NON-LTE LINE TRANSFER WITH PARTIAL REDISTRIBUTION

I. General Emission Profile

I. Hubený

Astronomical Institute, Czechoslovak Academy of Sciences, 251 65 Ondřejov, Czechoslovakia

Received 22 December 1980

I. Общий профиль испускания

An attempt to formulate, in a general way, the problem of the transfer of line radiation by multilevel atoms, allowing for the effects of partial redistribution, is presented. First, we try to analyse the available astrophysical partial redistribution studies in order to formulate basic physical assumptions underlying these studies. The physical description, called the classical approach, understood in this way, is discussed in detail, and its physical inconsistencies are pointed out. The classical approach is used to generalize Oenienius’ (1965) two-level treatment for the case of a multilevel atom. The main attention is devoted to the effects of the differences in the velocity distribution functions for atoms in different excited states. A suitable form of the emission profile is derived. Comparing its form to those considered in the studies of Milkey and Mihalas (1973) and Heasley and Kneer (1976), two points appeared to be different, namely the correct form of the stimulated emission term, and an extra term, which describes explicitly a response of differences in the velocity distribution functions for different excited states on the emission profile. The situations, in which these differences are of negligible importance, are briefly discussed.

1. Introduction

In recent years, it has been demonstrated that the departures from complete frequency redistribution can produce significant changes in the formation of spectral lines. Comprehensive reviews of work concerning the partial redistribution effects are given by Milkey (1976), Rybicki (1977), Mihalas (1978). A summary of older work can be found in Jeffersies (1968), Hummer and Rybicki (1971), and Athay (1972).

As it is well-known, a solution of the non-LTE line transfer problem requires, in general, a knowledge of the line source function. Two basic questions are to be solved here, namely a determination of the relevant atomic level populations, and a determination of the appropriate absorption and emission profiles. In a relatively simple case of complete redistribution, the absorption and emission profiles are supposed to be identical, and the problem reduces to the standard non-LTE line transfer problem (see Mihalas, 1978 and references cited therein). This consists in a simultaneous solution of the radiative transfer equation and the set of equations of statistical equilibrium.

Leaving out the assumption of complete redistribution, the problem becomes much more complicated, because the absorption and emission profiles are no longer identical. While a determination of the absorption profile is usually relatively simple, and is practi-
cally the same as in the former case, the emission profile now contains information about the manner in which photons are absorbed, emitted and redistributed within a line. In particular, the emission profile depends, in general, on the line radiation.

One can basically distinguish two types of difficulties involved in partial redistribution problems, namely i) the determination of the form of the emission profile, and ii) the solution of the highly non-linear set of equations for the radiation field (radiative transfer equation), atomic level populations (equations of statistical equilibrium), together with the definition of the emission profile.

As can be seen from the literature available on this topic, most attention is devoted to the second item. Several methods of treating this problem numerically have been developed. The most powerful are the sub-state-formalism of Milkey and Mihalas (1973) (and its generalization to allow for several coupled transitions with the common upper level — the so called cross-redistribution problem — of Milkey et al., 1975), and the method of Heasley and Kneer (1976). These methods are formulated in a way to enable the exploitation of a generalized complete linearization technique (Auer and Mihalas, 1969, 1970; Auer, 1968, 1973). A conceptually different method is the so called redistribution perturbation method (Cannon et al., 1975; Cannon, 1976). On the other hand, all the methods mentioned above take the form of the emission profile based on somewhat heuristic arguments.

The first item has been studied extensively in the important paper of Oxenius (1965). He showed, in the simple case of a two-level atom, that the emission profile depends, in a complicated way, not only on the radiation field and various transition rates involved, but also on the velocity distributions of atoms in the ground and in excited states. The latter dependence is due to the sensitivity of the absorption and emission processes to the particle velocity. While the velocity distribution of the ground-state atoms can be reasonably supposed to be nearly Maxwellian, the velocity distribution of the excited atoms depends, in general, on the microscopic parameters of the medium, in particular on the radiation field. He has derived the set of equations, which describe the level populations, the velocity distribution functions and the radiation field.

This formalism, even if physically more consistent (but see Section 2), is not formally very suitable for astrophysical applications, due to its great numerical complexity. Moreover, it is not clear, in the scope of this formulation, how one can take advantage of such properties of individual problems, which enable certain processes to be regarded as negligible.

On the other hand, the formulation of Milkey and Mihalas (1973), as well as that of Heasley and Kneer (1976), are very suitable for numerical calculations. However, it is not clear, what is their relation to the exact formulation of Oxenius (1965). In particular, both former formulations do not contain an explicit dependence of the emission profile on the velocity distribution of the excited atoms. This means, that either the Oxenius' formulation can be reformulated in a way to eliminate such a dependence analytically, or there are some tacit assumptions or simplifications (or even inconsistencies) in the studies of Milkey and Mihalas (1973) and Heasley and Kneer (1976).

The aim of this paper is then twofold:

1) to clarify, in the case of the two-level atom, to what extent the formulations of Milkey and Mihalas (1973) and of Heasley and Kneer (1976) are equivalent or restricted with respect to that of Oxenius (1965), and
2) to extend the idea of Oxenius' formulation to the more realistic case of the multilevel atom, and, in particular, find an explicit formulation, which is suitable for astrophysical applications (which means, particularly, a study of line formation in stellar atmospheres).

Before doing this, we feel it is first necessary to clarify the physics underlying these studies. We shall show that there are still several physical questions which are to be solved before any astrophysical formulation of the partial redistribution problem can be viewed as satisfactory.

This paper, which forms the first part of the present series, is devoted to a purely theoretical discussion. In the subsequent papers, we shall try to apply the formalism, developed here, to numerical solutions of some actual stellar atmospheric problems. In fact, this formalism has already been applied in order to treat the Lyman lines in calculating non-LTE model atmospheres of A-type stars properly (Hubeny, 1980, 1981a).

2. Basic Assumptions

The basic physical model is as follows: We consider a gas of several types of atoms and ions, and free electrons. We are interested in the transfer of line radiation by one given sort of atoms or ions, called active atoms or radiations, while the other particles are considered to be a perturber bath. Radiators are supposed to be mixed homogeneously with the
gas of perturbers. An interaction between radiators and perturbers is described by means of various collisional processes, like elastic (velocity changing) collisions, inelastic collisions (changes of the inner state of an atom), and interactions giving rise to pressure broadening of spectral lines. The whole system of particles is embedded in the radiation field.

The basic problem is the determination of the stationary distribution of the system of radiators + radiation field. But even a correct formulation of this problem, which has to be based on quantum statistical mechanics, is extremely difficult. Considerable progress has been made during recent years (see, e.g., Burnett et al., 1980; Burnett and Cooper, 1980a,b), but attention is mostly devoted to laser irradiated plasma.

On the other hand, it is believed that a simple "classical" description is capable of providing a basis for an understanding of the line transfer problems in stellar atmospheres. In this section, we shall summarize the concepts which yield the classical description and point out some of the questions concerning its applicability. The following analysis is by no means exhaustive or exact, we merely want to point out some of the problems which deserve further clarification.

We shall proceed as follows. First, we shall formulate explicitly two basic postulates, which determine the nature of the classical description.

1) The radiators are supposed to have both the classical degrees of freedom, described by the distribution function $F(r, v, t)$, ($r$ being the position, $v$ the velocity with respect to the laboratory frame, and $t$ the time); and the internal degrees of freedom, corresponding to different energy states (levels). The atoms at a certain energy level are considered as one sort of particles with the distribution function $F_i(r, v, t)$, where $i$ numerates the individual internal degrees of freedom. $F(r, v, t) = n_i(r, t) f_i(r, v, t)$, where $n_i$ is the corresponding number density (or level population), and $f_i$ is the velocity distribution function, normalized to unity (see Eq. 3.7). The steady-state distribution function $F(r, v)$ is assumed to obey the classical kinetic equation, generally written as (see Oxenius, 1979; Hubeny, 1981b)

$$v \cdot \frac{\partial F_i}{\partial r} = \left( \frac{\delta F_i}{\delta t} \right)_{el} + \left( \frac{\delta F_i}{\delta t} \right)_{inel} + \left( \frac{\delta F_i}{\delta t} \right)_{rad},$$

where the terms on the r.h.s. correspond to the elastic collision, inelastic particle collision, and the photon collision (or the radiative interaction) terms, respectively. Their explicit form will be discussed later.

2) The radiation field is assumed to behave like a gas of mutually noninteracting, massless particles — photons. Their distribution function $f_\nu(r, n, t)$, ($\nu$ being the frequency, $n$ is the unit vector in the direction of propagation), is usually expressed as the specific intensity of radiation $I_\nu(r, n, t)$. For simplicity, we shall consider unpolarized radiation. In the steady state, the radiation field is described by means of a kinetic equation for photons, i.e. the radiative transfer equation, which reads

$$n \cdot \frac{\partial I_\nu(n)}{\partial r} = -\kappa_f(n) I_\nu(n) + \eta_f(n),$$

where $\kappa_f(n)$ and $\eta_f(n)$ are the phenomenological (or macroscopic) absorption and emission coefficients, respectively. (The indication of their dependence on the position $r$ has been omitted.)

It is to be noted, that the above items express a degree of physical approximation, which is believed to be sufficient for stellar atmospheric applications. (In general, both above equations should be derived from the equation of motion for the global density matrix projected into the relevant subspaces.) The question, whether such an approximative description is indeed sufficient for the physical conditions prevailing in stellar atmospheres and for the degree of precision required by the appropriate spectroscopic diagnostics, remains to be solved. In particular, the question of the physical significance of the (instantaneous) distribution function $F_\nu(r, v, t)$ is, in our opinion, the most stringent one.

We shall not discuss these fundamental assumptions, as such an analysis would be beyond the scope of this work.

3) Further specification of the classical description is associated with the manner in which one specifies the absorption and emission coefficients. We shall analyze the classical procedure, as represented by the approach of Oxenius (1965) or its generalization presented here (Sections 3–6), in order to point out further concepts of the classical description.

3a) First, both coefficients are written in terms of the normalized profiles (see Eqs 3.1 and 3.2) and these are, in turn, written in terms of simple integrals of the microscopic (velocity-dependent) profiles over the relevant velocity distribution functions (see Eqs 3.4 and 3.5). This form is an approximation, because some correlations between the radiator motion and the interaction, giving rise to the broadening of spectral lines, can exist (see Ward et al., 1974). However, this effect can usually be neglected under stellar atmospheric conditions.

3b) The form of microscopic absorption and emission profiles is the most vulnerable point of the classi-
cal description. Let us turn to the absorption profile first. It is usually assumed that

\[(2.3) \quad Q_n(n, v) = q(\xi); \quad \xi = v - \frac{v_0}{c} n \cdot v,\]

where \(Q_n(n, v)\) is the microscopic absorption profile for atoms with velocity \(v\), \(q\) is the absorption profile in the rest-frame of the atom, and \(\xi\) is the frequency \(v\) measured in the atomic frame; \(v_0\) and \(c\) being the central frequency of the line and the light speed, respectively. The transformation (2.3) is valid for \(v/c \ll 1\), which is always the case in stellar atmospheric calculations.

An important assumption is that the form of \(q\) is taken as a given function of frequency (and, in general, of the physical parameters of the perturber gas, like temperature, perturber density, etc.). In other words, it is taken as an atomic parameter, following from independent quantum-mechanical calculations (see, e.g., Griem, 1974). Consequently, the form of \(q\) is given regardless of the previous history of the atom under consideration, while, at the same time, the microscopic emission coefficient is expressed in the form which takes the previous history of the atom into account (see Eqs. 3.8 and 3.9). In other words, whenever the emission profile is written in terms of the redistribution function (see 3.9), the "previous history" of the atom is considered. This means that the probability of emission depends, in some way, on the previous absorption of a photon.

As emphasized above, the astrophysical formulations of the partial redistribution problem allow for a different form for the absorption and emission profiles. The reasons for the difference of the (microscopic) emission and absorption profiles have been discussed several times (see, e.g., Jefferies, 1968). However, such explanations are normally based on the classical view. For the aims of subsequent discussion, we shall briefly repeat them. Let us consider a line transition connecting the lower level \(l\) and the upper level \(u\). If the rate of the radiative transition \(u \rightarrow l\) is much smaller than the total rate of transitions out from the level \(u\), then the absorption \(l \rightarrow u\) and the emission \(u \rightarrow l\) can be regarded as two independent one-photon processes, and the equality of both profiles (for absorption and emission) can be deduced from detailed balancing arguments (Jefferies, 1968). Otherwise, the whole process \(l \rightarrow u \rightarrow l\) has to be regarded as one two-photon process, and a correlation between the frequencies and directions of the absorbed and emitted photons can appear. If the energy spectrum of exciting particles (i.e. true particles or photons) is essentially constant over the "width" of the energy levels, the profiles for absorption and emission are again identical. We shall refer to this situation as the "natural excitation" (or natural population) — see Jefferies (1968). On the other hand, if an excitation is not natural, as for example the photo-excitation \(l \rightarrow u\) by radiation with an intensity not constant over the line, both profiles need not be identical.

It is usually assumed that the lower level \(l\) is populated naturally. This is practically always true for the ground level. If, however, \(l\) is an excited level, one can simply imagine a situation where the rate of a radiative transition, for example that of \(1 \rightarrow l\) (1 denotes the ground level) is much greater than the rate of other transitions into the level \(l\). If, moreover, the radiation field in the transition \(1 \rightarrow l\) is not constant over the line width of level \(l\), one cannot even suppose, in general, that the absorption profile for the transition \(l \rightarrow u\) is given by the "natural excitation" absorption profile, because the whole process \(1 \rightarrow l \rightarrow u\) has to be regarded as one two-photon process.

This is the intrinsic inconsistency of all the existing descriptions by means of the classical picture. One might, however, proceed along the same lines as for the emission profile and introduce the generalized "redistribution function". The latter would represent a combined probability of a general two-photon process (i.e. not only for absorption + emission, but also for others: absorption + absorption, emission + absorption). Both profiles, the emission and absorption ones, would then be written in a form (similar to 3.9) which allows for possible deviations from the natural excitation of a relevant level.

However, we shall not consider this phenomenon in the present study, as a correct quantum-mechanical treatment is needed before a suitable formulation of this problem for astrophysical purposes can be developed. Moreover, it seems that such an effect is rather small in the usual stellar atmospheric applications.

3c) Another fundamental problem is connected with the generally non-Markovian nature of the scattering process. In order to be able to formulate the kinetic equation in the form of (2.1), it is possible to recover the Markovian nature of the problem by considering the scattering process \(l \rightarrow u \rightarrow l\) as the following string of processes, which are supposed to be distinct in time: i) the absorption \(l \rightarrow u\), which excites an atom to the (broadened) upper level \(u\); ii) an "elastic" process, which changes the energy state of an atom still at the upper level \(u\); and iii) the emission \(u \rightarrow l\). One can visualize this concept by introducing a set of "sublevels" of each energy level. The Markovian nature of the scattering process is then
maintained, if the transition between two sublevels of the upper level depends only on the actual state of the atom, not on its previous history. (For purposes of this paper, we shall refer to this concept as the quasi-Markovian behaviour, even if this term is not quite correct.) One note seems to be pertinent: In this context, the inconsistency of the classical description, mentioned above, can simply be reformulated as follows: The form of the emission profile allows for the quasi-Markovian behaviour of the scattering process, while that of the absorption profile is taken in a completely Markovian way.

It is a highly schematized view of the redistribution problem. However, practically all the present studies assume, often tacitly, such an approximation. On the other hand, this a view can be justified by the quantum mechanical calculations of Omont et al. (1972), whose form of the redistribution function allows this interpretation. This is not surprising, as they assumed the impact approximation, which is closely related to our concept of the quasi-Markovian behaviour. However, this justification is valid only for absorbed and emitted frequencies within the domain of validity of the impact approximation. Recently, the theory of line scattering has been extended to include frequency regions outside the impact regime (Nienhus and Schuller, 1977; Cooper, 1979; Burnett et al., 1980; Burnett and Cooper, 1980a,b), but it is not yet clear, how to formulate this problem for astrophysical purposes.

4) All the above assumptions are substantial for determining the concepts of the classical picture. That is to say, their omission would yield a qualitatively different description. On the other hand, there are other, "less fundamental" assumptions, which are also to be mentioned explicitly. Their omission would yield a description which remains (at least under certain favorable physical conditions in a medium) within the frame of the classical picture, defined by the set of assumptions 1–3. Strictly speaking, they have a well-defined physical meaning only in the classical picture.

4a) This first question is connected with a relevant description of the scattering process in the laboratory frame. If we leave out the question of possible correlation between atomic velocity and pressure broadening mentioned above, there remains the question of the correlation between atomic velocities and scattering.

Oxenius (1965) introduced two limiting approximations, referred to as the correlated and non-correlated scattering. The former corresponds to the situation, where the velocity of the atom is not changed during the scattering process, while, in the latter case, it is assumed that, during the lifetime of the upper state, elastic collisions take place that change the velocity of the atom thus destroying the correlation between the absorbed and emitted photons completely. The intermediate case has been studied by Argyros and Mugglestone (1971a,b), within the scope of the quasi-Markovian, classical description.

However, their approach is physically inconsistent, due to the following reason. Even if velocity-changing collision could leave (but very rarely) an excited atomic state unchanged, they must change the phase of the emitted photon, thus destroying, to some extent, the correlation between the absorbed and emitted photons. This effect is entirely neglected by Argyros and Mugglestone (see also the discussion in Hummer and Rybicki, 1971). A correct description of such phenomena would require a more general approach than that provided by the classical picture. Nevertheless, these effects are rather small, as the elastic collision frequency is much smaller than the lifetime of the excited states in most astrophysical applications (see Oxenius, 1979). The assumption of the correlated scattering (i.e. no velocity changes during a scattering process) appears to be reasonable. This assumption will then be used throughout this paper.

4b) Another question is related to the form of the appropriate kinetic equation. If we adopt the quasi-Markovian, classical picture described above, the relevant collision terms in Eq. (2.1) can be written in the usual form (see, e.g., Oxenius, 1979; Hubeny, 1981b). From the point of view of further applications, it is necessary to estimate the importance of the individual terms, in particular, the so called streaming term — the l.h.s. of Eq. (2.1). This question is discussed in detail by both latter authors. According to (Hubeny, 1981b), the streaming term is practically always negligible under stellar atmospheric conditions. Moreover, we also consider the elastic collision term to be negligible, consistently with the above discussion. The kinetic equation (2.1) then reads

\[
\frac{\delta F_i}{\delta t} = 0.
\]

The explicit form of the collision terms will be given in the following sections.

To summarize, we shall adopt the quasi-Markovian, classical picture outlined above, in spite of several underlying physical inconsistencies, and try to develop a formalism which is suitable for stellar atmospheric applications. The present work thus represents a direct generalization of Oxenius' (1965) approach, but not an exact treatment. Further theoretical work is then
needed in order to clarify the questions pointed out in this section. We hope to attack this problem in the near future.

3. Two-Level Atom

The absorption and emission coefficients for line radiation at frequency \( \nu \) and propagating in a direction specified by the unit vector \( \mathbf{n} \) can be written, following Oxenius (1965), as

\[
\chi_{s}(\mathbf{n}) = \frac{h\nu}{4\pi} [n_1 B_{12} \varphi_{s}(\mathbf{n}) - n_2 B_{21} \varphi_{s}(\mathbf{n})]
\]

and

\[
\eta_{s}(\mathbf{n}) = \frac{h\nu}{4\pi} n_2 A_{21} \psi_{s}(\mathbf{n}),
\]

respectively. Here \( B_{12} \), \( B_{21} \) and \( A_{21} \) are the usual Einstein coefficients of absorption, stimulated emission and spontaneous emission, respectively; \( n_1 \) and \( n_2 \) are the populations of the lower and upper atomic energy levels, respectively. \( \varphi_{s}(\mathbf{n}) \) and \( \psi_{s}(\mathbf{n}) \) are the profiles of absorption and emission, respectively. More specifically, \( \varphi_{s}(\mathbf{n}) \, d\nu \, d\Omega/4\pi \) is the probability of an absorption of a photon with frequency in the range \( (\nu, \nu + d\nu) \) propagating in the solid angle \( d\Omega \) centered about the unit vector \( \mathbf{n} \) — hereinafter we refer to that photon as \( (\nu, \mathbf{n}) \). The physical interpretation of the emission profile is completely analogous. As stressed by Oxenius (1965), both the stimulated and spontaneous emissions have the same profile \( \psi_{s}(\mathbf{n}) \), which follows from the general quantum-mechanical results.

Both profiles are normalized

\[
\int \int \varphi_{s}(\mathbf{n}) \, d\nu \, d\Omega/4\pi = \int \int \psi_{s}(\mathbf{n}) \, d\nu \, d\Omega/4\pi = 1.
\]

According to Oxenius (1965), it is possible to write (see also item 3a of Section 2)

\[
\varphi_{s}(\mathbf{n}) = \int f_{1}(\nu) \, Q_{s}(\mathbf{n}, \nu) \, d\nu \, d\Omega/4\pi,
\]

\[
\psi_{s}(\mathbf{n}) = \int f_{2}(\nu) \, E_{s}(\mathbf{n}, \nu) \, d\nu \, d\Omega/4\pi,
\]

where \( Q_{s}(\mathbf{n}, \nu) \), \( [E_{s}(\mathbf{n}, \nu)] \), is the absorption [emission] profile for particles with velocity \( \nu \), \( f_{1}(\nu) \) and \( f_{2}(\nu) \) are the velocity distribution functions for atoms in the lower and upper states, respectively. Both profiles \( Q_{s} \) and \( E_{s} \), as well as \( f_{1} \) and \( f_{2} \) are normalized, i.e.

\[
\int \int Q_{s}(\mathbf{n}, \nu) \, d\nu \, d\Omega/4\pi = \int \int E_{s}(\mathbf{n}, \nu) \, d\nu \, d\Omega/4\pi = 1,
\]

and

\[
\int f_{1}(\nu) \, d\nu \, d\Omega = \int f_{2}(\nu) \, d\nu \, d\Omega = 1.
\]

As discussed above, one has to determine the profiles \( \varphi_{s}(\mathbf{n}) \) and \( \psi_{s}(\mathbf{n}) \), i.e., the velocity-dependent profiles \( \tilde{Q}_{s}(\mathbf{n}, \nu, \mathbf{v}) \), \( E_{s}(\mathbf{n}, \nu, \mathbf{v}) \), together with the velocity distribution functions \( f_{1}(\nu) \) and \( f_{2}(\nu) \). This is a very complicated problem which requires a coupled set of the radiative transfer and the kinetic equations (2.1) and (2.2).

The problem can be simplified by assuming that the velocity distribution function \( f_{1}(\nu) \) is given a priori. If the lower level is sufficiently sharp (i.e., the lifetime of an atom in the ground state is sufficiently large), the velocity distribution can be assumed to be Maxwellian. In fact, this is always assumed in astrophysical applications.

Under this assumption, the absorption profile \( \varphi_{s}(\mathbf{n}) \) is given a priori (i.e., without a dependence on the radiation field and/or populations), as

\[
\varphi_{s}(\mathbf{n}) = \int f_{1}(\nu) \, Q_{s}(\mathbf{n}, \nu) \, d\nu \, d\Omega/4\pi = \int f_{1}(\nu) \, q \left( \nu - \frac{\nu_0}{c} \cdot \mathbf{v} \right) \, d\nu \, d\Omega/4\pi,
\]

where \( q \) is the absorption profile in the rest-frame of an atom.

On the contrary, one cannot assume the velocity distribution \( f_{2}(\nu) \) to be also given. The kinetic equation (2.4) must, in general, be solved. Nevertheless, it is possible to write down an expression for the emission profile, where the velocity distribution \( f_{2}(\nu) \) plays a minor role in usual astrophysical situations as, e.g., for radiative transfer in resonance lines.

We start with the general expression for the velocity-dependent emission profile (Oxenius, 1965)

\[
E_{s}(\mathbf{n}, \nu) = \alpha(\mathbf{n}) \, n_{1} \, f_{1}(\nu) \, B_{12} \left[ 1 - R(\nu, \nu', \mathbf{n}, \mathbf{n}) \right] \, d\nu \, d\Omega/4\pi + C_{12} \, Q_{s}(\mathbf{n}, \nu),
\]

where \( R(\nu', \nu, \mathbf{n}, \mathbf{n}) \) is the (normalized) redistribution function for atoms with velocity \( \mathbf{v} \),

\[
\int \int \int \int R(\nu', \nu, \mathbf{n}, \mathbf{n}) \, d\nu' \, d\Omega'/4\pi \, d\nu \, d\Omega/4\pi = 1,
\]

and \( C_{12} \) is the rate of collisional excitation, supposed to be independent of the velocity \( \mathbf{v} \).
The redistribution function is defined so that
\[ R(v', n', v, n, v) \, dv' \, d\Omega' / 4\pi \, dv \, d\Omega / 4\pi \] is the joint probability that an atom with velocity \( v \) absorbs a photon \( (v', n') \) and emits a photon \( (v, n) \), provided that no process destroying the excited atomic state intervenes (Hummer and Rybicki, 1971).

The normalization parameter \( \alpha(v) \) is given by
\[
\alpha(v) = \left\{ n_1 f_1(v) \left[ B_{12} \int I_{sc}(n') \, Q_{sc}(n', v) \, dv' \, d\Omega' / 4\pi + \right. \right. \\
\left. \left. + \, C_{12} \right] \right\}^{-1}.
\]

The equations (3.8)–(3.11) represent the quasi-Markovian, classical view of the redistribution problem, as discussed in section 2. The first term in Eq. (3.9) corresponds to an emission of a photon by an atom excited by a previous photo-excitation (i.e. a non-natural excitation), the second term corresponds to an emission following collisional excitation (i.e. a natural excitation).

The definition expressions for the emission profile, viz Eqs (3.5), (3.9) and (3.11), are not very suitable for numerical calculations. In the following, we shall transform these equations to a more suitable form. Let us first introduce the auxiliary quantities
\[
\varrho_n(n, v) = \int I_{sc}(n') R(v', n', v, n, v) \, dv' \, d\Omega' / 4\pi,
\]
\[
\mu(v) = \int I_{sc}(n') Q_{sc}(n', v) \, dv' \, d\Omega' / 4\pi,
\]
\[
\lambda(v) = \int I_{sc}(n') E_{sc}(n', v) \, dv' \, d\Omega' / 4\pi.
\]

The emission profile given by Eq. (3.5) contains integration over the velocity distribution \( f_2(v) \). In order to eliminate the explicit dependence of the emission profile on \( f_2(v) \), we first write down the kinetic equation for the excited atoms. In the steady state, this equation represents the balance between creations and destructions of excited atoms with velocity \( v \). According to Oxenius (1965) this reads
\[
n_1 f_1(v) \, d^3v [B_{12} \mu(v) + C_{12}] = \]
\[= n_2 f_2(v) \, d^3v \left\{ \int A_{21} E_{sc}(n', v) \left[ 1 + \frac{B_{21}}{A_{21}} I_{sc}(n') \right] \right. \]
\[\left. \cdot \, dv' \, d\Omega' / 4\pi + C_{21} \right\}.
\]

The first term on the l.h.s. represents the creations due to photo-excitations from the lower level, the second term creations due to collisional excitations; the first (integral) term on the r.h.s. represents destructions of the upper states with velocity \( v \) due to emissions (spontaneous and stimulated), and the second term destructions by collisional deexcitations.

Combining equations (3.9) and (3.11), and using the notations (3.12), (3.13), we obtain
\[
E_n(n, v) n_1 f_1(v) [B_{12} \mu(v) + C_{12}] = n_1 f_1(v) [B_{12} \varrho_n(n, v) + C_{12} \varrho_n(n, v)].
\]

Now, multiplying equation (3.15) by \( E_n(n, v) \) and substituting into Eq. (3.16), one arrives at
\[
n_1 f_1(v) \, d^3v [B_{12} \varrho_n(n, v) + C_{12} \varrho_n(n, v)] = n_2 f_2(v) \, d^3v [A_{21} + B_{21} \lambda(v) + C_{21}].
\]

This equation can be interpreted as follows: The number of atoms in the excited state with velocity \( v \) (more precisely, in the volume \( d^3v \) around \( v \) in the velocity space), which emit photons \( (v, n) \), is \( n_2 f_2(v) \). \( E_n(n, v) \, d^3v \, dv \, d\Omega / 4\pi \); equation (3.17) thus represents a steady-state number conservation equation for such defined particles.

Integrating over the velocities we obtain the following equation
\[
n_1 [B_{12} \mathcal{R}_n(n) + C_{12} \varphi_n(n)] = n_2 \psi_n(n) \left[ A_{21} + B_{21} \mathcal{J}_n(n) + C_{21} \right],
\]
which represents the steady-state number conservation equation for atoms in the excited state, regardless of their velocities, which emit photons \( (v, n) \). Here the simplifying notations
\[
\mathcal{R}_n(n) = \int f_1(v) \varrho_n(n, v) \, d^3v =
\]
\[= \int \int f_1(v) I_{sc}(n') R(v', n', v, n, v) \, d^3v \, dv' \, d\Omega' / 4\pi,
\]
and
\[
\mathcal{J}_n(n) = \frac{\int f_2(v) E_n(n, v) \lambda(v) \, d^3v}{\int f_2(v) E_n(n, v) \, d^3v},
\]
are used.

The emission coefficient is then
\[
n_1 \mathcal{R}_n(n) + n_1 C_{12} \varphi_n(n) \frac{hv}{4\pi},
\]
where
\[
a_n(n) = \frac{A_{21}}{A_{21} + B_{21} \mathcal{J}_n(n) + C_{21}}.
\]
represents the probability of a spontaneous emission of a photon \((v, n)\).

In order to compare the expression (3.21) with other studies, we substitute the expression following from the velocity-integrated equation for the steady-state number conservation of excited atoms (3.15), namely

\[
(3.23) \quad n_1(B_{12}J_e + C_{12}) = n_2(A_{21} + B_{21}J_e + C_{21}),
\]

for the second term on the r.h.s. of Eq. (3.21). In equation (3.23),

\[
(3.24) \quad J_e = \int f_1(v) \, \mu(v) \, d^3v = \int I_e(n) \, \varphi_e(n) \, dv \, d\Omega/4\pi,
\]

and

\[
(3.25) \quad J_e = \int f_2(v) \, \lambda(v) \, d^3v = \int I_e(n) \, \psi_e(n) \, dv \, d\Omega/4\pi.
\]

We then finally obtain

\[
(3.26) \quad \eta_e(n) = \frac{\hbar v}{4\pi} \, a_e(n) \, \{n_2(A_{21} + B_{21}J_e + C_{21}) \cdot \varphi_e(n) + n_1B_{12}[\varphi_e(n) - J_e \, \varphi_e(n)]\}.
\]

Expression (3.26) represents the simplest and most suitable form of the emission coefficient for astrophysical applications. Nevertheless, it is not completely independent of \(f_2(v)\), as the term \(\mathcal{J}_e(n)\) in \(a_e(n)\) still contains an integral over the velocity distribution of the excited states \(f_2(v)\). Moreover, the term \(J_e\) contains explicitly the emission profile \(\psi_e(n)\). Equation (3.26) is thus suitable particularly in cases, where the stimulated emission rate is much smaller than the spontaneous emission rate, i.e. for \(\hbar v/kT > 1\) (\(h\) and \(k\) being the Planck and Boltzmann constants, respectively, and \(T\) the temperature).

Let us now turn to the discussion of the angular dependence of the absorption and emission profiles. If the velocity distribution function \(f_1(v)\) is isotropic, i.e. \(f_1(v) = f_1(|v|)\), the absorption profile \(\varphi_e(n)\) is then also independent of direction, as follows from Eq. (3.4). On the other hand, the emission profile need not, in general, be independent of direction, even in the static case, namely due to the term \(\mathcal{J}_e(n)\).

If, however, the radiation field is supposed to be isotropic, then

\[
(3.27) \quad \mathcal{J}_e = \int \mathcal{J}_e(n) \, d\Omega/4\pi = \int J_e(R', v) \, dv' \, d\Omega.
\]

where

\[
(3.28) \quad R'(v', v) = \int R(v', n', v, n) \, d\Omega'/4\pi \, d\Omega/4\pi
\]

and

\[
(3.29) \quad R(v', n', v, n) = \int f_1(v) \, R(v', n', v, n, v) \, d^3v.
\]

Here, \(R(v', n', v, n)\) is the velocity-averaged redistribution function, defined in the usual way; \(R(v', v)\) is the velocity- and angle-averaged redistribution function, commonly used in stellar atmospheric work. The mean intensity of radiation

\[
(3.30) \quad J_e = \int I_e(n) \, d\Omega/4\pi
\]

is equal to \(I_e\) in the case of an isotropic radiation field.

In the case of an isotropic radiation field, the angle-averaged form of equation (3.18) is then

\[
(3.31) \quad n_1(B_{12}\overline{\mathcal{J}}_e + C_{12}\overline{\varphi}_e) = n_2\psi_e(A_{21} + B_{21}\overline{\mathcal{J}}_e + C_{21}),
\]

where the angle-averaged emission profile \(\overline{\varphi}_e\) is defined as

\[
(3.32) \quad \overline{\varphi}_e = \int \psi_e(n) \, d\Omega/4\pi = \int f_2(v) \, E_e(v) \, d^3v,
\]

and

\[
(3.33) \quad E_e(v) = \int E_e(n, v) \, d\Omega/4\pi.
\]

As follows from Eqs. (3.17), (3.18), (3.32), (3.33), \(\overline{\mathcal{J}}_e\) is given by

\[
(3.34) \quad \overline{\mathcal{J}}_e = \frac{\int \int f_2(v) \, E_e(v) \, J_e(v) \, dv' \, d^3v}{\int f_2(v) \, E_e(v) \, d^3v}.
\]

Analogously as in Section 3, we can derive the angle-averaged form of the emission coefficient

\[
(3.35) \quad \eta_e = \frac{\hbar v}{4\pi} \, a_v \left[n_2(A_{21} + B_{21}J_e + C_{21}) \, \varphi_e \right. - \left. n_1B_{12}J_v\varphi_v + n_1B_{12}\overline{\mathcal{J}}_e\right],
\]

where the term \(a_v\) is defined analogously as in (3.22), with the angle-averaged \(\overline{\mathcal{J}}_e\) instead of \(\mathcal{J}_e(n)\).

4. The Stimulated Emission Term

We see that our equation (3.35) is analogous to that used by Heasley and Kneer (1976) — except the back-
ground emissivity — their Eq. (2). However, there is a difference between these equations. They write down the quantity $a_v$ (in our notation) as

$$a_v = \frac{A_{21}}{A_{21} + B_{21}J_v + C_{21}},$$

arguing that the probability of spontaneous emission at frequency $\nu - a_v$ is given by the ratio between the number of spontaneous emissions $A_{21}$ and the number of all transitions from the upper level, i.e. spontaneous emissions $A_{21}$, collisional de-excitations $C_{21}$, and stimulated emissions at frequency $\nu - B_{21}J_v$. They justified the form of the latter contribution by the argument that the stimulated emission is coherent both in the atom's and observer's frame. The same argument has also been used by Milkey and Mihalas (1973).

However, as follows from Eq. (3.17) and the subsequent discussion, this probability has to be given as

$$a_v = \frac{A_{21}}{A_{21} + B_{21}J_v + C_{21}}.$$

Generally, $J_v$ is not necessarily equal to $J_v$. A close inspection of equation (3.17), especially the stimulated emission term, shows that the depopulation rate of excited atoms with velocity $v$ which emit photons $(\nu, n)$ as seen by the observer, due to stimulated emissions, has to be given as $\int [B_{21} E_v(n', v) l_v(n')] dv' d\Omega'/4\pi$, rather than $B_{21} I_v(n)$.

Let us discuss this point in detail. We interpret $n_2 f_3(v) d\nu$ as the number of excited atoms with velocity $v$. The rate of their destruction due to stimulated emission is $\int [B_{21} E_v(n', v) l_v(n')] dv' d\Omega'/4\pi$. Let us further similarly interpret $n_2 f_3(v) E_v(n, v) d\nu$ as the number of those excited atoms with velocity $v$ which emit photons $(\nu, n)$. The number of their destructions by stimulated emissions (per unit volume and time) is then $n_2 f_3(v) E_v(n, v) d\nu d\Omega'/4\pi$. The integral term accounts for the destructions of all excited atoms with velocity $v$ by all stimulated emissions, while the first part (i.e. $n_2 f_3(v) E_v(n, v) d\nu$) accounts for the proper portion of those excited atoms with velocity $v$ which emit photons $(\nu, n)$. The coherence of the stimulated emission appears in the first $E_v(n, v)$, as that is the same for both stimulated and spontaneous emissions.

The number of destructions of excited atoms with velocity $v$ by spontaneous emissions (per unit volume and time) is $n_2 f_2(v) A_{21} d\nu$. Should we analogously consider the number of corresponding destructions by stimulated emissions as $n_2 f_3(v) B_{21} l_v(n) d\nu$, arguing that the stimulated emission is frequency-coherent (as in Milkey and Mihalas, 1973; Heasley and Kneer, 1976), we did not include the overall destruction of excited atoms with velocity $v$ by all stimulated emissions.

Very recently, this point has been discussed independently by Baschek, Mihalas and Oxenius (1981); their conclusions are practically the same as ours.

5. Multilevel Atom

In this section, we try to generalize the classical approach, used in Sections 3 and 4, to the multilevel case. Although this seems, at first sight, to be straightforward, we shall repeat the above applied procedure step by step again, thus pointing out the physical inconsistencies involved and visualizing the assumptions, which are usually made tacitly.

We shall, therefore, start with an introductory example of a three-level atom with continuum. This example allows us to encounter all the basic physical mechanisms involved and various types of coupling between different transitions. As soon as the physics of this three-level case is clarified, a further generalization to more realistic model atoms is rather a matter of a complicated but formal handling of mathematical expressions.

a) First Resonance Transition

Let us first consider the resonance line 1 $\leftrightarrow$ 2. In order to determine the emission profile, it is necessary to take into account all possible mechanisms populating the first excited level.

The first mechanisms are, as in the case of a two-level atom, the spontaneous emissions following previous excitation — both radiative and collisional one — of an atom in the ground state (as before, it is sufficient to restrict oneself to considering the spontaneous emissions only, because the stimulated emissions behave identically). Then the first part of the microscopic emission profile is the same as in the case of the two-level atom, namely (generalizing the notation of Section 3)

$$(5.1) \quad n_1 f_1(v) \left[ B_{12} q_{12}(v, n, v) + C_{12} Q_{12}(v, n, v) \right].$$

On the other hand, there are the additional mechanisms here, namely the emission following the previous transition 3 $\rightarrow$ 2, both radiative and collisional. It is reasonable to suppose that the collisional part is given by

$$(5.2) \quad n_3 f_3(v) C_{32} Q_{12}(v, n, v),$$
since de-excitation collisions occur at random (i.e.,

the level 2 naturally). As before, it is supposed

that the rate coefficient \( C_{32} \) is independent of velocity,

and that a collisional de-excitation does not change an

atomic velocity.

The situation is much more complicated in the case

of the radiative contribution. This term should,

in general, be given by

\[
(5.3) \quad n_3 f_3(v) \int \left[ A_{32} + B_{32} \delta_{32}(n') \right] \cdot R_{321}(v', n', v, n, v) \, dv' \, d\Omega' / 4\pi ,
\]

where \( R_{321} \) gives the probability amplitude that two

photons, \((v', n') - \) in the line 2 \( \leftrightarrow 3 \), and \((v, n) - \)
in the line 1 \( \leftrightarrow 2 \), are successively (spontaneously)

emitted. From the quantum-mechanical viewpoint,

such correlation between frequencies and directions

of both photons can exist if the combined transition is

regarded as one two-photon process. However, we

have already neglected similar effects (of the two-

photon absorption) in the form of the absorption

profile – see Section 2. We shall therefore suppose,

for simplicity, that both photons are completely uncorrelated,

and, moreover, that the “redistribution” function \( R_{321} \)
given by the product of the relevant profiles,

\[
(5.4) \quad R_{321}(v', n', v, n, v) = E_{32}(v', n', v, n, v) .
\]

This equation can be understood as follows: The photon

\((v', n') - \) is emitted according to the emission

profile \( E_{32} \), while the photon \((v, n) - \) is emitted

according to the absorption profile \( Q_{12} \), as it seems reasonable
to suppose that transition 3 \( \rightarrow 2 \) populates the

level 2 naturally. The corresponding part of the microscopic emission profile can thus be written as

\[
(5.5) \quad n_3 f_3(v) \left[ A_{32} + B_{32} \lambda_{32}(v) + C_{32} \right] Q_{12}(v, n, v) .
\]

The next two mechanisms are the emissions following the previous recombination \( k \rightarrow 2 \), both radiative

and collisional. Again, a reasonable approximation of the collisional part is

\[
(5.6) \quad n_k f_k(v) C_{k2} Q_{12}(v, n, v) .
\]

We use the same approximation even for the radiative part,

\[
(5.7) \quad n_k f_k(v) R_{k2} Q_{12}(v, n, v) ,
\]

where \( R_{k2} \) and \( C_{k2} \) are the radiative recombination

and collisional recombination rates, respectively. The latter approximation is justified if recombinations

can be assumed to populate the bound atomic levels

naturally, which is usually a good approximation.

Moreover, we take the velocity distribution of ions
to be Maxwellian, i.e., \( f_i(v) = f_1(v) \).

The total microscopic emission profile for the

transition 2 \( \rightarrow 1 \) is then

\[
(5.8) \quad E_{21}(v, n, v) =
\]

\[
= \alpha_{21}(v) \left[ n_1 f_1(v) \left[ B_{12} q_{121}(v, n, v) + C_{12} Q_{12}(v, n, n, v) \right] + n_3 f_3(v) Q_{12}(v, n, v) \right] \cdot \left[ A_{32} + B_{32} \lambda_{32}(v) + C_{32} \right] +
\]

\[
+ n_k f_k(v) Q_{12}(v, n, n, v) \left[ R_{k2} + C_{k2} \right] ,
\]

where the normalization parameter \( \alpha \) is given by

\[
(5.9) \quad [\alpha_{21}(v)]^{-1} = n_1 f_1(v) \left[ B_{12} \mu_{12}(v) + C_{12} \right] +
\]

\[
+ n_3 f_3(v) \left[ A_{32} + B_{32} \lambda_{32}(v) + C_{32} \right] +
\]

\[
+ n_k f_k(v) \left[ R_{k2} + C_{k2} \right] .
\]

We now follow the procedure outlined in Section 3,

which consists in rewriting the rather complicated
definition expressions for the emission profile, Eqs

(5.8) and (5.9), in a more suitable form. We start with

the steady-state number conservation equation for

excited atoms with velocity \((v, d^3v) - \) i.e., the kinetic equation. Generalizing Eq. (3.15) and using the approximation (5.5), we obtain

\[
(5.10) \quad n_1 f_1(v) \left[ B_{12} \mu_{12}(v) + C_{12} \right] +
\]

\[
+ n_3 f_3(v) \left[ A_{32} + B_{32} \lambda_{32}(v) + C_{32} \right] +
\]

\[
+ n_k f_k(v) \left[ R_{k2} + C_{k2} \right] =
\]

\[
= n_2 f_2(v) \left[ Q_{21} + B_{21} \lambda_{21}(v) + C_{21} +
\]

\[
+ B_{23} \mu_{23}(v) + C_{23} + R_{2k} + C_{2k} \right] ,
\]

where \( R_{2k} \) and \( C_{2k} \) are the rates for the radiative and

collisional ionizations, respectively.

Using the same procedure as for deriving Eq. (3.17),

we now obtain its generalized form

\[
(5.11) \quad n_1 f_1(v) \left[ B_{12} q_{121}(v, n, v) + C_{12} Q_{12}(v, n, v) \right] +
\]

\[
+ n_3 f_3(v) \left[ A_{32} + B_{32} \lambda_{32}(v) + C_{32} \right] .
\]

\[
. Q_{12}(v, n, v) + n_1 f_1(v) \left[ R_{k2} + C_{k2} \right] Q_{12}(v, n, v) =
\]

\[
= E_{21}(v, n, v) n_2 f_2(v) .
\]

\[
\left[ A_{21} + B_{21} \lambda_{21}(v) + C_{21} + B_{23} \mu_{23}(v) +
\]

\[
+ C_{23} + R_{2k} + C_{2k} \right] ,
\]

which again can be viewed as a steady-state number

conservation equation for atoms in the second level

with velocity \( v \), which emit photons \((v, n) \).
In order to derive the macroscopic emission profile, one has to integrate Eq. (5.11) over all velocities. In the case of a two-level atom, there was a problem with the stimulated emission term on the r.h.s. only; other types of integrals, i.e. \( \int f_2(v) Q_{12}(v, n, v) \, dv \) or \( \int f_3(v) E_{21}(v, n, v) \, dv \) have led immediately to \( \varphi_{12}(v, n) \) or \( \psi_{21}(v, n) \), respectively. In the present case, we encounter further complicated terms. Let us postpone their discussion, and first introduce, quite formally, the following definitions

\[(5.12a) \quad \varphi_{12}^3 = \int f_3(v) Q_{12}(v, n, v) \, dv, \]

\[(5.12b) \quad \varphi_{12}^1 = \int f_1(v) Q_{12}(v, n, v) \, dv, \]

\[(5.13) \quad J_{32}^{12} = \frac{\int f_3(v) Q_{12}(v, n, v) \lambda_{32}(v) \, dv}{\int f_3(v) Q_{12}(v, n, v) \, dv}, \]

\[(5.14) \quad J_{23}^{21} = \frac{\int f_2(v) E_{21}(v, n, v) \mu_{23}(v) \, dv}{\int f_2(v) E_{21}(v, n, v) \, dv}, \]

and, similarly to Eqs (3.19) and (3.20),

\[(5.15) \quad J_{21}^{21} = \frac{\int f_2(v) E_{21}(v, n, v) \lambda_{21}(v) \, dv}{\int f_2(v) E_{21}(v, n, v) \, dv}, \]

\[(5.16) \quad R_{121} = \int f_1(v) Q_{121}(v, n, v) \, dv. \]

The integrated form of equation (5.11) is then

\[(5.17) \quad n_1(B_{12}R_{121} + C_{12} \varphi_{12}^1) +
+ n_3(A_{23} + B_{32}J_{32}^{12} + C_{32}) \varphi_{12}^3 +
+ n_4(R_{k2} + C_{k2}) \varphi_{12}^2 =
= n_2\psi_{21}[A_{21} + B_{21}J_{21}^{21} + C_{21} + B_{23}J_{23}^{21} +
+ C_{23} + R_{k2} + C_{k2}]. \]

The resulting form of the emission coefficient, using the definition (3.2), is now

\[(5.18) \quad \eta_{21}(v, n) = \frac{hv}{4\pi} a_{21}(v, n) [\text{l.h.s. of eq (5.17)]}, \]

where

\[(5.19) \quad a_{21}(v, n) = A_{21} + B_{21}J_{21}^{21}(v, n) + C_{21} +
+ B_{23}J_{23}^{21}(v, n) + C_{23} + R_{k2} + C_{k2}], \]

which generalize the two-level-atom results (3.21) and (3.22).

The final step consists in rewriting Eq. (5.18) by means of the velocity-integrated steady-state equation of number conservation of atoms in the second level,

\[(5.20) \quad n_1(B_{12}J_{12} + C_{12}) +
+ n_3(A_{32} + B_{32}J_{32} + C_{32}) +
+ n_4(R_{k2} + C_{k2}) = n_2P_2, \]

where

\[(5.21) \quad P_2 = A_{21} + B_{21}J_{21}^* + C_{21} +
+ B_{23}J_{23}^* + C_{23} + R_{k2} + C_{k2}, \]

which represents the total rate of transitions out from the second level; \( J_{21}^* \), \( J_{12}^* \), \( J_{23}^* \), and \( J_{32}^* \) are given by the corresponding generalized form of Eqs (3.24) and (3.25) (see also the list of general expressions in Section 6).

Finally, using Eqs. (5.18), (5.17), and (5.21), we obtain

\[(5.22) \quad \eta_{21} = \frac{hv}{4\pi} a_{21} [n_2P_2\varphi_{12}^1 + n_1B_{12}]. \]

\[(5.23) \quad corr_{21} = n_3(A_{32} + B_{32}J_{32}^{12} + C_{32}) \varphi_{12}^3 -
- n_3(A_{32} + B_{32}J_{32}^* + C_{32}) \varphi_{12}^1. \]

The form of the emission coefficient given by Eq. (5.22) allows a simple physical interpretation, which is based on the comparison with the equivalent equation (3.26) for a two-level atom.

Provided that all mechanisms populate the second level naturally, i.e. yield the emission profile equal to the absorption profile, the emission coefficient would be given by the first term in square brackets in Eq. (5.22) only. Note, however, that even if the microscopic emission profile \( E_{21} \) is equal to the microscopic absorption profile \( Q_{12} \) under such an assumption, the similar equality of the macroscopic (velocity-averaged) profiles is not, in general, achieved, due to different velocity distributions.

In this artificial case, the emission coefficient is proportional (as can also be expected from a rather heuristic point-of-view) to:

\[(\text{probability of spontaneous emission}) \times (\text{total rate of transitions out from the second level}) \times (\text{absorption profile}).\]
In reality, natural excitation cannot, in general, be assumed for the scattering process 1 → 2 → 1, so that the rate of the radiative transition 1 → 2 must be subtracted and the correct rate of emission 2 → 1 following absorption 1 → 2 must be added. This yields the second term in Eq. (5.22).

Up to this point this interpretation is equivalent to that discussed by Heasley and Kneer (1976), and previously by Griem (1964), for two-level atom. But in our case, there is the extra term, designated corr$_{31}$, which describes the response of the differences in the velocity distribution functions $f_3$ and $f_1$ to the emission profile.

From the methodological point of view, it can be seen that a straightforward generalization of Eq. (3.26) for the multilevel problems by means of a trivial generalization of the term $a_{31}$ and $P_3$, would yield neither the additional term corr$_{31}$, nor the correct form of $a_{31}$. Note, particularly, that the radiative rate terms $B_{21}f_{21}^2$ and $B_{23}f_{23}^2$ in Eq. (5.19) do not, in general, agree with the corresponding total rates $B_{21}J_{21}^2$ and $B_{23}J_{23}^2$, respectively. The explanation of this behaviour is, in principle, equivalent to that discussed in Section 4.

b) Next Transitions and the Problem of Cross-Redistribution

In exactly the same manner as in the preceding subsection, we can derive similar expressions for the emission profiles, as well as for the macroscopic emission coefficients, for the remaining transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$.

Leaving out the explicit indication of their dependence on frequency and direction, these read

\begin{equation}
\eta_{31} = \frac{h\nu}{4\pi} a_{31} \left[ n_3 P_3 \varphi_{13}^3 + \right.
+ n_1 B_{13}(\not\mathcal{R}_{131} - J^a_{13} \varphi_{13}^1) +
+ n_2 B_{23}(\not\mathcal{R}_{231} - J^a_{23} \varphi_{13}^1) + \text{corr}_{31}, \]
\end{equation}

and

\begin{equation}
\begin{aligned}
\eta_{32} = \frac{h\nu}{4\pi} a_{32} \left[ n_3 P_3 \varphi_{13}^3 + n_2 B_{23} \right.
\left. \cdot (\mathcal{R}_{232} - J^2_{23} \varphi_{23}^2) + n_1 B_{13}(\not\mathcal{R}_{132} - J^a_{13} \varphi_{23}^1) + \text{corr}_{32}, \right]
\end{aligned}
\end{equation}

where

\begin{equation}
\text{corr}_{31} = n_2 C_{23}(\varphi_{13}^2 - \varphi_{13}^1),
\end{equation}

\begin{equation}
\text{corr}_{32} = n_1 C_{13}(\varphi_{13}^2 - \varphi_{13}^1),
\end{equation}

and

\begin{equation}
\begin{aligned}
a_{31} &= A_{31} \left[ \sum_{i=1}^{2} (A_{3i} + B_{3i} \mathcal{F}_{31}^{3i} + C_{3i}) + R_{3k} + C_{3k} \right] \\
&= A_{32} \left[ \sum_{i=1}^{2} (A_{3i} + B_{3i} \mathcal{F}_{31}^{3i} + C_{3i}) + R_{3k} + C_{3k} \right] \\
P_3 &= \sum_{i=1}^{2} (A_{3i} + B_{3i} J_{31}^{3i} + C_{3i}) + (R_{3k} + C_{3k})
\end{aligned}
\end{equation}

and where, in general, $(i, j = 1, 2)$

\begin{equation}
\begin{aligned}
\mathcal{F}_{31}^{3i}(v, n) &= \left. \frac{\int f_3(v) E_{31}(v, n, \nu) \varphi_{31}(v) \, dv}{\int f_3(v) E_{31}(v, n, \nu) \, dv} \right|_0, \\
\varphi_{31}^{3i}(v, n) &= \left. \int f_3(v) Q_{31}(v, n, \nu) \, dv \right|_0.
\end{aligned}
\end{equation}

The other symbols have their usual meaning (see also Section 6).

The physical interpretation of equations (5.24) and (5.25) is quite similar to that of the first resonance line. The only difference is the occurrence of two terms of type $H - J_0$ instead of the one in the case of the first resonance line. Taking the transition $1 \leftrightarrow 3$ as an example, the term proportional to $\mathcal{R}_{131} - J^a_{13} \varphi_{13}^1$ describes a contribution of the direct scattering in the line, while the term proportional to $\mathcal{R}_{231} - J^a_{23} \varphi_{13}^1$ describes a contribution due to the so-called cross-redistribution $2 \leftrightarrow 3 \rightarrow 1$. The interpretation of the equivalent expression for $\eta_{32}$ is analogous.

An inspection of Eqs (5.24) and (5.25) shows that the emission profiles for the transitions with the common upper level are, in general, coupled via the cross-redistribution terms. To illustrate this effect, let us briefly discuss the two following cases:

i) The energy separation between levels 1 and 2 is of the same order of magnitude as that between levels 2 and 3 (as for example, the hydrogen atomic structure). If one arbitrarily assumes that $R_{231} = Q_{23} Q_{13}$ (i.e., no correlation between the processes $2 \leftrightarrow 3$ and $3 \rightarrow 1$, and a natural excitation of the level 3), the cross-redistribution term in Eq. (5.24) is given by

\begin{equation}
n_2 B_{23}(\mathcal{R}_{231} - J^a_{23} \varphi_{13}^1) = n_2 B_{23} \varphi_{13}^2 (\mathcal{F}_{23}^{13} - J_{23}^a) + n_2 B_{23} J_{23}^a (\varphi_{13}^3 - \varphi_{13}^1),
\end{equation}

where the term $\mathcal{F}_{23}^{13}$ is given by Eq. (6.6) of Section 6. We shall discuss the nature of the $\mathcal{F}$-terms in Section 6; it will be shown that under a reasonable approximation
(6.13) \( \mathcal{J}_{23}^{3} = J_{23}^{2} \), and the first term in Eq. (5.33) vanishes. The second term in Eq. (5.33) is similar to the correction term \( \text{corr}_{31} \). If, moreover, \( n_{2}(C_{23} + B_{23}J_{23}^{3}) < n_{1}P_{3} \), the two last terms in eq. (5.24) are negligible, and the emission coefficient is given by a simple expression equivalent to the two-level case. The only weak coupling with other transitions arises due to (generally small) stimulated emission terms in the expression for \( a_{31} \) — Eq. (5.28). On the contrary, if some of these assumptions are not fulfilled, it is necessary to solve the general coupled set of the radiative transfer equation and the kinetic equations.

ii) The energy separation between levels 1 and 2 is much less than that between levels 2 and 3. Both lines, 3 \( \leftrightarrow \) 1 and 3 \( \leftrightarrow \) 2, can then be regarded as resonance lines, with a common upper level. A conceptually similar situation arises, if the transition 1 \( \leftrightarrow \) 2 is radiatively forbidden, regardless of the separation of energy levels. In such cases, the radiative transition between levels 1 and 2 is of minor importance, and, consequently, the velocity distribution functions of levels 1 and 2 can be reasonably supposed to be identical (and, moreover, Maxwellian). In that case, the correction terms \( \text{corr}_{31} \) and \( \text{corr}_{32} \) vanish, and the problem is reduced to the usual cross-redistribution problem, first discussed by Milkey et al. (1975), and in a somewhat different formulation by Heasley and Kneer (1976). As in the preceding cases, our formulation differs from those mentioned by the correct stimulated emission terms \( \mathcal{J}_{31}^{2} (i, j = 1, 2) \), which enter the spontaneous emission probability \( a_{3i} \) (\( i = 1, 2 \)). The reason for this difference is the same as explained in Section 4. However, as already emphasized, this difference is of negligible importance for the ultraviolet lines in which \( hv/kT \) is sufficiently large compared to unity to ensure that the stimulated emission rate is small compared to the spontaneous emission rate.

### 6. General Emission Coefficient

By applying the same procedure as in Section 5, we obtain the emission coefficient in the case of a general multilevel atom, \( (j > i) \)

(6.1) \( n_{ji} = \frac{hv}{4\pi} a_{ji} \left[ n_{j}P_{ji}^{i} + \sum_{i<j} n_{j}B_{ji}(\mathcal{R}_{ij}) - J_{ji}^{i}(\phi_{ij}) + \text{corr}_{ji} \right] \),

where

(6.2) \( a_{ji} = A_{ji} \left[ \sum_{i<j} (A_{ji} + B_{ji}J_{ji}^{*} + C_{ji}) + \right. \)

\[ \left. + \sum_{i>j} (B_{ji}J_{ji}^{i} + C_{ji}) \right] P_{3} + \mathcal{R}_{ij}^{i}(\phi_{ij}) \] + \text{corr}_{ji},

and

(6.3) \( P_{3} = \sum_{i<j} (A_{ji} + B_{ji}J_{ji}^{i} + C_{ji}) + \)

\[ + \sum_{i>j} (B_{ji}J_{ji}^{i} + C_{ji}) + R_{3} + C_{3} \],

and

(6.4) \( \text{corr}_{ji} = \sum_{i<j} n_{i}C_{ij}(\phi_{ij} - \phi_{ij}) + \sum_{i>j} n_{i} \cdot \left[ (A_{ij} + B_{ij}J_{ij}^{i} + C_{ij}) \phi_{ij} - (A_{ij} + B_{ij}J_{ij}^{i} + C_{ij}) \phi_{ij} \right] \).

Furthermore, \((i, j, l, m)\) arbitrary

(6.5) \( \phi_{ij}(v, n) = \int f(v) Q_{ij}(v, n, v) \, d^{3}v \),

(6.6) \( \mathcal{R}_{ij}^{i}(v, n) = \frac{A}{\int f(v) \sigma_{im}(v, n, v) \, d^{3}v} \),

\[ A = \int f(v) \sigma_{im}(v, n, v) \cdot \left[ \int \int \sigma_{ij}(v', n', v) I_{v}(n') \, dv' \, d\Omega'/4\pi \right] \, d^{3}v \]

where

(6.7) \( \sigma_{im} = \begin{cases} \sqrt[4]{E_{im}} & \text{for } l > m \\ \sqrt[4]{Q_{lm}} & \text{for } l < m \end{cases} \)

For clarity, we recall the explicit form of the remaining terms too, even if they are given by a straightforward generalization of the previous formulas. The redistribution integral is given by \((l, i < j)\)

(6.8) \( \mathcal{R}_{ij}(v, n) = \int f(v) \varrho_{ij}(v, n, v) \, d^{3}v \)

\[ = \int f(v) \left[ \int \int R_{ij}(v', n', v, n, v) I_{v}(n') \, dv' \, d\Omega'/4\pi \right] \, d^{3}v \]

\[ = \int \int R_{ij}(v', n', v, n, v) I_{v}(n') \, dv' \, d\Omega'/4\pi \]

where

(6.9) \( R_{ij}(v', n', v, n, v) = \)

\[ = \int f(v) \mu_{ij}(v) \, d^{3}v \]

is the velocity-averaged redistribution function. Finally, the absorption and emission integrals are given respectively by \((j > i)\)

(6.10) \( J_{ij}^{i} = \int f(v) \mu_{ij}(v) \, d^{3}v \)

Publishing House of the Czechoslovak Academy of Sciences • Provided by the NASA Astrophysics Data System
\[ \int f_j(\mathbf{v}) \left[ \int \phi_j(v, \mathbf{n}, \mathbf{v}) I_j(n) \ d\Omega / 4\pi \right] d^3v = \int f_j(\mathbf{v}) \lambda_j(\mathbf{v}) \ d^3v = \int \int E_j(v, \mathbf{n}, \mathbf{v}) I_j(n) \ d\Omega / 4\pi \ d^3v = \int \psi_j(\mathbf{v}, \mathbf{n}) I_j(n) \ d\Omega / 4\pi . \]

There should be no confusion between \( R \) defined in Eq. (6.9) (redistribution function) and \( R_{jk} \) (photionization rate).

Let us discuss the individual terms entering Eq. (6.1). The most complicated terms to evaluate are the quantities \( F_j^{lm} \). First, we shall examine their behaviour in some specific cases:

i) If, for a given line \( i \leftrightarrow j \ (i < j) \), the radiation field is supposed to be independent of frequency and direction (the so-called isotropic white-light approximation), then \( I_j(n) = I_{ij} \), and

\[ \tilde{F}_j^{lm} = \tilde{I}_{ij} \quad \text{for all} \ l, m . \]

ii) If we assume that the quantities \( \mu_{ij}(\mathbf{v}) \) and \( \lambda_{ij}(\mathbf{v}) \) are weakly dependent on the velocity, it is reasonable to replace them by their relevant averages over the velocity distribution of atoms in the \( i \)-th state, i.e. \( \mu_{ij}(\mathbf{v}) \rightarrow \mu_{ij}^*, \lambda_{ij}(\mathbf{v}) \rightarrow \lambda_{ij}^* \), and thus

\[ \tilde{F}_j^{lm} = \begin{cases} \mu_{ij}^* & \text{for} \ i < j \\ \lambda_{ij}^* & \text{for} \ i > j \end{cases} \quad \text{for all} \ l, m . \]

The approximation ii) has also been used by Baschek et al. (1981) in discussing the correct stimulated emission term in a two-level atom (in our notation the term \( \tilde{F}_j^{21} \)).

As follows from Eqs (6.2) and (6.3), both latter approximations lead to the exact validity of the following useful identity

\[ a_{ij} P_j = A_{ij} \quad (j > i) . \]

It is interesting to note that Heasley and Kneer (1976) argued that the term \( a_{ij} P_j / A_{ij} \) can strongly deviate from unity if the stimulated emission term is important. They used, however, an incorrect stimulated emission term, as discussed above (see also Baschek et al., 1981). The more exact formulation then yields, under the reasonable approximation ii), the exact validity of Eq. (6.14).

On the other hand, the stimulated emission rate is negligible with respect to the spontaneous emission rate for many applications in the stellar atmospheres theory. A typical example is the formation of ultraviolet resonance lines (with frequency \( v_{ij} \sim 10^{15} \)) in a typical stellar atmosphere (where the temperature \( T \sim 10^6 \)), where \( \exp(\frac{-h\nu}{kT}) \ll 1 \), and then the stimulated emission is negligible.

Finally, if all transition rates are very small compared to the corresponding spontaneous emission rates, the quantity \( a_{ij} \) is given (approximately) by the branching ratio

\[ a_{ij} \approx A_{ij} / \sum_{i \neq j} A_{ij} . \]

Let us now turn to the correction term \( \text{corr}_{ij} \). This term presents large numerical difficulties, because it contains the (unknown) velocity distribution functions for all atomic energy levels. If they are assumed to be Maxwellian, i.e. \( f_j(\mathbf{v}) = f_1(\mathbf{v}) \) for all \( i \), then (in approximation 6.13), \( \text{corr}_{ij} = 0 \). If the velocity distribution functions of excited states differ from their equilibrium values, the correction term can still be of second-order importance. Let us examine it for a resonance line \( i \leftrightarrow j \), \( i = 1, j \) arbitrary. Now, if a) for \( l < j \) (where the distributions can differ substantially from \( f_1 \)), \( C_{ij} \ll B_{ij} \), and b) for \( l > j \), the distribution functions are not very different from \( f_1 \) (due to a strong coupling with the continuum), then the correction term \( \text{corr}_{ij} \) would be very small compared to the first two terms in eq. (6.1), and can then be neglected in the first-order solution.

Let us now briefly discuss the terms \( \phi_{ij} \) and \( \mathcal{R}_{ij} \). In both cases, we are faced with two problems, namely a) to determine the rest-frame absorption profile and/or the redistribution function, and b) to perform their velocity averages over the corresponding velocity distribution function.

In the classical picture discussed in Section 2, the first problem is relatively simple: Both the rest-frame absorption profile and redistribution function are given as atomic parameters. However, from the computational point of view, they can still be quite complicated, as they depend, in general, of the radiation field. The latter dependence is due to the fact that the corresponding radiative damping factors contain, in general, both spontaneous and stimulated emission rates. Nevertheless, if the spontaneous emission probability from the upper level is much larger than the stimulated emission probability, the latter dependence is of negligible importance.

The second item, viz. a corresponding velocity
averaging procedure, is the basic limiting factor in the actual applications at present. If the velocity distribution function of the lower level is given a priori (e.g., it is Maxwellian), which is only the case of resonance lines, the problem presents none but numerical difficulties. In this case, the averaging procedure can be performed independently of the solution of the line transfer problem. In the case of the absorption profile, it yields the well-known Doppler or Voigt profiles and, in the case of the redistribution function, the velocity-averaged redistribution functions $R_i - R_v$ and their combinations (Hummer, 1962; Heinzel, 1981).

However, if the velocity distribution is not known a priori, the solution of the whole set of steady-state equations for the number densities (both velocity-dependent and velocity-averaged), simultaneously with the radiative transfer equation and the definition equation for the emission profile, is, in general, obligatory.

To our knowledge, no numerical solution of this type has been performed yet. It is thus not clear, what its astrophysical significance is in actual applications; in particular, what is the magnitude of the differences in the radiative transfer solutions by means of the exact and simplified treatments. We plan to deal with this problem elsewhere.

7. Conclusion

We can summarize the results of this paper as follows:

1) We attempted to analyse the available astrophysical partial redistribution studies. We have introduced the term quasi-Markovian, classical view, where the basic physical concepts of the current astrophysical approach are summarized. Its physical uncertainties, and even inconsistencies, are discussed in detail.

The quasi-Markovian, classical treatment has been used to generalize the Oxenius (1965) approach. In view of our goals outlined in the Introduction, two points appeared to be interesting:

2) The reformulation of the Oxenius' (1965) approach, in the two-level-atom case, to a form similar to that of Milkey and Mihalas (1973) and Heasley and Kneer (1976) showed, that both formulations yield almost identical results. However, as indicated by a detailed discussion, the latter authors used an incorrect form of the stimulated emission term. This point has also recently been discussed independently by Baschek, Mihalas and Oxenius (1981); their explanation of the behaviour of the stimulated emission term is basically the same as ours.

3) Using the same approach as in reformulating the two-level-atom case, we have derived a suitable form of the emission coefficient in the case of the multilevel atom. Comparing its form to that following from a heuristic derivation, two points appeared to be different: a) the correct form of the stimulated emission term, as in the case of the two-level atom, and b) the correction term, which describes explicitly the response of the differences in the velocity distribution functions for different excited states on the emission profile. Moreover, we have briefly discussed the possible situations, in which these extra terms are of negligible importance, i.e., the form of the general emission coefficient yields a form identical with previous studies.

Acknowledgments

The author is grateful to Dr B. Baschek for communicating his results before publication, and to Drs P. Heinzel and M. Macháček for reading the manuscript and making several useful remarks.

REFERENCES