THE THERMAL STATICS OF CORONAL LOOPS

B. ROBERTS

Department of Applied Mathematics, University of St. Andrews, St. Andrews, Fife, Scotland

and

S. FRANKENTHAL

Tel Aviv University, School of Engineering, Ramat Aviv, Israel

(Received 26 October, 1978; in final form 4 March, 1980)

Abstract. The thermal statics of constant pressure coronal loops is discussed, with particular emphasis on non-equilibrium and scaling relations. An analytical solution showing explicitly the occurrence of non-equilibrium in radiation dominated loops is presented. In addition, the general scaling law for hot loops is given. However, in view of the uncertainties in the coronal heating function and the observational determined loop parameters, it is suggested that scaling laws are currently of limited value.

1. Introduction

The theory of coronal loops is of much current interest, as is witnessed by the number of recent publications dealing with the thermal statics (see Rosner et al., 1978; Hood and Priest, 1979; Wragg and Priest, 1980; and references therein), siphon flows (Cargill and Priest, 1980; Noci, 1980), and instabilities (Antiochos, 1979; Hood and Priest, 1980) in loops. We are concerned here with the thermal statics of loops and, in particular, with the feature of thermal non-equilibrium.

Non-equilibrium is distinct from instability. Instability is the breakdown of an equilibrium; the equilibrium exists, even if it is unlikely to occur in nature. On the other hand, non-equilibrium means the total absence (and not the breakdown) of a possible static configuration; motions necessarily occur in the ‘lowest order’ state.

The topic of non-equilibrium has received some previous attention in the solar literature. Indeed, Parker (1975, 1977) has suggested that magnetostatic non-equilibrium may explain the ceaseless activity in the corona. Thermal non-equilibrium in current sheets has been explored by Smith and Priest (1977) and, more recently, Milne et al. (1979) have considered the coupling between static and thermal non-equilibria in determining the structure of a quiescent prominence. In coronal loops, despite the large number of published investigations, only Hood and Priest (1979) have discussed thermal non-equilibrium. We pursue this topic here, principally to demonstrate an analytical solution of the equation of thermal statics. The solution exhibits in a compact form the occurrence of non-equilibrium in loops, and so provides a convenient mathematical demonstration of the feature of non-equilibrium. We relate our analytical solution to the existence theorems described in Birn et al. (1978).
In addition to describing non-equilibrium, we briefly consider the use of scaling laws (see, for example, Rosner et al., 1978) in comparing observations and theory of loops, concluding that such use is currently of limited value.*

2. Thermal Statics

Consider a coronal plasma permeated by a uniform magnetic field. Under static conditions the energy equation is simply a balance between thermal conduction (principally along the field), radiation in an optically-thin atmosphere, and mechanical heating. Thus,

\[
\frac{d}{dz} \left( \kappa_0 T^{5/2} \frac{dT}{dz} \right) = Xn^2 T^\alpha - En^s T^t,
\]

where \( z \) is measured along the magnetic field, and \( T(z) \) and \( n(z) \) are the temperature and electron number density of the plasma. The term on the left-hand-side is representative of thermal conduction (\( \kappa_0 \approx 10^{-11} \) in mks units); \( Xn^2 T^\alpha \) is the radiation term (with \( X \) and \( \alpha \) given by a number of authors; e.g. Hildner, 1974), and \( En^s T^t \) the assumed form of the (unknown) mechanical heating term. There are a number of possible choices for powers \( s \) and \( t \) for given constant \( E \) (see Rosner et al., 1978), though, for simplicity, either \( s = t = 0 \) or \( s = 1 \) and \( t = 0 \) has been generally assumed. We will retain the general form of \( s \) and \( t \).

To illustrate the occurrence of thermal non-equilibrium it is sufficient to consider constant pressure loops (see Rosner et al., 1978; Hood and Priest, 1979), though the feature is also apparent when gravity is allowed for (Wragg and Priest, 1980) and even in the presence of steady flows (Cargill and Priest, 1980). With pressure \( p \) constant along the loop, the ideal gas law may be used to eliminate the density from (1), giving a second order, non-linear, ordinary differential equation for \( T(z) \).

To model a loop we will suppose it to be of total length \( 2L \), measured from footpoint to footpoint, symmetrically placed in the corona so that the footpoints are at a prescribed temperature \( T_1 \). The top of the loop is therefore at a temperature maximum or minimum. With this definition of \( T_1 \) and \( L \), the half-length of the loop is simply the distance \( L \), measured from the loop’s apex, at which the assigned temperature \( T_1 \) is attained.

Now inspection of Equation (1) reveals that the natural dependent variable is not \( T \) but its \( \frac{7}{2} \) power. Introduce, then, the dimensionless variables

\[
\tau = \left( \frac{T}{T_1} \right)^{7/2}, \quad x = \frac{z}{L},
\]

representing a dimensionless ‘temperature’ \( \tau(x) \) as a function of ‘distance’ \( x \). Then, for constant pressure loops, the energy equation may be written in the convenient form

\[
\frac{d^2 \tau}{dx^2} = R^2 (\tau^{-1+\lambda} - \epsilon \tau^{-1+\nu}),
\]

* See also Chiuderi et al. (1980).
where $\lambda$ and $\nu$ are the radiation and heating powers, given by

$$\lambda = 1 + \frac{2}{3}(\alpha - 2), \quad \nu = 1 + \frac{2}{3}(t-s),$$

and

$$R = \left( \frac{7Xn_1^2T_1^{\alpha}}{2\kappa_0 T_1^{1/2}} \right)^{1/2} L, \quad \varepsilon = \frac{E n_1^4 T_1^}{X n_1^2 T_1^\alpha}.$$

The parameter $\varepsilon$ is a measure of the importance of heating to radiation at the base of the loop, and $R$ a measure of the length $L$. Note that in writing (3) we have supposed, for simplicity, that $X$ and $\alpha$ are constants over the temperature range covered by coronal loops. This is permissible if, for example, we choose $T_1 \approx 10^6$ K, and then (Hildner, 1974) $\alpha = -1$.

Equation (3) is supplemented by the boundary conditions

$$\tau = 1 \quad \text{at} \quad x = 1, \quad \frac{d\tau}{dx} = 0 \quad \text{at} \quad x = 0, \quad (4)$$

representing the attaining of temperature $T = T_1$ at the base of the loop ($z = L$), and zero conductive flux at the apex ($z = 0$).

Equations (3) and (4) together constitute a two-point boundary value problem and, since (3) is non-linear, we have no guarantee of the uniqueness or existence of a solution of the system. Indeed, multiple solutions or no solution may arise, depending upon the magnitudes of the parameters $R$ and $\varepsilon$.

To illustrate the feature of non-equilibrium (i.e. a lack of a solution to the above boundary value problem) it is convenient to consider the special case when heating is negligible, so that $\varepsilon$ is very small. Such a case, it may be argued, is of limited physical interest. However, our point here is to demonstrate, in an explicit fashion, the feature of non-equilibrium and we may do this most simply by neglecting the heating. Mathematically, the non-equilibrium characteristics of the full problem ($\varepsilon \neq 0$) and the reduced problem ($\varepsilon = 0$) are similar and so such a simplification is justified; we will return to the case $\varepsilon \neq 0$ shortly.

If in (3) we choose $\varepsilon = 0$ and $\lambda = 0$ (consistent with $\alpha = -\frac{3}{2}$, compared with the value $\alpha = -1$ adopted by Hildner which gives $\lambda = \frac{1}{2}$), we obtain

$$\frac{d^2\tau}{dx^2} = \frac{R^2}{\tau}. \quad (5)$$

It may be shown that Equation (5) has solution

$$Rx = \sqrt{2} \tau D(u), \quad (6)$$

where $u^2 = \log_e (\tau/\tau_0)$, $\tau_0 = \tau(x = 0)$, and $D(u)$ is Dawson's integral:

$$D(u) = e^{-u^2} \int_0^u e^{t^2} dt, \quad u \geq 0.$$
Dawson's integral is tabulated (see Abramowitz and Stegun, 1967, p. 319), and has the following properties: \( D(u) \sim u \) as \( u \to 0 \); \( D(u) \sim 1/2u \) as \( u \to \infty \); the maximum in \( D(u) \) occurs at \( u = u_{\text{max}} = 0.924 \), for which \( D_{\text{max}} = 1/2u_{\text{max}} = 0.541 \).

The solution given by (6) satisfies the condition that \( d\tau/dx = 0 \) at \( x = 0 \) (where \( u = 0 \)). It remains to require that \( \tau = 1 \) at \( x = 1 \), and this gives

\[
R = \sqrt{2} D(-\log \tau_0)^{1/2}, \quad \tau_0 \leq 1. \tag{7}
\]

Equation (7) determines \( \tau_0 \) for a given \( R \), and then the temperature profile follows from (6).

Now it is clear from the fact that the function \( D(u) \) has a maximum that if \( R > \sqrt{2} D_{\text{max}} \), then there is no equilibrium solution under the conditions described. On the other hand, if \( R < \sqrt{2} D_{\text{max}} \) the form of the function \( D(u) \) makes it clear that there are two possible equilibria satisfying the energy equation (3) and the boundary conditions (4). (We may show that the solution with \( \tau_0 \) close to unity is stable and the other is unstable.) Thus, increasing \( R \) (by, for example, increasing the loop's length \( L \) or base pressure \( p \)) beyond the critical value \( \sqrt{2} D_{\text{max}} \) will force an existing (stable) equilibrium into a region of thermal non-equilibrium. As a numerical illustration, we note that for a base temperature of \( T_1 = 10^6 \) K and base density \( n_1 = 10^{15} \) m\(^{-3} \) we obtain a critical loop length of some \( 10^7 \) m, which is comparable with that found from the observations of active region loops (see summary by Priest, 1978).

The feature of non-equilibrium, explored in detail in the above, is not peculiar to the circumstances \( \lambda = 0, \varepsilon = 0 \). To see this, consider first the possibility \( \lambda > 0 \), with again \( \varepsilon = 0 \). Equation (3) may then be solved in terms of hypergeometric functions. But the existence of non-equilibrium is readily seen by constructing the integral solution of (3). With \( \varepsilon = 0 \), Equation (3) has integral

\[
R = (\frac{1}{2} \lambda)^{1/2} \int_{\tau_0}^{1} (t^\lambda - \tau_0^\lambda)^{-1/2} \, dt, \tag{8}
\]

giving \( \tau_0 \leq 1 \) in terms of \( R \). Now, for \( 0 < \lambda < 2 \), the integral in (8) is bounded, so if \( R \) exceeds this bound no equilibrium solution exists. More generally, we may apply the existence theorems quoted in Birn et al. (1978) to the original system, Equation (3) with \( \varepsilon \neq 0 \) and \( 0 < \lambda < \nu < 1 \). Such an application shows that non-equilibrium arises for sufficiently large \( R \) provided that radiation is greater than heating at the base of the loop. Thus, non-equilibrium may occur even in the presence of heating. (Notice that this criterion for the occurrence of non-equilibrium gives no indication of how large 'sufficiently large \( R \)' actually is: for this we must turn to an analytical (or numerical) solution of the problem, as we have given, for example, in our solution in terms of Dawson's integral.)

Non-equilibrium, then, is absent only if heating is dominant over cooling; in such circumstances, the temperature profile is uniquely determined by (3) and (4). To see this, and also to illustrate some feature of the heating dominant temperature profiles, suppose that the cooling term on the right-handside of (3) is negligible compared
with the heating. Then the temperature structure is determined by balancing conduction against mechanical heating:

$$\frac{d^2 \tau}{dx^2} = -\varepsilon R^2 \tau^{-1+\nu}.$$  (9)

Equation (9) may be integrated to give

$$\left(\frac{1}{2} \varepsilon R^2\right)^{1/2} x = \tau_0^{1-\nu/2} \left(1 - \frac{\tau}{\tau_0}\right)^{1/2} _2F_1\left(1 - \frac{1}{\nu}, \frac{1}{2}, \frac{3}{2}; 1 - \frac{\tau}{\tau_0}\right),$$  (10)

where \( _2F_1 \) is the Gaussian hypergeometric function (see Gradshteyn and Ryzhik, 1965, p. 284). Applying the condition that \( \tau = 1 \) at \( x = 1 \) then gives \( \tau_0 \) uniquely in terms of \( R \); with \( \tau_0 \) determined, \( \tau(x; R) \) follows from (10).

The circumstance of \( \tau_0 \gg 1 \) is of particular interest, for it corresponds to hot active region loops. Indeed, the condition \( \tau_0 \gg 1 \) is likely to be the rule rather than the exception – this is one of the mathematical advantages of working in terms of \( \tau \) rather than \( T \). (For example, a loop with apex temperature \( T_0 = 2.5 \times 10^6 \) K and ‘base’ temperature \( T_1 = 10^5 \) K gives \( \tau_0 \approx 25 \).) With \( \tau_0 \) large, (10) may be greatly simplified; the result is a scaling law for hot loops, relating the apex temperature \( T_0 \) to the pressure \( p \), length \( L \) and heating constant \( E \). In the original physical variables the scaling law for hot loops is

$$T_0^{(7+2s-2s)/4} \sim p^{1/4} L^{1/2} E^{-1/2}.$$  (11)

For example, with uniform heating (\( s = t = 0 \)) Equation (11) gives (see also Hood and Priest, 1979) \( T_0 \sim E^{2/7} L^{4/7} \) which, we note, gives \( T_0 \) as moderately insensitive to variations in \( E \) and \( L \).

3. Discussion

In the previous section we have demonstrated explicitly the form of non-equilibrium in a radiation dominated loop. What are the consequences of such non-equilibrium? It would appear that the thermal structure in the loop would evolve until a new equilibrium was once more possible, and such an equilibrium would correspond to a cooler apex temperature (see the numerical solutions of the energy equation given by Hood and Priest, 1979). We note that no ‘evolution calculations’, following such a thermal development, have yet been given. However, it seems reasonable to suppose that the cool cores observed in sunspot loops (see Foukal, 1975, 1976) are formed as a result of thermal non-equilibrium (Priest, 1978; Hood and Priest, 1979), brought about by a gradual change in loop length or gas pressure.

Now the onset of non-equilibrium is likely to be a catastrophic event. It is thus of interest to consider the possibility of observing the occurrence of non-equilibrium. Clearly the difficulties are considerable, but in this context it is tempting to conjecture
(with Hood and Priest, 1979) that the curious event of loop evacuation described by Levine and Withbroe (1977) occurred as a result of non-equilibrium, brought about by a gradual increase in loop length (or gas pressure) or decrease in heating – and not, as Levine and Withbroe suggest, by a sudden cessation of heating. However, in view of the numerous theoretical and observational uncertainties – and the complex, possibly post-flare, nature of this event (Rosner, 1979) – this interpretation must remain speculative.

We turn now to a consideration of scaling laws in loops. In this context, we recall the relation obtained by Rosner et al. (1978), namely

$$T_0 \sim (pL)^{1/3},$$  \hspace{1cm} (12)

where in deriving (12) the radiation power $\alpha = -\frac{1}{3}$ was chosen. This relation applies to a thermally isolated loop (which, in addition to satisfying (4), has zero conductive flux at its base). In such loops the heating coefficient $E$ is related to $p$ and $L$: for the case of uniform heating ($s = t = 0$) this relationship is $E \sim p^{7/6}L^{-5/6}$ (Rosner et al., 1978).

Now, in general, hot loops have the scaling law given by Equation (11) and this includes the isolated loop scaling (12) (when the condition of zero base conductive flux is used to eliminate the dependence upon the heating coefficient $E$). In our discussion we have preferred not to adopt the boundary condition of zero conductive base flux, for the use of such a condition demands a proper study of transition region modelling; instead, by selecting the ‘base’ temperature at, say, $10^6$ K we exclude the intricate effects in this complicated region (see also the discussion in Hood and Priest, 1979).

It is clearly of interest to compare the predictions (e.g. (11) or (12)) from the theory of static loops with the available observations. For example, if we compare (12) with the data quoted in Rosner et al., perhaps excluding the post-flare loops (the physics of which is likely to be somewhat different from active region loops), then the present authors conclude that the ‘agreement’ is unconvincing: the observations offer no support for the use of (12) as opposed to any other relation that one could dream up having a weak dependence upon $p$ and $L$. The difficulty lies in the lack of sufficiently accurate density, temperature and length measurements for a wide variety of loops, both within a single active region and from several different active regions. Until such improved observations become available, it is pointless to attempt to compare theory and observation in any detailed way: the agreement or lack of it may be entirely specious, masked by the errors in the data and the weak parameter dependence in the predictions of thermal statics.

The more general scaling (11) is at first sight a more attractive candidate for comparing theory and observations. However, the same considerations of accuracy in data apply and, additionally, the heating coefficient $E$ may vary from loop to loop (due to variations in magnetic field, for example) and from active region to active region, thus further limiting the comparison of theory and observations. To make such a comparison, it would appear that we must await developments in understand-
ing the nature of coronal heating, as well as the obtaining of an extensive observational study of loops from a single active region.

Altogether, then, the present authors would urge that the use of scaling laws is currently an inappropriate and perhaps misleading instrument for the understanding of the structure of coronal loops. We must await further developments in coronal physics to consider revising this opinion.

Acknowledgements

This work was begun while the authors were Visiting Scientists at American Science and Engineering. Work at AS&E was supported by NASA contract NA58-27758. We are happy to acknowledge the hospitality afforded to us by the Solar Physics Group at AS&E. Also, B. Roberts would like to thank his colleagues at St. Andrews, A. W. Hood and E. R. Priest, with whom much of the present work has been discussed. He would also like to acknowledge a valuable correspondence with R. Rosner and D. F. Webb.

References