Solitons in solar magnetic flux tubes

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Summary. It is suggested on theoretical grounds that solar intense magnetic flux tubes can support the propagation of solitons. Under photospheric conditions, a 'tube soliton' may propagate with a speed of about 7 km s\(^{-1}\) and an 'external soliton' with a speed of some 11 km s\(^{-1}\). We speculate that the tube soliton may be manifest in the chromosphere as a spicule.

1 Introduction

The distribution of magnetic flux on the surface of the Sun occurs not in a diffuse form but as isolated magnetic flux tubes (see reviews by Stenflo 1976; Harvey 1977; Parker 1979; Spruit 1981). Intense flux tubes have photospheric field strengths of 1–2 kG and diameters of 100–300 km. It seems likely, too, that stars of similar type to the Sun (e.g., dwarf main-sequence stars) will also possess a magnetic flux distribution (intense-tubes) similar to that found on the Sun. The small-scale nature of solar magnetic flux tubes makes their observational properties difficult to ascertain; flux tubes are a prime candidate for study from space.

The theoretical properties of intense flux tubes are now beginning to emerge (see, for example, Defouw 1976; Galloway et al. 1977; Parker 1978, 1979; Roberts & Webb 1978; Webb & Roberts 1978, 1980; Spruit & Zweibel 1979; Spruit 1979, 1981; Wilson 1980), and reveal a number of features that are likely to prove important for advancing our understanding of the solar atmosphere and convection zone. For example, it seems plausible that spicules, elongated jets of cool, dense gas seen to emerge from the chromospheric layers and penetrate into the hot corona (Beckers 1972), are actually vertical motions inside intense flux tubes; the tubes act as ducts for motions generated in the dense (and energetic) photosphere (Parker 1974; Roberts 1979; Hollweg 1981; Rae & Roberts 1982).

Here we point out a new feature of magnetic flux tubes: their ability to support solitons. Solitons are solitary waves that propagate without suffering any permanent change in form, and are naturally of much current interest in physics (cf. Bullough & Caudrey 1980). A soliton arises as a result of competing effects, typically the tendency of non-linear terms to become increasingly important, as a wave's amplitude grows and steepens, balanced by the
tendency for dispersive effects to spread the wave. In the case of a magnetic flux tube, the 
non-linearities arise in much the same way as would occur for an acoustic wave (in a field-
free medium), and the dispersive effects arise from the natural inertia of the tube’s 
surroundings.

2 Soliton theory

To be more specific, consider a uniform, two-dimensional, slab of magnetic field \( B \), with 
width \( 2x_0 \), embedded in a field-free atmosphere. The effects of gravity will be ignored. We 
are interested in symmetric pulsations of the slab (tube). Using the linear equations of ideal 
Mhd, and requiring that motions decline exponentially as we move away from the magnetic 
slab, it is easy to find the normal modes of the system (Roberts 1981). For the intense 
magnetic flux tubes of the solar photosphere, we are interested in the special circumstances 
of waves that are very much longer than the lateral dimensions of the field. In this case, the 
phase-speed \( c(k) \) of a wave of wavenumber \( k \) propagating along a thin magnetic slab is given 
approximately by (Roberts 1981)

\[
c(k) = c_T - \alpha |k|, \tag{1}
\]

where \( c_T \) is the sub-sonic, sub-Alfvénic tube speed, given in terms of the sound speed 
\( c_s = (\gamma p/\rho)^{1/2} \) and Alfvén speed \( v_A = B/(\mu \rho)^{1/2} \) by

\[
c_T^2 = c_s^2 + v_A^2.
\]

The gas pressure and density inside the slab are \( p \) and \( \rho \), respectively. The constant \( \alpha \), a 
measure of the environment’s inertia on the slab, is given by

\[
\alpha = \frac{1}{2} \left( \frac{\rho_e}{\rho} \right) x_0 \left( \frac{c_T}{v_A} \right)^3 c_T, \tag{2}
\]

where \( \rho_e \) is the gas density in the field-free medium. The expression (2) for \( \alpha \) has been 
written under the assumption that the equilibrium temperature in the slab is the same as that 
in the environment.

The form of (1), exhibiting the modulus of wavenumber \( k \), is a little unusual. Fortunately, 
it has arisen previously, in the study of water waves on an internal density interface 
(Benjamin 1967; Davis & Acrivos 1967; Ono 1975). The question we now ask is: what is 
the corresponding wave equation associated with (1) when non-linearities are allowed for? A 
general discussion of non-linear waves by Whitham (1967, 1974) suggests that the equation 
for a longitudinal motion \( v(z, t) \) along the slab \( Bz \) will be of the form

\[
\frac{\partial v}{\partial t} + \beta v \frac{\partial v}{\partial z} + \frac{\partial}{\partial z} \int_{-\infty}^{z} v(z', t) G(z' - z) \, dz' = 0, \tag{3}
\]

where \( \beta \) is a measure of the importance of non-linearity (assumed to arise quadratically), and

\[
G(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) \exp \left( -ikz \right) \, dk \tag{4}
\]

is the generalized Fourier transform of the phase-speed \( c(k) \).

Equation (3), known as Whitham’s equation, is a partial differential integral equation. No 
general discussion of its properties is available. However, for the case under investigation 
here, with \( c(k) \) given by (1), it is easy to show that

\[
G(z) = c_T \delta(z) + \frac{\alpha}{\pi} \left( \frac{1}{z^2} \right),
\]
where $\delta (z)$ is the Dirac delta function. Substituting in (3) then yields

$$\frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \beta v \frac{\partial v}{\partial z} + \alpha \frac{\partial^2 v}{\partial z^2} + H [v(z, t)] = 0,$$

where $H(v)$ is the Hilbert transform of $v$:

$$H [v(z, t)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(z', t) \, dz'}{z' - z},$$

the integral being a Cauchy principal value.

Equation (5) describes the longitudinal propagation of a weakly non-linear, dispersive sound pulse in a magnetic slab. An equation of similar form has arisen before, in the context of internal water waves (Benjamin 1967; Ono 1975), and is commonly referred to as the Benjamin—Ono equation. The above somewhat heuristic derivation of (5) may be made rigorous by a detailed multiple-scale analysis (Roberts & Mangeney 1982), and this in fact yields the value of $\beta$:

$$\beta = \frac{1}{2} v_s^2 \left[ 3c_e^2 + (\gamma + 1) \nu_A^2 \right] / \left[ c_e^2 + \nu_A^2 \right].$$

Consider now a wave $v(z, t)$ of permanent form, travelling along the slab of magnetic field with speed $s$. For such a waveform, an appropriate solution of (5) is (Benjamin 1967)

$$v(z, t) = \frac{a}{1 + \lambda^2 (z - st)^2},$$

where $a$ is the velocity amplitude, and the speed $s$ and scale $\lambda$ are given by

$$s = c_T + \frac{1}{4} \beta a, \quad \lambda = \frac{a \beta}{4 \alpha}.$$

Thus we see that the speed $s$ of the solitary wave exceeds the (linear) tube speed $c_T$ by an amount directly proportional to the velocity amplitude $a$. Note, too, that the speed $s$ depends purely upon the parameters of the slab (and not those of the environment). It is appropriate, then, to term this wave the ‘tube soliton’.

The above analysis pertains to the case of equal temperatures within and external to the slab. If, instead, the slab is slightly cooler than its surroundings, then we find that the slab can support another mode and with it another soliton. The second mode has a phase-speed $c(k)$ given, in linear theory, by (Roberts 1981)

$$c(k) = c_e - \alpha_e k^2,$$

valid for long wavelength propagation ($kx_0 \ll 1$). Here $c_e$ is the sound speed in the slab’s environment and the coefficient $\alpha_e$, given by

$$\alpha_e = \frac{1}{2} \left( \frac{\rho_e}{\rho} \right) x_0 \left( \frac{c_e^2}{c_e^2 + \nu_A^2} \right) \left( \frac{c_e^2 - c_s^2}{c_e^2 - c_T^2} \right) c_e,$$

is a measure of the effect of the slab’s inertia on propagation in the external medium.

The form of (8) is well-known and in fact leads, through (3), to the Korteweg-de Vries equation:

$$\frac{\partial v}{\partial t} + c_e \frac{\partial v}{\partial z} + \beta e v \frac{\partial v}{\partial z} + \alpha_e \frac{\partial^3 v}{\partial z^3} = 0,$$

valid for long wavelength propagation ($kx_0 \ll 1$). Here $c_e$ is the sound speed in the slab’s environment and the coefficient $\alpha_e$, given by

$$\alpha_e = \frac{1}{2} \left( \frac{\rho_e}{\rho} \right) x_0 \left( \frac{c_e^2}{c_e^2 + \nu_A^2} \right) \left( \frac{c_e^2 - c_s^2}{c_e^2 - c_T^2} \right) c_e.$$

$\beta = \frac{1}{2} v_s^2 \left[ 3c_e^2 + (\gamma + 1) \nu_A^2 \right] / \left[ c_e^2 + \nu_A^2 \right].$
where $\beta_e$ is a measure of the (assumed quadratic) non-linearity. Equation (10) possesses a soliton solution (e.g., Whitham 1974)

$$v(x, t) = a_e \text{sech}^2 \left[ \lambda_e (z - s_e t) \right],$$

(11)

where $a_e$ is the velocity amplitude, and the soliton speed $s_e$ and scale $\lambda_e$ are given by

$$s_e = c_e + \frac{1}{3} \beta_e a_e, \quad \lambda_e = \left( \frac{a_e\beta_e}{12a_e} \right)^{1/2}. \quad (12)$$

We shall term this second solitary wave the 'external soliton'.

3 Discussion

The above analysis has demonstrated that a magnetic flux tube of the type occurring in the solar photosphere can support the propagation of two types of solitary wave, one — the tube soliton — governed by the Benjamin-Ono equation and the other — the external soliton — governed by the Korteweg-de Vries equation.

Consider a numerical illustration. For intense magnetic flux tubes at the surface of the Sun we may take the sound and Alfvén speeds equal, say $c_s = v_A = 9 \text{ km s}^{-1}$, and the sound speed in the environment as $c_e = 10 \text{ km s}^{-1}$. Then the tube speed $c_T$ is $6.4 \text{ km s}^{-1}$ and the non-linear measure $\beta = 0.7$; thus the tube soliton propagates with a speed $s = 6.4 + 0.17a \text{ km s}^{-1}$. Velocity amplitudes at photospheric levels are unlikely to exceed those generated by granules. With granules producing speeds of up to $3 \text{ km s}^{-1}$, this suggests a soliton speed of about $7 \text{ km s}^{-1}$. In any case, it is unlikely that it will be so large that $s$ equals the sound speed, so the tube soliton moves with a speed between $c_T$ and (probably) $c_s$, with $7 \text{ km s}^{-1}$ as a reasonable estimate.

Similarly, the external (or fast) soliton is likely to propagate with a speed slightly in excess of the external sound speed, say at a speed of $11 \text{ km s}^{-1}$.

What evidence of such motions is there? Unfortunately there is no direct evidence, simply because of the extreme difficulties in observing such small-scale structures as intense flux tubes. Looking for indirect evidence, it is tempting to regard spicules as a possible soliton feature (though, here, we must remember that the important effects of gravity have so far been ignored in our treatment). We speculate that the tube soliton may become manifest in the chromosphere as a spicule.

In conclusion, then, our analysis predicts the possibility of two solitons propagating along an isolated flux tube, a tube soliton with a speed of some $7 \text{ km s}^{-1}$ (for solar photospheric conditions) and an external soliton with a speed of about $11 \text{ km s}^{-1}$. Direct observational evidence for such phenomena would be most interesting.

References


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