{-ny} \, dy = - \int_{-nz}^\infty w^{-1} e^{-(w + nz)} \, dw = e^{-nz} Ei(nz), \hfill (B7)$$

where $Ei$ is the exponential integral as defined in Abramowitz and Stegun (1965).

For $\mathcal{F}(z)$ we then obtain the expression

$$\mathcal{F}(z) = \frac{z^3}{\pi} \sum_{n=1}^\infty \left\{ e^{-nz} Ei(nz) - \frac{1}{nz} \right\}.$$  \hfill (B8)
which has the asymptotic expansions

\[ \mathcal{F}(z) \sim \frac{\pi}{6} \frac{z^3}{z} \]

\[ \mathcal{F}(z) \sim \frac{z^2}{\pi} \ln z \]  \hspace{1cm} (B9)

and has a zero value (\( \mathcal{F}(z_0) = 0 \)) for \( z_0 = 0.6359 \ldots \).

Finally, for the ratio \( R = \frac{\mathcal{F}(y)}{\tilde{J}(y)} \), we have:

\[ R(z) = \frac{1}{\pi} (e^z - 1) \sum_{n=1}^{\infty} \left\{ e^{-nz} \text{Ei}(nz) - \frac{1}{nz} \right\}. \]  \hspace{1cm} (B10)

References


