PULSE PROPAGATION IN A MAGNETIC FLUX TUBE

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ABSTRACT

The linear development of a pulse as it propagates adiabatically along an isothermal magnetic flux tube embedded in a gravitationally stratified atmosphere is studied. It is shown that, for a quiescent environment, longitudinal disturbances in the tube are governed by an equation of the Klein-Gordon type. An impulsively generated disturbance results in a wave front propagating at the subsonic and subAlfvénic tube speed; the wave front trails a wake oscillating at the tube frequency. The results are illustrated for solar photospheric conditions.

Subject headings: hydromagnetics — Sun: magnetic fields

I. INTRODUCTION

The Sun's magnetic field is distributed at its surface not in a diffuse and uniform form, but in a concentrated state, comprising the small scale, subtelescopic intense flux tubes at one extreme to the readily visible sunspots at the other. The structured nature of the solar atmosphere persists through the chromosphere and into the corona, where field lines are essentially delineated by magnetic loops. By analogy, it is likely that many magnetic stars are equally structured by the presence of magnetic field.

The magnetohydrodynamical nature of the coupling between the solar photosphere and chromosphere is of considerable current interest. The photospheric layers exhibit a number of examples of energetic phenomena, such as the supergranules and granules. Flows of 0.5–3.0 km s\(^{-1}\), typical of supergranules and granules in the high density medium of the photosphere, carry an energy density that is large in comparison with typical chromospheric values, where gas densities are perhaps \(10^{-5}\) times smaller than those in the photosphere. Thus, large amplitude chromospheric disturbances are to be expected if they occur as a result of dynamical coupling with disturbances in the photosphere. A question of interest, then, is: how does a disturbance at one level in the solar atmosphere communicate with another level?

An obvious communication channel is provided by the intense flux tubes, which constitute the bulk of the photospheric magnetic flux outside of sunspots (see reviews by Stenflo 1977; Harvey 1977; Zwann 1978; Parker 1979b). The tubes link the photosphere to the chromosphere in a direct fashion. It is of interest, then, to investigate how a disturbance in the photosphere, perhaps arising as a result of buffeting on the sides of a flux tube, propagates up an intense tube into the relatively tenuous atmosphere above.

To carry out such an investigation we shall use the slender flux tube equations (see, for example, Roberts and Webb 1978) in linearized form. Nonlinear terms will undoubtedly become important at some stage in the propagation of a disturbance, but our results indicate that linear theory should provide an adequate description of the evolution of an initial disturbance for propagation over several scale heights, say, from the photosphere at \(\tau_{\odot} = 1\) to the temperature minimum and perhaps higher. The use of slender flux tube theory allows a completely analytical description in a complicated wave propagation problem and may thus provide a useful means of comparison when nonlinear numerical codes become available (see Hollweg 1981 and references therein for results of some preliminary nonlinear computations).

As a further simplification in the use of the slender flux tube equations, we shall suppose that the atmosphere of the tube is isothermal and propagation is adiabatic. The assumption of isothermality is reasonable over the 0–1000 km (say) range above \(\tau_{\odot} = 1\) with which we shall be concerned. Nonadiabatic effects, however, are of some importance over this height range (Webb and Roberts 1980) and will tend to prevent the buildup of wave amplitudes to shock strengths (and the associated breakdown of linear theory). Indeed, in this context we note that the only available observational study of wave motions in slender flux tubes (see Giovanelli, Livingston, and Harvey 1978) shows that shocks do not occur in the first 1000 km above \(\tau_{\odot} = 1\). Linear, adiabatic slender flux tube theory is likely to provide, then, a useful qualitative guide to the structure of pulse propagation from the photosphere to the low chromosphere.

We should note, too, that for the most part we shall assume that the tube is embedded in a "quiescent" environment, so that the tube is only weakly coupled to...
its surroundings. The role of the environment on wave propagation is discussed in detail for the special case of no gravity.

To begin our study, it is convenient to recall that the characteristic speed of propagation of symmetric pulsations in a uniform magnetic flux tube embedded in an unstratified medium (see Defouw 1976; Roberts and Webb 1978) is the (slow) tube speed $c_T$, given in terms of the sound speed $c_s$ and Alfvén speed $v_A$ by

$$c_T = c_s v_A / \left( c_s^2 + v_A^2 \right)^{1/2}. \quad (1)$$

For a wave of frequency $\omega$ and longitudinal wave number $k$, the dispersion relation is

$$\omega^2 = k^2 c_T^2, \quad (2)$$

showing that propagation is nondispersive; both the phase speed $c_p \equiv \omega/k = \pm c_T$ and the group velocity $c_g \equiv \partial \omega / \partial k = \pm c_T$ are constants. An initial compression of such a tube, then, will propagate as a pulse with wave front speed $c_T$; as the pulse propagates, it leaves behind it an undisturbed atmosphere.

Propagation in a stratified atmosphere is more complicated. For linear disturbances in a slender tube, the slow tube wave has dispersion relation (Defouw 1976; Roberts and Webb 1978)

$$\omega^2 = k^2 c_T^2 + \omega_T^2, \quad (3)$$

where $\omega_T$, the cutoff frequency, is a constant in an isothermal atmosphere and depends upon $c_0$, $v_A$, and the gravitational acceleration $g$; for details, see §II. Propagation is now dispersive, the longer wavelengths (smaller $k$) having faster phase speeds but slower group velocities. The maximum group velocity is $c_T$, arising as $k \rightarrow \infty$. Thus, the effect of stratification (nonzero $g$) is to increase the speed of propagation of an individual mode above $c_T$; the wave front of an initial disturbance, however, travels at the maximum group speed, namely, $c_T$.

A dispersion relation of the form of equation (3) arises also for the vertical propagation of acoustic gravity waves in a field-free medium (Lamb 1932). The theory for acoustic gravity waves indicates that the propagation of a pulse results in the formation of a wake (Lamb 1932; see also Schmidt and Zirker 1963; Kato 1966; Stein and Schwartz 1972; Leibacher 1977). By analogy, then, an initial compression of a slender flux tube will result in a disturbance propagating along the tube with wave front speed $c_T$ (the maximum in the group velocity), behind which trails a wake which oscillates with frequency $\omega_T$. Thus, unlike the case of a uniform ($g = 0$) medium, the atmosphere behind the pulse is disturbed by the passage of the pulse and remains in motion long after the wave front of the pulse has passed.

Under photospheric conditions within an intense tube in temperature equilibrium with the external atmosphere, $\omega_T \sim 0.70 s^{-1}$ (corresponding to a cutoff period of 210 s) for isothermal conditions; nonisothermality results in a drop in the cutoff period to $\sim 186 s$ (Roberts and Webb 1978). An initial compression of an intense flux tube, then, results in the propagation of a pulse with wave front speed $c_T \sim 6 \text{km s}^{-1}$ under photospheric conditions) and the creation of a wake which oscillates with a period of 210 s, say (depending upon conditions). In the cooler atmosphere of a sunspot, if regarded as made up of individual flux tubes (see the suggestion by Parker 1979a), the cutoff frequency $\omega_T$ is somewhat higher and the corresponding period is lower, all of which is consistent with observed umbral oscillation characteristics.

II. THE KLEIN-GORDON EQUATION FOR PROPAGATION IN A TUBE

To discuss the propagation of longitudinal, isentropic motions in a slender magnetic tube it is convenient to treat first the general case of a slender elastic tube, specifying in a later section to a magnetic flux tube embedded in a field-free environment.

We suppose that the undisturbed state of the gas within an elastic tube of cross-sectional area $A_0(z)$ is one of hydrostatic equilibrium, with the pressure $p_0(z)$ and density $\rho_0(z)$ being related by

$$p_0(z) = -\rho_0(z) g \quad (4)$$

(here $z$ is the vertical axis, and the dash denotes differentiation of equilibrium quantities with respect to $z$).

Small amplitude disturbances about the equilibrium state (4) are supposed describable by the linear equations for longitudinal motion in a tube, namely, continuity, longitudinal momentum, and isentropic energy:

$$\frac{\partial}{\partial t} \left( \rho A_0 + \rho_0 A \right) + \frac{\partial}{\partial z} \left( \rho_0 A \phi \right) = 0, \quad (5)$$

$$\rho_0 \frac{\partial \phi}{\partial t} = -\frac{\partial p}{\partial z} - \rho g, \quad (6)$$

$$\frac{\partial p}{\partial t} + p_0 \phi = c_s^2 \left( \frac{\partial ^2}{\partial t^2} + \frac{\partial ^2}{\partial z^2} \right). \quad (7)$$

Here $p(z,t)$ and $\rho(z,t)$ are the perturbations in gas pressure and density at time $t$ and height $z$ within the tube of cross-sectional area $A(z,t)$, and $c_s(z) = (\gamma \rho_0(p_0)^{\gamma})^{1/2}$ is the sound speed of an ideal gas with (constant) adiabatic exponent $\gamma$. The velocity $\phi(z,t)$ along the tube is assumed to dominate over any transverse motions, and we consider only symmetrical (sausage mode) pulsations of the tube.
Introducing the speed $c(z)$ through the relation (Lighthill 1978, chap. 2)

$$\frac{1}{c^2} = \frac{1}{c_0^2} + \frac{\rho_0}{A_0} \frac{\partial A}{\partial p} \bigg|_{p = 0}$$

(8)

and the scaled velocity $Q(z, t)$ through

$$Q(z, t) = \left[ \frac{\rho_0(z) A_0(z) c^2(z)}{\rho_0(0) A_0(0) c^2(0)} \right]^{1/2} v(z, t),$$

(9)

allow us to reduce equations (5)-(7) to the equation (for details see Roberts 1981b)

$$\frac{\partial^2 Q}{\partial t^2} - c^2(z) \frac{\partial^2 Q}{\partial z^2} + \omega^2_0(z) Q = 0,$$

(10)

where

$$\omega^2_0 = N_0^2 + c^2 \left[ \frac{1}{2} \left( \frac{\rho_0'}{\rho_0} + \frac{A_0'}{A_0} + \frac{c_2'}{c^2} \right) + \frac{1}{4} \left( \frac{\rho_0'}{\rho_0} + \frac{A_0'}{A_0} + \frac{c_2'}{c^2} \right)^2 \right] + \left( \frac{g}{c_0^2} - \frac{A_0^2}{A_0} \right) \left( \frac{\rho_0'}{\rho_0} + \frac{c_2'}{c^2} + \frac{g}{c_0^2} \right),$$

(11)

and $N_0^2$ is given by

$$N_0^2 = g \left( \frac{\rho_0'}{\rho_0} + \frac{g}{c_0^2} \right),$$

is the square of the Brunt-Väisälä (buoyancy) frequency.

Equation (10), which describes the propagation of acoustic gravity waves in an elastic tube, is of the Klein-Gordon type (see Morse and Feshbach 1953). As such, we may anticipate that an initial disturbance in the tube will result in the propagation of a wave front followed by an oscillating wake. To exhibit this in some detail we consider the case of a rigid tube, leaving that of an (elastic) magnetic flux tube to the next section.

Suppose, then, that the tube is rigid, for which equation (8) immediately yields $c(z) = c_0(z)$. If, further, the gas within the rigid tube $A_0(z)$ is isothermal, then $c = c_0$ is a constant, as is the buoyancy frequency $N_0$, and $\omega^2_0(z)$ reduces to

$$\omega^2_0(z) = 1 + \frac{1}{2} \left( \frac{A_0'}{A_0} \right)^2 + \frac{1}{2} \Lambda_0^2 - \left( \frac{1}{\gamma} \right) \frac{A_0'}{A_0} \frac{1}{\Lambda_0} - \left( \frac{A_0'}{A_0} \right)^2 \right] \gamma,$$

(12)

where $\Lambda_0 = \rho_0 / \rho_0 g$ is the (isothermal) scale height of the atmosphere within the tube.

As an illustration of the above, consider the propagation of waves in a rigid, straight tube ($A_0(z) = 0$). Then $\omega = \omega_0 = c_0 / 2 \Lambda_0$, the cutoff frequency for vertically propagating acoustic gravity waves, and equation (10) becomes

$$\frac{\partial^2 Q}{\partial t^2} - c_0^2 \frac{\partial^2 Q}{\partial z^2} + \omega_0^2 Q = 0,$$

(13)

with $v(z, t) \propto \rho_0^{-1/2}(z) Q(z, t)$.

Normal mode solutions of equation (13) may be readily written down. For $Q(z, t) \propto e^{i(\omega - k z)}$, equation (13) yields the dispersion relation

$$k^2 = \frac{\omega^2 - \omega_0^2}{c_0^2}, \quad \omega = \pm \omega(k) = \pm \left( \omega_0^2 + k^2 c_0^2 \right)^{1/2},$$

(14)

showing that low frequency ($\omega < \omega_0$) disturbances are evanescent. Wave propagation ($\omega > \omega_0$) is dispersive, with the phase speed $\omega / k$ being in excess of, and the group speed $d\omega / dk$ being below, the sound speed.

This feature of propagation cutoff is manifest in the evolution of an initial disturbance by the occurrence of a wake. To see this, note that the formal solution of equation (13) may be written in the form (see, for example, Whitham 1974)

$$Q(z, t) = \int_{-\infty}^{\infty} f_+ (k) e^{i[kz + \omega(k)t]} dk + \int_{-\infty}^{\infty} f_- (k) e^{-i[kz - \omega(k)t]} dk,$$

(15)

with the functions $f_{\pm}$ determined by initial conditions, and $\omega(k)$ given by equation (14).

As a simple illustration of equation (15), suppose that the initial conditions are

$$Q(z, 0) = 0, \quad \frac{\partial Q}{\partial t}(z, 0) = q_0 \delta(z / \Lambda_0),$$

(16)

representing a $\delta$-function pressure pulse, say. Then equations (15) and (16) readily yield

$$Q(z, t) = \frac{q_0}{\pi} \frac{\Lambda_0}{\omega(k)} \int_{0}^{\infty} \frac{1}{\omega(k)} \cos kz \sin \omega(k)t \, dk,$$

which may be evaluated (Erdelyi 1954, p. 24) to give

$$Q(z, t) = \left\{ \begin{array}{ll}
q_0 \Lambda_0 \frac{\omega_0}{2c_0} J_0 \left[ c_0^2 \left( c_0^2 t^2 - z^2 \right)^{1/2} \right], & 0 < z < c_0 t, \\
0, & z > c_0 t,
\end{array} \right.$$
in terms of the Bessel function $J_0$. Equation (17) illustrates the propagation of a wave front with speed $c_0$; ahead of the front the gas is undisturbed, behind the front a wake oscillates with a frequency $\omega_a$.

As a second illustration of the propagation of acoustic gravity waves in a rigid tube, suppose that the tube's profile is exponential, $A_0(z) = A_0(0)e^{-z/\Lambda_0}$, e-folding in $\alpha^{-1}$ scale heights. The governing equation is (10) with now $v \propto [\rho_0(z)A_0(z)]^{-1/2}Q$, $c = c_0$ and (see also Hollweg and Roberts 1981)

$$\omega_a^2 = \left[ (\frac{4}{\gamma} - 2\alpha - 1) \right] \omega_a.$$ (18)

As before, the wave front propagates at the acoustic speed, but the tube geometry modifies the frequency at which the wake oscillates. The wake frequency $\omega_a$, regarded as a function of $\alpha$, has a minimum value $(2/\gamma)(\gamma - 1)^{1/2} \omega_a$, corresponding to a maximum period of

$$\tau_m = \frac{\gamma}{2(\gamma - 1)^{1/2}} \left( \frac{2\pi}{\omega_a} \right)^2 = 2\pi \left[ \frac{\gamma \Lambda_0}{(\gamma - 1)g} \right]^{1/2},$$ (19)

arising when $\alpha = (2/\gamma) - 1$. Hence, in a rigid isothermal tube the propagation of an initial disturbance results in the creation of a wake oscillating with a period below $\tau_m$. In the photospheric region from $T_{\text{5000}} = 1$ to the temperature minimum, $\tau_m$ is 214 s for $\gamma = 5/3$, 331 s for $\gamma = 1.2$; in a sunspot the corresponding maximum periods are lower, 175 s for $\gamma = 5/3$ and 270 s for $\gamma = 1.2$.

### III. THE MAGNETIC FLUX TUBE

We turn now to a discussion of propagation in a magnetic flux tube. There are two sources of complexity: the intrinsic elasticity of the flux tube, and its interaction with the environment in which the tube resides. The former effect has been allowed for in the derivation of equation (10), but the latter effect—the influence of an environment—has yet to be examined.

The equilibrium state of the magnetic flux tube will be taken to be an isothermal hydrostatic atmosphere (so that eq. [4] applies with $\Lambda_0$ constant) in temperature balance with its surroundings (so $\Lambda_0$ is also the scale height in the environment of the tube). Then, for a flux tube of field strength $B_0(z)$, cross-sectional area $A_0(z)$, gas pressure $p_0(z)$, and density $\rho_0(z)$, the equilibrium state is described by (see, for example, Roberts and Webb 1978)

$$p_0(z) = p_0(0)e^{-z/\Lambda_0}, \quad \rho_0(z) = \rho_0(0)e^{-z/\Lambda_0}, \quad B_0(z) = B_0(0)e^{-z/2\Lambda_0}, \quad A_0(z) = A_0(0)e^{z/2\Lambda_0}.$$ (20)

Note that the sound speed $c_0 = (\gamma p_0/\rho_0)^{1/2}$, Alfvén speed $v_A = B_0/(\rho_0 p_0)^{1/2}$, and the tube speed $c_T$ (see eq. [1]) are all constants. The magnetic permeability is $\mu$.

Linear perturbations about the equilibrium state (eq. [20]) are described by equations (5)-(7), supplemented by the equation of flux conservation,

$$B_0A + A_0B = 0,$$ (21)

and the assumption of transverse pressure balance

$$p + B_0/B = \pi_e,$$ (22)

for external pressure perturbation of value $\pi_e(z, t)$ on the boundary of the flux tube (Eq. [22] has been discussed by a number of authors, e.g., Parker 1979 a, b; Roberts and Webb 1979; Spruit 1981; Roberts 1981a). For the above system, we find (for details, see Roberts 1981b) that $Q(z, t)$ satisfies

$$\frac{\partial^2 Q}{\partial t^2} - c_T^{-2} \frac{\partial^2 Q}{\partial z^2} + \omega_a^2 Q = -e^{-z/\Lambda_0} \frac{c_T}{\rho_0(z)} \left( v_A \right) \left( \frac{\partial \pi_e}{\partial z} + \frac{g}{c_T^2} \pi_e \right),$$ (23)

where

$$\omega_a^2 = N_0^2 + \left( \frac{3}{4} - \frac{1}{\gamma} \right) \frac{c_T^2}{\Lambda_0^2},$$ (24)

and

$$v(z, t) = e^{z/\Lambda_0} Q(z, t).$$ (25)

Equation (23) governs the linear behavior of longitudinal motions in a slender flux tube embedded in a field-free environment.

Except under those circumstances where the effect of the terms in $\pi_e(z, t)$ is slight, equation (23) is not of the Klein-Gordon form. In general, to solve equation (23) it is necessary to supplement it with a system of equations governing motions in the environment of the tube as well as the condition of continuity of the normal component of velocity across the tube. The form of the motions $Q(z, t)$ will then depend upon the assumed conditions far from the tube. Rather than embarking upon such an undertaking here, we shall confine our attention to the circumstances of a passive environment, when the right-hand side of equation (23) may be neglected. In so doing we are ignoring, then, any modulation of tube motions that the environment may bring about.

A more precise treatment of equation (23) is possible for the special case of zero gravity, and this will be given in § IIIb where it is shown that wave propagation is...
rendered weakly dispersive by the inertia of the tube’s surroundings. Such weak dispersion is probably of little importance in a linear treatment, such as we give here; hence, the discussion of a tube in a quiescent environment ($\tau = 0$) is valid if nonlinearities and dispersive effects are considered as unimportant. However, the effect of weak dispersion may well be important when nonlinearities are properly accounted for. Indeed, the balancing of nonlinearity and dispersion leads to the formation of a soliton, which may propagate, with finite amplitude and stationary form, along a flux tube at a speed in excess of $c_T$. This result will be discussed more fully elsewhere (Roberts and Mangeney 1982).

**a) The Tube in a Quiescent Environment**

Consider equation (23) under the assumption that the environment is essentially quiescent, in the sense that the right-hand side of equation (32) is negligible. Then motions are again described by the Klein-Gordon equation,

$$\Box Q - c_T^2 \omega_e^2 Q = 0, \quad (26)$$

and are therefore similar in form to those described earlier for a rigid tube. A disturbance initiated at $t = 0$ will generate, then, a wave front traveling at the subsonic (and subAlfvénic) speed $c_T$ behind which a wake oscillates at frequency $\omega_e$. Thus, the effect of the flux tube’s elasticity may be summarized as leading to a slower propagation speed (for the wave front) and to a reduced oscillation frequency (for the wake):

$$\omega_e^2 = \left( \frac{3 - 2}{4} \right) \omega_0^2 - \left( \frac{3 - 2}{2} \right) \frac{\beta}{\beta + 2/\gamma} \omega_0^2. \quad (27)$$

The first term on the right-hand side of the above corresponds to the rigid exponential tube (with $\alpha = 1/2$), and the second arises from the flexibility of the tube’s boundary (as measured by the nonzero plasma beta, $\beta = 2\mu p_0 / B_0^2$).

To illustrate further the nature of motions in a flux tube embedded in a quiescent atmosphere, suppose now that motions in the tube are generated by an oscillating “base” at $z = 0$, represented by

$$Q(0, t) = Q_0 e^{i\omega t}, \quad t > 0. \quad (28)$$

We are envisaging buffeting on the sides of the flux tube due to motions of granules in the convection zone (see Parker 1974a, b; Roberts 1979) and, for the sake of mathematical simplicity, have represented such motions as producing a vertical oscillation of (small) amplitude $Q_0$ and frequency $\Omega_0$.

The solution of equation (26) subject to equation (28) and zero initial conditions may be shown to be

$$Q(z, t) = \begin{cases} Q_0 e^{i\omega_{d}(z/c_T)} - Q_0 \omega_r \frac{z}{c_T} e^{i\omega_{d}t} I(z, t), & 0 < z < c_T t, \\ 0, & z > c_T t, \end{cases} \quad (29)$$

where the integral $I$ is given by

$$I(z, t) = \int_{0}^{t} e^{i(\omega_{d}t - z/c_T)} \frac{1}{\sqrt{\pi}} \left( \frac{\tau^2 + z^2}{c_T^2} \right)^{1/2} J_1(\omega_0 \tau) e^{-\omega_{d}t} \frac{1}{\sqrt{2\pi}} d\tau, \quad (30)$$

for Bessel function $J_1$.

Equation (29) shows, then, the presence of a distinct wave front propagating with speed $c_T$, beyond which the disturbance is zero. Examining equations (29) and (30) for large $t$ reveals the asymptotic result, for $0 < z < c_T t$, that

$$Q \approx \begin{cases} Q_0 e^{i(\omega_{d}(z/c_T) - \omega_r^{2/3})^{1/2}}, & \Omega_0 > \omega_r, \\ Q_0 e^{i\omega_0 t} e^{i(\omega_{d}^{2/3} - \omega_r^{2/3})^{1/2}}, & \Omega_0 < \omega_r. \end{cases} \quad (31)$$

Thus, a generator with frequency above the tube cutoff leads to propagation, as we would expect, the wave front moving with group velocity $\Omega_0 c_T / (\Omega_0^2 - \omega_r^{2/3})^{1/2}$; frequencies generated below cutoff simply result in a nonpropagating oscillation of rapidly declining (in $z$) amplitude. For a discussion of the motions close to the wave front ($z \approx c_T t$), see Thau (1974).

**b) Environmental Effects**

We return now to the general equation (23) and make some assessment of the contribution from the terms in $\pi_{k}(z, t)$. We shall restrict attention to the case for which motions in the environment arise purely as a result of disturbances in the tube. Furthermore, we shall be concerned only with the case of laterally evanescent motions, so that the energy of the disturbance is confined to the tube. We are thus excluding the case for which the tube acts as an acoustic radiator of waves, which propagate out to infinity.

Consider, then, the possibility that motions in the tube generate a “weak” response in the tube’s surroundings, with motions in the environment decaying exponentially fast as we move laterally away from the tube. In this case we expect that the waves in the tube...
are essentially governed by the Klein-Gordon equation (26), with solutions modified only slightly by the inertial properties of the surroundings.

To relate motions in the tube, as governed by equation (23), to those in the environment, we introduce the fluid equations holding outside the tube:

\[
\begin{align*}
\frac{\partial p^{(e)}}{\partial t} + \text{div} \rho_e \varphi^{(e)} &= 0, \\
\frac{\partial \rho_e^{(e)}}{\partial t} &= -\nabla p^{(e)} - \rho_e \varphi^{(e)}, \\
\frac{\partial \rho_e^{(e)}}{\partial t} + \rho_e^e \varphi^{(e)} &= \frac{3}{\rho_e} \left[ \frac{\partial p^{(e)}}{\partial t} + \rho_e \varphi^{(e)} \right].
\end{align*}
\]

(32)

Here \( p^{(e)}, \rho_e, \) and \( \varphi^{(e)} \) are the perturbations in the pressure, density and velocity about the equilibrium state for which \( \varphi^{(e)} = 0, \rho_e(z) = \rho_e. \) Across the boundary of the tube, total pressure and normal component of velocity are continuous. The pressure continuity condition has already been incorporated in equation (23) through the use of relation (22). Hence, only the continuity of normal velocity remains to be applied.

To simplify our discussion of the manner in which longitudinal motions in the tube are influenced by the inertia of the tube's surroundings, it is convenient to consider the unstratified case, for which \( \Lambda_0 \to \infty, \omega_e = 0, \) \( \nu = Q, \) and equation (23) reduces to

\[
\frac{\partial^2 \varphi}{\partial t^2} - c^2 \frac{\partial^2 \varphi}{\partial z^2} = -\frac{1}{\rho_0} c_T^2 \frac{\partial^2 \pi_e}{\partial z \partial t}.
\]

(33)

For this case, equation (32) yields the wave equation for perturbation pressure \( p^{(e)} \). Solving the wave equation for the field-free medium allows us to determine \( \pi_e(z,t) = p^{(e)} \) evaluated (in linear theory) on the unperturbed boundary of the tube. Finally, the continuity of normal velocity and the assumption (consistent with the slender flux tube theory used to derive eqs. [23] and [33]) that internal transverse motions are approximately linear in the coordinate across the tube allows us to determine the motions in the tube.

The precise result depends upon the assumed geometry of the magnetic field, be it taken as a cylinder or a Cartesian slab. In Fourier form, with \( \varphi \propto e^{i(\omega t-kz)} \), we obtain the dispersion relation (Roberts and Webb 1979; Roberts 1981a)

\[
\left( k^2 c_T^2 - \omega^2 \right) \left( c_0^2 + v_A^2 \right) = \left( \frac{\rho_e}{\rho_0} \right) \omega^2 \left( k^2 c_0^2 - \omega^2 \right) K,
\]

where \( K \) depends upon the tube's geometry, be it a cylinder of radius \( r_0 \) or a slab of width \( 2x_0 \):

\[
K = \begin{cases} 
-\frac{1}{r_0} \log \left( m_x r_0 \right), & \text{cylinder}, \\
x_0/m_x, & \text{slab},
\end{cases}
\]

with \( m_x = \left[ k^2 - (\omega^2/c_T^2) \right]^{1/2} \), the transverse wavenumber outside the tube, assumed positive.

A scrutiny of equation (34) reveals (see Roberts and Webb 1979; Roberts 1981a for details) a solution with \( \omega^2 = \mu k^2 c_T^2 \), for long waves in a slender flux tube \((kr_0 \ll 1, kr_0 \ll 1)\). This is precisely in accord with the solution given by neglecting \( \pi_e \) in equation (33). Thus, we may conclude that the effect of the inertia of the tube's surroundings is to render the propagation of the tube wave as weakly dispersive.

We should note, too, that equation (34) yields a mode with phase speed close to \( c_T \), the sound speed in the environment of the tube. This mode is not present if \( \pi_e(z,t) \) is neglected. Hence, the neglect of \( \pi_e \) has essentially two effects: first, it removes the dispersive form of the tube wave \((\omega = kc_T)\), rendering it nondispersive; and, second, it eliminates the external sound mode \((\omega = kc_e)\). In general, then, there are modes which we may regard as essentially external in character, albeit slightly modified by the presence of the tube, and there are modes of the tube, albeit slightly modified by the exterior. By neglecting \( \pi_e \), we confine our attention to the tube wave and ignore its weak dispersive character. Our assumption is that in a stratified atmosphere (nonzero \( g \)) the neglect of \( \pi_e \) will simply lose the dispersive feature associated with the inertia of the tube's surroundings. If the external pressure field generated by motions in the tube penetrates only weakly into the surroundings, then this assumption seems justified (see also Webb 1980). In this respect, then, the discussion in § IIIa is pertinent and provides a useful guide to pulse propagation in a tube.

IV. DISCUSSION

The communication channels provided by the intense magnetic flux tubes that link the solar photosphere to the chromosphere raise the question as to how information injected in the photosphere propagates along the tube to the atmosphere above. Such channels, it would seem, provide important electrodynamic links between the relatively dense photosphere and the more tenuous chromosphere above, ducting steady downdrafts (Giovanelli and Slaughter 1978), outwardly propagating waves (Giovanelli, Livingston, and Harvey 1981), and probably spicules, too (see discussion in Parker 1974a, b, 1979b; Roberts 1979; Hollweg 1981).

We have illustrated here how impulsive motions may propagate along an intense tube; information (in the form of a wave front) propagates at the subsonic and subAlfvénic tube speed \( c_T \) (the speed \( c_T \), which is constant in an isothermal flux tube, turns out to be an important characteristic speed, even in a hydrostatically stratified tube). As the wave front of an impulsively generated disturbance propagates along the tube, it trails behind it a wake, which oscillates at the tube frequency.
This structure, of a wave front and wake, is typical of the Klein-Gordon equation and is similar to that occurring in the vertical propagation of acoustic gravity waves. Characteristically, in the region of the temperature minimum $c_T$ is $\sim 6$ km s$^{-1}$ and $\omega_c$ is $\sim 0.03$ s$^{-1}$.

On the other hand, if motions are created by a sustained source, switched on at time $t = 0$ and oscillating thereafter with frequency $\Omega_0$, then only for those frequencies $\Omega_0$ above cutoff $\omega_c$ will disturbances propagate; a low frequency generator simply lifts the atmosphere up and down without propagation.

Thus, the significance of frequency cutoff depends upon what are the circumstances creating the motions. If disturbances are impulsively generated, then wave propagation occurs and the existence of cutoff is manifest in the creation of an oscillating wake—all frequencies are generated, some propagating and others evanescent, with the net disturbance being the Fourier-integrated sum over all frequencies. If disturbances are as a result of a sustained frequency generator, then only if the generating frequency is above cutoff will disturbances propagate, and so under these circumstances the existence of cutoff may serve as an effective filter of motions. If, for example, we regard spicules as being generated impulsively, then the above considerations (contrary to an argument in Spruit 1981) show that such motions—at least in the linear approximation—may arise in a flux tube, despite the existence of propagation cutoff.

Of course, the nonlinear, nonadiabatic development of such impulsively generated disturbances is of obvious importance—especially with regard to spicules—and already some explorations (e.g., Hollweg 1981) are beginning. It is hoped that the present study will provide a basis upon which a comparison of linear and nonlinear theories may usefully be made.

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REFERENCES


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