THE INFLUENCE OF DIFFERENTIAL ROTATION ON THE EQUATORIAL COMPONENT OF THE SUN'S MAGNETIC DIPole FIELD

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ABSTRACT

This paper examines the effect that solar differential rotation would have on a hypothetical large-scale equatorial dipole field. The evolving large-scale field pattern is expressed as a series of non-axisymmetric moments. As time increases, power is transferred to progressively higher order moments. In the 27$^\text{d}$ rotating coordinate system, each moment undergoes a small retrograde drift which remains nearly uniform until that mode begins to fade. The synodic rotation periods of the first few moments are comparable to the observed 28.5$^\text{d}$ period of the Sun's large-scale field near sunspot maximum. Differential rotation may be the source of this 28.5$^\text{d}$ period, but the eruption of new flux is necessary to keep the pattern going.

Subject headings: Sun: magnetic fields — Sun: rotation

I. INTRODUCTION

In an analysis of inferred interplanetary sector polarity during 1926–1973, Svalgaard and Wilcox (1975) discovered a long-lived recurrence pattern with a synodic period of ~ 28.5$^\text{d}$. This pattern was most prominent near the peak of each sunspot cycle but sometimes occurred well into the declining phase. Hundhausen (1977) has discussed the 28.5$^\text{d}$ pattern that occurred in 1972 and 1973 during the declining phase of the last sunspot cycle. Sheeley and Harvey (1980) have described the new 28.5$^\text{d}$ pattern that began with the increase of sunspot activity in late 1977.

A characteristic of this 28.5$^\text{d}$ recurrence pattern is its long lifetime compared to the lifetimes of the individual coronal holes and photospheric magnetic regions that produce it. Both Hundhausen (1977) and Sheeley and Harvey (1980) observed that this pattern resulted from an eastward migration of mid-latitude features relative to low latitude ones and suggested that solar differential rotation is the underlying source of this 28.5$^\text{d}$ period. The present paper is an attempt to explore this possibility further.

II. THE MODEL

It is well known that near sunspot minimum the Sun's large-scale field is very nearly a dipole field whose axis is aligned with the solar rotation axis. The strength of this vertical dipole decreases with the onset of sunspot activity, and its polarity reverses near sunspot maximum. At this time a dipole component, if it exists, must lie with its axis in the equatorial plane. In this sense, one might speculate that the Sun's dipole field tips sideways in a complicated way as the polar field reverses near sunspot maximum.

A tilted dipole can be decomposed into its horizontal and vertical components. The vertical component, being axisymmetric, is unaffected by differential rotation and simply decays exponentially with the characteristic diffusion time of $\frac{1}{\tau} \sim 10$ yr (cf. Leighton 1964). The horizontal component is strongly affected by differential rotation, and it is this part of the problem that we wish to study.

Let us begin with the transport equation that governs the subsequent evolution of this surface magnetic field. According to Leighton (1964), this transport is composed of two parts, one due to diffusion and the other due to differential rotation. In a rigidly rotating coordinate system (such as the 27$^\text{d}$ recurrent system), this equation takes the form

$$\frac{\partial B}{\partial t} = \frac{R^2}{\tau} \nabla^2 B + \omega \frac{\partial B}{\partial \phi},$$

where $\tau$ is a random walk diffusion time on the order of 20 yr (but whose actual value will not be of particular interest here), $R$ is the solar radius, and $\omega$ is the differential angular rotation velocity. Using the Newton and Nunn (1951) rotation formula (cf. Appendix C) and a spherical coordinate system, this equation becomes

$$\frac{\partial B}{\partial t} = \frac{1}{\tau} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial B}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 B}{\partial \phi^2} \right] + \frac{1}{\tau_0} \left[ \cos^2 \theta \frac{\partial B}{\partial \phi} \right],$$

where $\tau_0 = 360/[2\pi(2.77)] = 20^\text{d}$ and is the time that differential rotation takes to wind an initially meridional structure through 1 rad.

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SOLAR DIFFERENTIAL ROTATION

For an initial global dipole the $\nabla^2 B$ terms are not large, and because $\tau \gg \tau_0$, we can neglect diffusion compared to differential rotation. In this case, the transport equation becomes

$$\frac{\partial B}{\partial t} \approx \frac{1}{\tau_0} \cos^2 \theta \frac{\partial B}{\partial \phi}.$$  (3)

The solution to this equation is any function $B$ of the form

$$B = B(\theta, \phi) = B(\theta, \phi + \frac{L}{\tau_0} \cos^2 \theta).$$  (4)

Thus, $B$ will remain constant with time, provided that

$$\phi + \frac{L}{\tau_0} \cos^2 \theta = \phi(0).$$  (5)

Thus, for example, this equation describes the windup of the magnetic neutral line of an initial horizontal dipole. In the $27^\circ$ coordinate system, this boundary is unchanged at the equator ($\theta = \pi/2$) and drifts progressively eastward at higher latitudes. This is the conventional way of describing the windup effects of different rotation.

An alternate approach is to begin with a specific form of the initial horizontal dipole. One such form is

$$-\frac{i}{(2)^{1/2}} Y_1^{1} (\theta, \phi) - Y_1^{-1} (\theta, \phi) = \left(\frac{3}{4\pi}\right)^{1/2} \sin \theta \sin \phi,$$  (6)

where $Y_1^{\pm 1}(\theta, \phi)$ are normalized spherical harmonic functions. In this case, the neutral line is the great circle defined by the prime meridian ($\phi = 0$) and its extension ($\phi = \pi$). The direction of the dipole may be taken to be in the ($\phi = +\pi/2$) direction where the positive flux is located. For convenience, we have dropped a normalization factor $(4/3\pi)^{1/2}$$R^2$, where $R$ is the flux through each hemisphere.

Thus, from equation (4), the evolution of this flux distribution is given by

$$-\frac{i}{(2)^{1/2}} \{Y_1^{1} (\theta, \phi + \alpha \cos^2 \theta) - Y_1^{-1} (\theta, \phi + \alpha \cos^2 \theta)\},$$  (7)

where we have used $\alpha = \tau_0$ for simplicity. This expression can be rewritten as a series of higher order moments

$$\sum_{i=1}^{\infty} r_i(\alpha)[\frac{i}{(2)^{1/2}} \{Y_1^{1} (\theta, \phi + \beta_i(\alpha)) - Y_1^{-1} (\theta, \phi + \beta_i(\alpha))\}],$$  (8)

whose amplitudes $r_i(\alpha)$ and phases $\beta_i(\alpha)$ evolve with time according to

$$r_i(\alpha)e^{i\phi(\alpha)} = \int_{0}^{\pi} \int_{0}^{2\pi} Y_1^{1} (\theta, \phi) Y_1^{-1} (\theta, \phi) e^{i\phi \cos^2 \theta} \sin \theta d\theta d\phi.$$  (9)

Putting $\alpha = 0$ gives $r_1 e^{i\phi} = \prod Y_1 Y_1^{-1} d\Omega = \delta_{11}$, which is unity when $l = 1$ and zero otherwise. Thus, all of the power resides initially in the dipole term. Of course, we anticipate that this power will be transferred to higher order moments as time increases. Next we shall examine this transfer process in detail by considering $r_1$ and $\beta_1$ explicitly as functions of $\alpha$.

III. DISCUSSION

The integration in equation (9) has been carried out in Appendix A, and the resulting formulae have been used to calculate $r_1(\alpha)$ and $\beta_1(\alpha)$ for $l = 1, 3, 5, \ldots, 31$. The amplitudes $r_1(\alpha)$ are plotted in Figure 1a. The phases $\beta_1(\alpha)$ have not been plotted. Instead a more easily visualized quantity $\gamma_1(\alpha)$, defined by $\gamma_1(\alpha) = l(\pi/2) - \beta_1(\alpha)$, has been plotted in Figure 1b. The phase $\gamma_1$ is simply the direction in which the $l$th multipole is pointing in the $\phi$-coordinate system.

Figure 1a verifies our expectation that power is gradually transferred from the dipole to the higher order moments. The amplitude of the dipole term decreases quadratically at first, then linearly for a while, and eventually as $\alpha^{1/2}$, as $\alpha \to \infty$. One by one each higher order moment becomes dominant and then decays with its own $\alpha^{-1/2}$ dependence as $\alpha \to \infty$. (In general, this dependence is given by

$$\frac{(l - 1)!}{2^{3/2}(l - 1)!^2} \left[\frac{3l(2l + 1)}{2(l + 1)}\right]^{1/2} \left(\frac{\pi}{\alpha}\right)^{1/2}$$

as $\alpha \to \infty$. These individual curves crowd closer together for progressively larger values of $l$, and their limiting value is $(3/2\alpha)^{1/2}$ as $l \to \infty$.

Figure 1b shows that each multipole rotates at a nearly constant rate for a while and then decelerates as it approaches the common position $\gamma_1 = \pi/4$. Here all of the moments eventually line up so that their equatorial flux distributions reinforce one another. In this way the old moments produce a latitudinally shrinking equatorial flux distribution that is...
Fig. 1.—(a) The transfer of power from an initial horizontal dipole to progressively higher order multipoles, according to the Newton and Nunn differential rotation formula. One by one each amplitude \( r_i(t) \) reaches its peak and then falls with a characteristic \( t^{-1/2} \) dependence. The abscissa \( t \) is the elapsed time in units of \( t_0 = 20^d \). The vertical scale has been multiplied by a factor of 10 to facilitate plotting. (b) The influence of Newton and Nunn differential rotation on the orientation of a horizontal dipole and its descendant multipoles. Within the 26.9° rotating system, each multipole experiences a nearly uniform retrograde drift for a while and then decelerates as its orientation \( \gamma_i(t) \) approaches the common longitude of 45°. The uniform synodic rotation periods range from 28.1° for the dipole to a limit of 30.0° for the higher order multipoles.

stationary in the 27° coordinate system. A detailed comparison with the curves in Figure 1a shows that the dipole begins to decelerate near \( t = 4 \) where the \( l = 3 \) multipole becomes strong. In addition, the uniform rotation of each higher order moment seems to stop as the amplitude of that moment falls below the dipole's amplitude.

The fact that the phases \( \gamma_i \) decrease with time means that the corresponding multipoles undergo retrograde rotation in the 27° coordinate system. Thus, as one might have anticipated from the Newton and Nunn formula, the multipoles have synodic rotation periods greater than 27°. The precise periods are easily determined in the limit of \( \alpha \ll 1 \). From equations (A5a), (A5b), and (A6b) of Appendix A one obtains

\[
\beta_i(t) \approx \left( \frac{l-1}{2} \right) \frac{\pi}{2} + \left( \frac{l}{2l+3} \right)^{\alpha},
\]

so that

\[
\gamma_i(t) \approx \left( \frac{l+1}{2} \right) \frac{\pi}{2} - \left( \frac{l}{2l+3} \right)^{\alpha}.
\]

Consequently, the initial slopes \( \gamma_i(0) \) are given by

\[
\gamma_i(0) = - \left( \frac{l}{2l+3} \right),
\]

and the synodic rotation periods are

\[
T_i = \frac{26.9^d}{1 - \left[ 26.9/(2\pi t_0) \right] \left[ (2l+3) \right]}.
\]

The synodic period of the dipole moment is 28.1°, and, remarkably, the periods of the higher order moments converge toward 30.0°. These values are sufficiently close to the observed 28.5° period to encourage us to inquire what process might keep the rotation going rather than terminating, as all the moments do, at the common value of \( \phi = \pi/4 \).
Ideally, one would like to begin with the dipole, and then let it rotate at its 28° period until \( \tau \approx 4 \tau_0 = 2.8 \) months. At this point, one would "freeze" the motion, somehow wipe out the higher order moments, and start over with a new but weaker dipole. This process would restore the initial conditions so that the weaker dipole could then rotate uniformly for another 2.8 months before we next "freeze" the motion, wipe out the new higher order moments, and begin again.

Of course the problem is how to wipe out the higher order moments. At first one might suppose that diffusion would produce this effect because it causes individual modes to decay as \( e^{-lt/(l+1)} \), where \( \tau \approx 20 \) yr (cf. Leighton 1964). Thus, the higher order modes would be weakened relative to the lower order ones. However, the substitution of our multipole expansion (eq. [8]) into equation (2) shows that diffusion does not compete with differential rotation unless \( l(l+1)\tau > \tau_0 \). That is, diffusion is effective for only those multipoles with \( l > (\tau/\tau_0)^{1/2} \approx 19 \). In particular, diffusion cannot prevent the \( l = 3 \) multipole from becoming dominant after only \( t \approx 4\tau_0 \approx 2.8 \) months. As Appendix B shows, the angular rotation rate of the dipole moment is given by

\[
\gamma_1(\alpha) = -\frac{1}{5} \left[ 1 + \left( \frac{8}{7} \right)^{1/2} \frac{r_3(\alpha)}{r_1(\alpha)} \right] \cos [\beta_3(\alpha) - \beta_1(\alpha)].
\] (14)

This rotation rate starts out at \( \gamma_1(0) = -\frac{1}{5} \) and remains nearly constant until \( r_3(\alpha) \approx r_1(\alpha) \), after which \( \gamma_1(\alpha) \) approaches zero as \( (\frac{8}{7})^{1/2} r_3/r_1 \to 1 \) and \( \beta_3 - \beta_1 \to \pi \). The fact that diffusion cannot keep \( r_3(\alpha) \ll r_1(\alpha) \) means that diffusion cannot prevent the deceleration of the dipole's angular rotation rate.

Thus, differential rotation can lead to rotation periods that are comparable to the observed 28°5 recurrence period, but differential rotation alone cannot maintain these periods indefinitely. Evidently a continual replenishment of the dipole field is necessary to keep the 28°5 pattern going. The fact that this pattern occurs predominantly in the years near sunspot maximum supports this assumption and suggests that new bipolar magnetic regions are the source of this replenishment. Sheeley and Harvey (1980) reached the same conclusion from a direct examination of synoptic observations during 1978 and 1979. The remaining problem is to see whether new bipolar magnetic regions can sustain these slowly rotating modes. If they can, it will be equally important to establish whether or not the new bipolar regions must erupt systematically in longitude in order to maintain the phase of the long-lived 28°5 pattern.

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APPENDIX A
THE EVALUATION OF \( r_1(\alpha) \) AND \( \beta_1(\alpha) \)

In this appendix analytic expressions for \( r_1(\alpha) \) and \( \beta_1(\alpha) \) will be derived from equation (9) of the text. Performing the \( \phi \) integration and recognizing that only the odd values of \( l \) contribute to the \( \theta \) integration, one obtains

\[
r_1(\alpha) \exp [i\beta_1(\alpha)] = \left[ \frac{3(2l + 1)}{2(l + 1)} \right]^{1/2} P_1, x(\alpha) x P_1, x(\alpha) \exp i x^2 d x,
\] (A1)

where \( P_1, x(\alpha) \) and \( P_1, x(\alpha) \) are Legendre's associated functions of the first kind (cf. Jahnke and Emde 1945). Using the identity

\[
P_1, \chi = (1 - \chi^2) \frac{dP_1, \chi}{d\chi} = \chi P_{-1, \chi} - \chi P_1, \chi)
\]

one can simplify equation (A1) to the form

\[
r_1(\alpha) \exp [i\beta_1(\alpha)] = \left[ \frac{3(2l + 1)}{2(l + 1)} \right]^{1/2} \int_0^1 [P_{-1, \chi} - \chi P_1, \chi] \exp [i x^2] d x.
\] (A2)

At this point, it is convenient to evaluate the real and imaginary parts of \( r_1 \) \( \exp (i\beta_1 \) separately. These respective parts are:

\[
p_1(\alpha) = \left[ \frac{3(2l + 1)}{2(l + 1)} \right]^{1/2} \int_0^1 [P_{-1, \chi} - \chi P_1, \chi] \cos (\alpha x^2) d x,
\] (A3a)

\[
q_1(\alpha) = \left[ \frac{3(2l + 1)}{2(l + 1)} \right]^{1/2} \int_0^1 [P_{-1, \chi} - \chi P_1, \chi] \sin (\alpha x^2) d x.
\] (A3b)
Although $p_t$ and $q_t$ can each be expressed as a finite series of Fresnel integrals, it is simpler to use power series expansions for $\cos(\alpha x^2)$ and $\sin(\alpha x^2)$ and then evaluate the resulting integrals exactly (cf. Stegun 1964). In this way, one obtains

$$p_t(a) = 2^\nu \frac{(3l + 1)(2l + 1)}{2} \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n}}{(2n)!} \frac{1}{(2n+1)!} \frac{1}{(4n + l + 2)!} [2n + (l + 1/2)!],$$  \hspace{1cm} (A4a)

$$q_t(a) = 2^\nu \frac{(3l + 1)(2l + 1)}{2} \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n+1}}{(2n+1)!} \frac{1}{(2n+1)!} \frac{1}{(4n + l + 4)!} [2n + (l + 3/2)!].$$  \hspace{1cm} (A4b)

Although the factors in braces are not particularly useful for calculating $p_t$ and $q_t$, they are useful for recognizing the first nonzero term of each series. Thus, to keep the arguments of the factorials nonnegative, we must have $n > (l - 1)/4$ in equation (A4a) and $n > (l - 3)/4$ in equation (A4b). With these conditions in mind, we then simplify fractions to obtain

$$p_t(a) = \frac{3l + 1}{2} \frac{(2l + 1)}{2} \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n}}{(2n)!} \frac{1}{(4n + l + 2)!} \frac{1}{(4n + l + 3 - 2k)!},$$  \hspace{1cm} (A5a)

$$q_t(a) = \frac{3l + 1}{2} \frac{(2l + 1)}{2} \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n+1}}{(2n+1)!} \frac{1}{(4n + l + 3 - 2k)!} \frac{1}{(4n + l + 5 - 2k)!}. $$  \hspace{1cm} (A5b)

Once $p_t(a)$ and $q_t(a)$ have been evaluated, $r_t(a)$ and $\beta_t(a)$ can be determined using

$$r_t(a) = \left[ p_t^2(a) + q_t^2(a) \right]^{1/2},$$  \hspace{1cm} (A6a)

and

$$\tan \beta_t(a) = \frac{q_t(a)}{p_t(a)}.$$  \hspace{1cm} (A6b)

### APPENDIX B

**THE EVALUATION OF $\gamma_1'(\alpha)$**

As an aid to studying the maintenance of the $28^{d}1$ rotation period, we shall derive an exact expression for $\beta_t'(\alpha)$ and thus $\gamma_1'(\alpha)$. The basic idea is to trace the evolution of an initial horizontal dipole in two steps. The first step begins at $t = 0$ and ends at $t = \alpha \tau_0$. The second step begins at $t = \alpha \tau_0$ and ends at the infinitesimally greater time of $t = (\alpha + \epsilon) \tau_0$. The net result of going from $t = 0$ to $t = (\alpha + \epsilon) \tau_0$ is then compared with the result that one would obtain by going from $t = 0$ to $t = (\alpha + \epsilon) \tau_0$ directly in a single step.

After the first step, the initial dipole becomes

$$\sum_{l=1}^{\infty} r_l(\alpha) \left[ -\frac{i}{(2)^{1/2}} \{ Y_l^1[\theta, \phi + \beta_t(\alpha)] - Y_l^{-1}[\theta, \phi + \beta_t(\alpha)] \} \right],$$  \hspace{1cm} (B1)

and after the second step it becomes

$$\sum_{l=1}^{\infty} r_l(\alpha + \epsilon) \sum_{k=1}^{\infty} r_{kl}(\epsilon) \left[ -\frac{i}{(2)^{1/2}} \{ Y_k^1[\theta, \phi + \beta_t(\alpha + \epsilon)] - Y_k^{-1}[\theta, \phi + \beta_t(\alpha + \epsilon)] \} \right],$$  \hspace{1cm} (B2)

where $r_{kl}$ and $\beta_{kl}$ are obtained by replacing $Y_l^1(\theta, \phi)$ by $Y_l^1(\theta, \phi)$ in equation (9) of the text. Alternatively, in going directly from $t = 0$ to $t = (\alpha + \epsilon) \tau_0$ in a single step, the dipole becomes

$$\sum_{l=1}^{\infty} r_l(\alpha + \epsilon) \left[ -\frac{i}{(2)^{1/2}} \{ Y_l^1[\theta, \phi + \beta_t(\alpha + \epsilon)] - Y_l^{-1}[\theta, \phi + \beta_t(\alpha + \epsilon)] \} \right].$$  \hspace{1cm} (B3)

Equating expressions (B2) and (B3) and using the orthonormal properties of the spherical harmonic functions, one obtains

$$r_t(\alpha + \epsilon) e^{i\beta_t(\alpha + \epsilon)} = \sum_{k=1}^{\infty} r_{kl}(\epsilon) e^{i\beta_{kl}(\epsilon)}.$$  \hspace{1cm} (B4)

Putting $l = 1$, we can drop the extra indices because $r_{1k} = r_k$ and $\beta_{1k} = \beta_k$. Then equation (B4) reduces to

$$r_1(\alpha + \epsilon) e^{i\beta_t(\alpha + \epsilon)} = \sum_{k=1}^{\infty} r_k(\epsilon) e^{i\beta_{kl}(\epsilon)}.$$  \hspace{1cm} (B5)
The remaining derivation consists of expanding equation (B5) in powers of $e$, retaining only first order terms, and using the approximations for $r_k(e)$ and $\beta_k(e)$ that are derived easily from equations (A4) and (A6) of Appendix A. These approximations are

$$r_k(e) \approx 2^k \left[ \frac{3k(k+1)(2k+1)}{2} \right]^{1/2} \frac{(k-1)!k!}{[(k-1)/2]!(2k+1)!} e^{k-1/2}, \quad (B6a)$$

$$\beta_k(e) \approx \left( \frac{k-1}{2} \right) \pi + \left( \frac{k}{2k+3} \right) e^k. \quad (B6b)$$

Substituting these expressions into equation (B5) and retaining terms up to first order, the right side of (B5) becomes

$$r_1(x)e^{i\beta_1(x)} + e^i \left[r_1(x)e^{i\beta_1(x)} + \left( \frac{8}{7} \right)^{1/2} r_3(x) e^{i\beta_3(x)} \right], \quad (B7)$$

and the left side is simply

$$r_1(x)e^{i\beta_1(x)} + e^i [r_1(x)e^{i\beta_1(x)} + ir_1(x)\beta_1(x)e^{i\beta_1(x)}]. \quad (B8)$$

The zero-order terms provide an identity, but the first order terms provide two equations when the real and imaginary parts are compared separately. These equations are

$$\frac{r_1(x)}{r_1(x)} = -\frac{1}{5} \left[ \frac{8}{7} \right]^{1/2} \frac{r_3(x)}{r_1(x)} \sin \left[ \beta_3(x) - \beta_1(x) \right], \quad (B9a)$$

$$\beta_1'(x) = \frac{1}{5} \left[ 1 + \left( \frac{8}{7} \right) \frac{r_3(x)}{r_1(x)} \right] \cos \left[ \beta_3(x) - \beta_1(x) \right]. \quad (B9b)$$

Using equations (B6a, b), one can see that, for $\alpha \ll 1$,

$$\left( \frac{8}{7} \right)^{1/2} \frac{r_3(x)}{r_1(x)} \approx \frac{8\alpha}{35} \quad \text{and} \quad \beta_3(x) - \beta_1(x) \approx \frac{\pi}{2} + \frac{2\alpha}{15}.$$
This Stanford solar rotation formula has two simplifying features: it matches the Newton and Nunn formula closely for latitudes less than 45°, and its coefficients for the $\sin^2 L$ and $\sin^4 L$ terms are equal. This latter simplifying feature allows us to define a new characteristic time $\tau_0 = 360/[2n(1.98)] = 28^d9$ and express the solutions in terms of the new dimensionless variable $\alpha = t/\tau_0$.

With this modification, equations (9) and (A2) become

$$r_\alpha(\alpha)e^{i\phi(x)} = \int_0^\infty \int_0^{2\pi} Y_1^1(\theta, \phi)Y_1^{-1}(\theta, \phi)e^{i\alpha(\cos^2 \theta + \cos^4 \theta)} \sin \theta \theta \theta \theta \theta$$

and

$$r_\alpha(\alpha)e^{i\phi(x)} = \left(\frac{3(2l + 1)}{2(l + 1)}\right)^{1/2} \int_0^1 [P_{l-1}(x) - xP_l(x)]e^{i\alpha(x^2 + x^4)}dx,$$

respectively. Similarly, the expressions for $p_l$ and $q_l$ become double sums:

$$p_l(\alpha) = 2\left(\frac{3(2l + 1)(2l + 1)}{2}\right)^{1/2} \sum_{n=0}^\infty (-1)^n 2^{2n} \sum_{j=0}^{2n} \frac{1}{j!(2n - j)!} \frac{1}{(4n + 2j)! [2n + j + (l + 1)/2]!} \frac{1}{(4n + 2j + l + 2)! [2n + j - (l - 1)/2]!},$$

$$q_l(\alpha) = 2\left(\frac{3(2l + 1)(2l + 1)}{2}\right)^{1/2} \sum_{n=0}^\infty (-1)^n 2^{2n+1} \sum_{j=0}^{2n+1} \frac{1}{j!(2n + 1 - j)!} \frac{1}{(4n + 2j + 2)! [2n + j + 1 + (l + 1)/2]!} \frac{1}{(4n + 2j + l + 4)! [2n + j + 1 - (l - 1)/2]!},$$

where again the arguments of the factorials must be nonnegative. Thus, in equation (C5a), once $l$ and $n$ have been established, the only nonzero terms of the series $\sum_{n=0}^\infty$ will be those for which $j \geq (l - 1)/2 - 2n$. Similarly, in equation (C5b) we must have $j \geq (l - 1)/2 - 2n - 1$. With these conditions in mind, we can rewrite equations (C5) in a more useful form:

$$p_l(\alpha) = \left(\frac{3(2l + 1)(2l + 1)}{2}\right)^{1/2} \sum_{n=0}^\infty (-1)^n 2^{2n} \sum_{j=0}^{2n} \frac{1}{j!(2n - j)!} \frac{1}{(4n + 2j + 1)! (4n + 2j + 2)!} \frac{1}{(4n + 2j + l + 3 - 2k)!} \prod_{k=1}^l \left(\frac{4n + 2j + l + 3 - 2k}{4n + 2j + 3} \right),$$

$$q_l(\alpha) = \left(\frac{3(2l + 1)(2l + 1)}{2}\right)^{1/2} \sum_{n=0}^\infty (-1)^n 2^{2n+1} \sum_{j=0}^{2n+1} \frac{1}{j!(2n + 1 - j)!} \frac{1}{(4n + 2j + 2)! (4n + 2j + 4)!} \frac{1}{(4n + 2j + l + 5 - 2k)!} \prod_{k=1}^l \left(\frac{4n + 2j + l + 5 - 2k}{4n + 2j + l + 5} \right),$$

where $r_\alpha(\alpha)$ and $\beta_\alpha(\alpha)$ are still determined from $p_l(\alpha)$ and $q_l(\alpha)$, using equations (A6a, b) of Appendix A. Finally, we still have $\gamma_l = l(n/2) - \beta_\alpha(\alpha)$.

We have used these equations to calculate $r_\alpha(\alpha)$ and $\beta_\alpha(\alpha)$ for $l = 1, 3, 5, \ldots, 31$. The functions $r_\alpha(\alpha)$ and $\gamma_l(\alpha)$ are plotted in Figures 2a and 2b, respectively. Again we emphasize that the independent variable $\alpha$ differs from the Newton and Nunn $\alpha$ that we used in Figures 1a and 1b. In fact, for a given value of $t$, the Stanford $\alpha$ is smaller than the Newton and Nunn $\alpha$ by the factor $20.7/28.9 = 1.98/2.77 = 0.72$.

The Stanford (SSO) solutions differ from the Newton and Nunn (NN) solutions in several respects. First, in the NN solution only $r_3(\alpha)$ begins to increase linearly with $\alpha$, but in the SSO solution both $r_3(\alpha)$ and $r_5(\alpha)$ begin to increase linearly. This behavior is a direct consequence of the extra $\cos^4 \theta$ term in equation (C3) which effectively couples spherical harmonics for which $\Delta l = 4$, as well as $\Delta l = 2, 0$, for terms of first order in $\alpha$. Second, unlike the higher order moments of the NN solution, none of the SSO higher order moments ever becomes stronger than the dipole moment. However, for large $\alpha$, the extra $\cos^4 \theta$ term has little effect, and the SSO solutions and NN solutions approach the same limiting curves. Third, the differing behavior of the NN and SSO solutions for small $\alpha$ and large $\alpha$ is illustrated dramatically by the starting points of the $\gamma_l$ curves. Whereas the NN curves were uniformly spaced at intervals of $\pi/2$, the SSO curves skip every third multiple of $\pi/2$. © American Astronomical Society • Provided by the NASA Astrophysics Data System
Fig. 2. (a) The transfer of power from an initial horizontal dipole to progressively higher order multipoles, according to the Stanford differential rotation formula. The dipole amplitude decreases steadily but is always dominant. One by one the amplitudes of the higher order multipoles each reach a peak and then fall with characteristic $\alpha^{-\frac{1}{2}}$ dependences. The abscissa, $\alpha$, is the elapsed time in units of $t_0 = 2859$. The ordinate, $r_i(\alpha)$, has been multiplied by a factor of 10 to facilitate plotting. (b) The influence of Stanford differential rotation on the orientation of a horizontal dipole and its descendant multipoles. Within the 26°8 rotating system the initial retrograde motions eventually disappear, and the individual multipoles all become coaligned at 45° longitude. The initial orientations of these multipoles are not uniformly spaced at intervals of 90° like their Newton and Nunn counterparts but skip every third multiple of 90°.

One can derive an exact expression for the SSO value of $\gamma_1(\alpha)$ using the same technique that we have already employed in Appendix B. In this case we find

$$\frac{r_i'(\alpha)}{r_i(\alpha)} = -2 \left( \frac{14}{9} \right)^{1/2} \frac{r_3(\alpha)}{r_1(\alpha)} \sin (\beta_3 - \beta_1) + \left( \frac{16}{495} \right)^{1/2} \frac{r_5(\alpha)}{r_1(\alpha)} \sin (\beta_5 - \beta_1), \quad \text{(C7a)}$$

$$\gamma_1(\alpha) = -\beta_1'(\alpha) = -2 \left[ 1 + \left( \frac{14}{9} \right)^{1/2} \frac{r_3(\alpha)}{r_1(\alpha)} \cos (\beta_3 - \beta_1) + \left( \frac{16}{495} \right)^{1/2} \frac{r_5(\alpha)}{r_1(\alpha)} \cos (\beta_5 - \beta_1) \right]. \quad \text{(C7b)}$$

Here the initial linear variation of both $r_3(\alpha)$ and $r_5(\alpha)$ has led to one more term than we found in the corresponding NN equations (cf. Appendix B) for which $r_5(\alpha)$ varied linearly but $r_3(\alpha)$ varied quadratically with $\alpha$.

One can see in equation (C7b) that the initial rotational velocity of the SSO dipole is $\frac{1}{2}$ and thus differs significantly from the NN value of $\frac{1}{4}$. However, this difference is compensated for almost exactly by the differing values of $\alpha$ and $t_0$. Using an expression similar to equation (13) of the text, we find that the SSO dipole period is 28°90 compared to the NN period of 28°91. For $l = 3$, the SSO period is 29°93 compared to a NN period of 28°99. For higher order moments the periods differ significantly, and in particular the SSO periods do not approach a simple limit as the NN ones did.

Table 1 summarizes the SSO rotation rates and synodic periods for $l = 1, 3, 5, \ldots, 25$. The modes separate into two categories, fast and slow, depending on whether $l$ is one greater than or one less than a multiple of 4, respectively. Examination of the fast modes shows that as $l$ increases, the angular rotation velocity increases without limit. As $l$ passes through the values 13 and 17, the fast modes become nearly stationary and then increasingly retrograde in an inertial coordinate system. Evidently the slow modes become stationary at a much higher $l$ value, perhaps on the order of 39.

Finally, while the NN and SSO formulas share the advantage that they do not change significantly with time, they also share the disadvantage that they were derived from observations below ~ 45° latitude. Consequently, the differences between the resulting NN and SSO multipoles must reflect differences between the high-latitude extrapolations of these two formulae. Thus, in retrospect, it might have been more instructive to use the Mount Wilson (Howard, Boyden, and Labonte 1980) and Kitt Peak (Livingston and Duvall 1969) differential rotation measurements, which, because of their relatively high spatial resolutions, extend to ~ 80° latitude.
TABLE 1

SSO ROTATIONAL MODES

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REFERENCES


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