MAGNETO-OPTICAL EFFECTS AND THE DETERMINATION
OF VECTOR MAGNETIC FIELDS FROM STOKES PROFILES

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Abstract. The analysis procedure proposed by Auer et al. (1977) for deducing magnetic field vectors from
Stokes profiles has been tested to investigate the influence of magneto-optical effects on the deduced field
parameters. The quality of the fit between synthetic profiles generated with the inclusion of magneto-optical
effects and the profiles returned by the inversion routine is also investigated. The results show that
magneto-optical effects should be included in the inversion routine especially to increase the accuracy of the
deduced azimuth of the magnetic field.

1. Introduction

The problem of the determination of vector magnetic fields in the solar atmosphere is
a crucial one for the correct understanding of various phenomena typical of solar active
regions. A major step toward the solution of this problem has been achieved with the
construction and operation of good quality polarimeters able to measure the four Stokes
parameters profiles across spectral lines. One of these instruments, the HAO Stokes
polarimeter, is described in Baur et al. (1980, 1981). Once a complete set of Stokes
profiles has been observed for a given line, the problem arises of deducing a ‘measurement’
of the magnetic field from such data. A solution to this problem has been proposed
by Auer et al. (1977) who suggested a non-linear least square fit of the observed Stokes
profiles to the analytical solutions of the radiative transfer equations for polarized
radiation given by Unno (1956). This method has the advantage of being sufficiently
simple so that it can be applied to the interpretation of large amounts of data; in this
sense it can be preferred to more sophisticated approaches based on spectral synthesis
from existing model atmospheres. However, the method suffers from several limitations
which are implicit in the Unno’s solutions, namely:(a) Unidimensional plane-parallel
atmosphere with a constant magnetic field; (b) linear dependence of the source function
with optical depth; (c) constant ratio of the line and continuous absorption coefficients;
(d) magneto-optical effects neglected.

Once the real data are presented to the inversion routine based on the method
suggested by Auer et al. (1977), several difficulties are generally met. One is the presence
of asymmetric Stokes profiles which are believed to be due to velocity gradients in the
line forming region or to velocity shifts between unresolved magnetic elements. Even in
those cases where minor asymmetries are measured or when symmetrized profiles are
presented to the inversion routine, the resulting ‘best fit’ Stokes profiles are often
considerably different from the experimental ones (Querfeld, 1980). This fact is not
surprising due to the number of approximations contained in the Unno’s solutions.
However, what is more important to understand is to what extent the value of the magnetic field, returned from the inversion routine, is representative of the real physical conditions of the solar atmosphere and which is the role of the single approximations contained in the Unno’s solutions in determining the signature of the resulting differences between the ‘best fit’ and the experimental profiles.

In a preliminary attempt to clarify this argument, we have undertaken a detailed analysis of the single approximations contained in the Unno’s solutions. This paper is concerned with the influence of magneto-optical effects and represents an extension of a reduced analysis previously given by Auer et al. (1977). The importance of magneto-optical effects has already been pointed out in several aspects of the physics of transfer of polarized radiation in the solar atmosphere (Wittmann, 1971; Calamai et al., 1975; Landi Degl’Innocenti, 1979). A theoretical deduction of the magneto-optical terms in the transfer equations for polarized radiation can be found in Landi Degl’Innocenti and Landi Degl’Innocenti (1972, 1975).

2. Formulation

To test the influence of magneto-optical effects on the results obtained through the inversion routine based on the method of Auer et al. (1977), we have presented to the routine synthetic profiles computed including magneto-optical effects. Retaining the approximations (a), (b), and (c) specified in the introduction, but dropping approximation (d), the transfer equations for polarized radiation can be solved to give:

\[
I = B_0 + \mu B_1 A^{-1} \left[ (1 + \eta_I) (1 + \eta_I)^2 + \rho_Q^2 + \rho_U^2 + \rho_V^2 \right],
\]

\[
Q = -\mu B_1 A^{-1} \left[ (1 + \eta_I)^2 \eta_Q + (1 + \eta_I) (\eta_U \rho_U - \eta_U \rho_V) + \rho_Q (\eta_Q \rho_Q + \eta_U \rho_U + \eta_V \rho_V) \right],
\]

\[
U = -\mu B_1 A^{-1} \left[ (1 + \eta_I)^2 \eta_U + (1 + \eta_I) (\eta_V \rho_U - \eta_V \rho_Q) + \rho_U (\eta_Q \rho_Q + \eta_U \rho_U + \eta_V \rho_V) \right],
\]

\[
V = -\mu B_1 A^{-1} \left[ (1 + \eta_I)^2 \eta_V + \rho_V (\eta_Q \rho_Q + \eta_U \rho_U + \eta_V \rho_V) \right],
\]

where

\[
A = (1 + \eta_I)^2 \left[ (1 + \eta_I)^2 - \eta_Q^2 - \eta_U^2 - \eta_V^2 + \rho_Q^2 + \rho_U^2 + \rho_V^2 \right] - (\eta_Q \rho_Q + \eta_U \rho_U + \eta_V \rho_V)^2,
\]

\[\mu\] has its usual meaning, \(B_0\) and \(B_1\) define the linear source function versus the continuum optical depth at line wavelength: \(B(\tau) = B_0 + B_1 \tau\), and, finally,

\[
\eta_I = \frac{1}{2} \left[ \eta_p \sin^2 \psi + \frac{1}{2} (\eta_r + \eta_b) (1 + \cos^2 \psi) \right],
\]

\[
\eta_Q = \frac{1}{2} \left[ \eta_p - \frac{1}{2} (\eta_r + \eta_b) \right] \sin^2 \psi \cos 2\phi,
\]

\[
\eta_U = \frac{1}{2} \left[ \eta_p - \frac{1}{2} (\eta_r + \eta_b) \right] \sin^2 \psi \sin 2\phi,
\]

\[
\eta_V = \frac{1}{2} (\eta_r - \eta_b) \cos \psi,
\]

\[\]
\[
\rho_Q = \frac{1}{2} \left[ \rho_p - \frac{1}{2} (\rho_r + \rho_b) \right] \sin^2 \psi \cos 2\phi , \\
\rho_U = \frac{1}{2} \left[ \rho_p - \frac{1}{2} (\rho_r + \rho_b) \right] \sin^2 \psi \sin 2\phi , \\
\rho_V = \frac{1}{2} (\rho_r - \rho_b) \cos \psi ,
\]
where the angles \( \psi \) and \( \phi \) are defined as in Figure 1, and where the absorption and anomalous dispersion profiles relative to the parallel, red, and blue components are given by (for a normal Zeeman triplet)

\[
\begin{align*}
\eta_p &= \eta_0 H(a, v) , \\
\rho_p &= 2 \eta_0 F(a, v) , \\
\eta_{b,r} &= \eta_0 H(a, v \pm v_H) , \\
\rho_{b,r} &= 2 \eta_0 F(a, v \pm v_H) ,
\end{align*}
\] (3)

\( \eta_0 \) being the ratio between the line and continuous absorption coefficients, \( v \) and \( v_H \) being the wavelength separation from line center and the Zeeman splitting both normalized to the Doppler broadening and \( a \) being the damping constant. The functions \( H(a, v) \) and \( F(a, v) \) are the Voigt and Faraday–Voigt profiles respectively; in the limiting case of negligible damping they reduce to

\[
\lim_{a \to 0} H(a, v) = \exp (-v^2) ,
\]

\[
\lim_{a \to 0} F(a, v) = \frac{1}{\sqrt{\pi}} D(v) ,
\] (4)

where \( D(v) \) is the Dawson integral:

\[
D(v) = \exp (-v^2) \int_0^v \exp (t^2) \, dt .
\] (5)
Neglecting magneto-optical effects, Equations (1) reduce to the more familiar Unno's solutions which have the well known property of producing an identically zero value for the profile of the third Stokes parameter $U$ (for $\phi = 0^\circ$). This property is at the basis of the algorithm employed in the routine proposed by Auer et al. (1977) for obtaining the azimuth of the magnetic field vector. According to this algorithm the azimuth of the magnetic field is found by rotating the coordinate system which defines the polarization axes in such a way to minimize the quantity:

$$\chi^2 = \sum_i U_i^2 ,$$

where the sum runs over the wavelength points where the Stokes profiles are observed.

This procedure is justified only if the presence of noise in the observations is uniquely responsible for producing a non-zero value for $U$. As magneto-optical effects are present, this algorithm introduces a systematic error in the determination of the azimuth of the magnetic field vector. This error can be quantified through the angle $\Phi$ defined as in Figure 1. A positive value of $\Phi$ means that the value of the azimuth returned by the routine is displaced in a counterclockwise direction with respect to the real azimuth of the magnetic field. The angle $\Phi$ can be considered as a measurement of the importance of magneto-optical effects.

3. Results

We have computed the value of $\Phi$ presenting to the routine synthetic profiles as given by Equations (1) for a grid of values of $\eta_0, v_H, \nu, \psi$; the azimuth $\phi$ of the magnetic field has been assumed zero since $\Phi$ is independent of $\phi$. The values $B_0$ and $B_1$ appearing in Equations (1) are inessential due to the normalization of the profiles as described in Auer et al. (1977). In Figures 2a, b, c the value of $\Phi$ is plotted against $v_H$ for given values of $\psi$ and $\eta_0$. The values of $\Phi$ corresponding to $\psi > 90^\circ$ can be obtained by the symmetry relation

$$\Phi(180^\circ - \psi) = -\Phi(\psi) .$$

The results of Figures 2 show that the angle $\Phi$ is considerably large in several cases. The behaviour of $\Phi$ versus $v_H$ can be understood considering the fact that, for $v_H \rightarrow 0$ magneto-optical effects can be neglected, as shown in Landi Degl'Innocenti and Landi Degl'Innocenti (1973), while, for $v_H \rightarrow \infty$, the $U$ profile asymptotically goes to zero as can be easily proven through Equations (1).

In fact, when the splitting among the various Zeeman components $p, b, r$ is sufficiently large that the cross products of the profiles pertaining to different components are negligible, the quantity $(\eta_Q \rho_V - \eta_V \rho_Q)$, which appears in the expression for $U$, vanishes asymptotically.

For given $\eta_0$ and $v_H$, $\Phi$ decreases as the inclination angle $\psi$ increases from $0^\circ$ to $90^\circ$. This behaviour can be qualitatively understood noticing that the functional behaviour of $Q$ and $U$ with respect to $\psi$ is approximately given by

$$Q \sim \sin^2 \psi ,$$

$$U \sim \sin^2 \psi \cos \psi$$
Fig. 2a. The 'error' angle is plotted against the Zeeman splitting for several values of the inclination angle and for $\eta_0 = 1$.

Fig. 2b. Same as Figure 2a for $\eta_0 = 3$. 

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so that, for the angle $\Phi$, we obtain
\[
\Phi \sim \frac{1}{2} \arctan \left( \frac{U}{Q} \right) \sim \frac{1}{2} \arctan \left( \cos \psi \right).
\]  

Finally, the behaviour of $\Phi$ with $n_0$ is such that $\Phi$ increases monotonically with $n_0$. This fact is consistent with the idea that the importance of magneto-optical effects should increase with increasing line strength.

The results obtained by Auer et al. (1977) were restricted to the $\psi$ value of 60°. The maximum $\Phi$ value of 3° quoted in their paper is consistent with our results but we have to remark that $\Phi$ increases considerably for lower $\psi$ values.

Together with the determination of $\Phi$ we have also investigated the agreement of the input parameters $n_0$, $\psi$, and $v_H$ with the corresponding values returned by the routine and the quality of the fit obtained for the Stokes profiles. In each of the cases considered, the values returned by the routine were found in close agreement with the input parameters, with few exceptions, found especially for the parameter $n_0$. This means that
magneto-optical effects are more important for the determination of the azimuth of the magnetic field than for its amplitude or inclination.

Our analysis of the quality of the fit for the Stokes profiles, shows that the first Stokes parameter, $I$, is always well reproduced by the fitting routine. For what the circular polarization is concerned, we obtain in general a fairly good agreement except for those cases where the input profile shows a typical reversal near line center which cannot be reproduced by the fitting routine. This is no surprise, as the reversal is known to be due to magneto-optical effects (Wittmann, 1971). An example of this behaviour is shown in Figure 3. The largest disagreement is found for the linear polarization $Q$ profile and is fairly well correlated with the amplitude of the error angle $\Phi$.

![Graph](image)

**Fig. 3.** 'Observed' (solid line) and best fit (dashed line) circular polarization profiles plotted against the reduced wavelength $\nu$. The profiles are normalized to the central line depression $I_c - I_0$. The relevant parameters are $\eta_0 = 10$, $v_H = 3$, $\psi = 30^\circ$.

In Figure 4 some particular examples are shown. The solid line is the 'observed' linear polarization $Q$ profile in the rotated coordinate system which minimizes the quantity $\chi$ in Equation (6), while the dashed line is the result of the fit calculated neglecting magneto-optical effects. Two general features can be noticed in Figure 4, namely: (a) The profile which gives the best fit is always lower at line center than the 'observed' profile, and (b) its maximum value is somewhat displaced toward the wing of the line. Both these features have sometimes been remarked when applying the inversion routine to real observations (Querfeld, 1980).

To improve the quality of the fit for the linear polarization profile, one can think of assigning a greater weight $\omega$ to the $Q$ profile in the chi-square expression which is minimized in the inversion routine to obtain the best fit parameters (cf. Auer et al., 1977).
Fig. 4. ‘Observed’ (solid line) and best fit (dashed line) linear polarization profiles plotted against the reduced wavelength $\nu$ for different combinations of the relevant parameters. The profiles are normalized as in Figure 3.
It is reasonable to expect that, by so doing, some improvement may be obtained although for some peculiar profile, as the one shown in the left lower corner of Figure 4 (which is negative at all wavelengths), no close agreement can be expected to be found. We have indeed checked the results of the inversion routine assigning different weights to the Q-profile. While the quality of the fit of the Q curve is slightly improved, the values returned by the inversion routine for the other parameters are found to differ more and more from the input values as the weight is increased. An example of such behaviour is shown in Table I.

| TABLE I |
| Comparison between input and deduced values according to the weight ω assigned to the Q-profile (1 is assigned to I and V') |
|---|---|---|---|---|
| Input values | Deduced values | ω = 0 | ω = 1 | ω = 10 | ω = 50 |
| νH | 1.5 | 1.496 | 1.48 | 1.45 | 1.46 |
| θ | 15° | 151 | 133 | 813 | 72 |
| n₀ | 10 | 10.0 | 8.8 | 5.7 | 6.7 |

The results of Table I show that the best agreement between input and deduced values for the parameters is obtained for ω = 0 or, in other words, when the Q profile is not taken into account in the evaluation of the chi-square. This fact, together with the worsening of the agreement for increasing ω, indicates that the Q profile is the most affected by magneto-optical effects.

In the results presented above we have not specified the value a assigned to the damping constant which appears in Equations (3). Detailed calculations show that the above results hold independently of the value of a (at least for 0 ≤ a ≤ 0.1).

4. Conclusions

We have shown that the analysis procedure due to Auer et al. (1977) for deducing the magnetic field vector from observed Stokes profiles can produce large errors in the determination of the azimuth of the magnetic field. This error is due to the fact that magneto-optical effects are neglected in the quoted procedure. The error in the azimuth is larger for small inclination angles of the magnetic field to the line of sight and for intermediate values of the Zeeman splitting (0.5 ≤ νH ≤ 2.5). The analysis of the agreement between the Stokes profiles computed including magneto-optical effects and the ones returned by the inversion routine, shows that magneto-optical effects have a larger influence on the linear polarization profiles than on the other Stokes parameters. This fact has to be taken into account when applying the inversion routine to real experimental data.
In the light of these arguments we consider that the analysis procedure proposed by Auer et al. (1977) has to be improved by the inclusion of magneto-optical effects. Even if this can lead to some complication in the inversion routine or to some increase in computer time, we consider the inclusion of magneto-optical effects an essential step toward the achievement of a better diagnostic of solar magnetic fields.

References