ELECTROMAGNETIC MULTIPLE FIELDS OF NEUTRON STARS

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ABSTRACT

There is now indisputable evidence that some pulsars possess space velocities so high that internal asymmetries in the dynamics of their formation are strongly implied. We develop in this paper a complete formalism for the calculation of the only such mechanism that has yet been subjected to quantitative analysis: electromagnetic recoil radiation. To make the general problem tractable without doing violence to the physics, we have made the following simplifying assumptions: (1) the magnetic induction $B$ in a thin shell enclosing the surface can be satisfactorily approximated by a sum of vacuum multipole fields; (2) the star is spherical, and all parts are in good electrical contact; (3) $|\Omega \times r| \ll c$ everywhere within the star; and (4) the star is surrounded by a vacuum. Our qualitative conclusions hold even if these assumptions are violated, but corrections to our quantitative results required by a relaxation of our assumptions are not easily computed.

Given this simple electrodynamic model of a neutron star, we solve the following problems: (1) What electric multipoles are induced by each magnetic multipole? (2) What is the general formula for the recoil produced by the projection on the rotational axis of a net linear momentum flux produced by the rotation of any two magnetic multipoles? (3) What is the set of centered multipoles that represents the field of an arbitrary off-centered multipole? We use these general results to perform a detailed analysis of the linear momentum radiated by an off-centered dipole. We find a force larger by a factor 6 than that obtained for the special case treated in the best previous calculation. In spite of this considerable increase in the computed strength of the effect, we still believe it to be too weak to produce the large space velocities observed for pulsars. For the mechanism to be effective, the pulsar must be born rotating near the breakup velocity. The observational evidence on pulsar spin-down rates and lifetimes is inconsistent with such a history.

In a separate section we examine the Deutsch solution, which contained some, but not all, of the terms proportional to $(\Omega R/c)^3$. For a particular assumption of magnetic field configuration we show in detail how his solution can be made consistent to third order and we give the results. There are no changes to Deutsch's conclusions.

Subject headings: stars: magnetic — stars: neutron

I. INTRODUCTION

Deutsch (1955) considered the electric field induced in the vacuum by the rotation of a conducting ball possessing a fixed internal magnetic field configuration. Uniform and dipolar fields are special cases of the functional form he chose, but both produce identical external electromagnetic fields, namely, magnetic dipole and electric quadrupole. The magnetic fields of cosmic bodies can often be crudely represented as centered magnetic dipoles. But there is growing interest in more general internal field configurations. In particular, the off-centered magnetic dipole has been proposed (Harrison and Tademaru 1975, hereafter HT75) as a mechanism by which electromagnetic recoil radiation might affect the space velocities of pulsars. To properly analyze this possibility as well as address other questions that may arise from time to time, it is useful to have at hand the solution to the induction problem for arbitrary internal magnetic multipole fields.

Because we envision the chief application of our work to be to neutron stars, our physical motivation is rather different from that of Deutsch (1955), who assumed the magnetic field to be (1) frozen into the stellar material and (2) not subject to correction by the circulation of induced charge. But because of the quantal structure of the crust of a neutron star, the magnetization of the crust is negligible compared to the magnetic field strength; that is, the magnetic field is imposed from below the crust and cannot be frozen into it. Thus we can justifiably represent the magnetic field in the crust as a superposition of vacuum multipoles. In the case of a normal star such a simplification of the analysis cannot be made.

Further, the magnetic field's not being frozen into the crust implies that the circulation of induced charge in the crust produces corrections to the magnetic multipole moments of the star. These induced magnetic moments produce fields of order $(\Omega R/c)^2$ times the intrinsic field at the surface. We have obtained expressions for these induced magnetic moments (Roberts 1976), but since they are of limited interest, we do not include them here.

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II. ELECTRIC MULTipoLES

The electric field of any solid body of revolution rotating about its axis must satisfy the boundary condition $E_i = E_o$, that is, the tangential components of the inside and outside electric fields must be equal. Outside our ball is a vacuum, so the electric field there is completely described by the insertion of its multipole moments into the general solution of Maxwell’s equations in free space (Jackson 1975, p. 746, eq. [16.46]). We assume $\Omega R/c = q_s << 1$, where $\Omega$ is the rotational angular speed and $R$ is the radius of the ball, so we ignore relativistic corrections to the magnetic field (Goldreich and Julian 1969). This assumption is not really necessary; Campbell, Macek, and Morgan (1977) show that a multipole formalism can be developed for rapidly varying sources by suitably weighting the charge and current distributions. We shall not use their methods in this paper because our aim is to exhibit the logical structure of the induction problem in its simplest possible form.

Because the magnetic field at the surface is some arbitrary superposition of magnetic multipoles, our analysis is complete once we have obtained the electric multipoles generated by any magnetic multipole $M_{lm}$. We derive the axisymmetric multipoles ($m = 0$) from the $E_e$ equation, but we derive the nonaxisymmetric multipoles from the $E_e$ equation because it is simpler that way, although we could do both from the $E_o$ equation.

The electric field within the rotating star is $E = -v \times B/c = -\left(\mathbf{\Omega} \times r\right) \times B/c$ because we assume the star to be perfectly conducting. In the crust the conductivity may be very small perpendicular to the magnetic field lines but will still be very large along the magnetic field lines. Electrons are free to migrate across the field lines beneath the crust, so all parts of the crust are in mutual electrical contact. We have not analyzed the multipolar induction problem for a dielectric sphere. The interested reader can find the problem of an unmagnetized dielectric sphere rotating in a uniform external magnetic field discussed in detail in Landau and Lifshitz (1960, p. 246). This is a similar problem to that of a dielectric sphere with an intrinsic magnetic dipole moment rotating in its own field.

Consider an arbitrary axisymmetric harmonic magnetic multipole $M_{l0}$. The magnetic field is

$$B = \frac{4\pi}{2l + 1} \frac{M_{l0}}{r^{l+2}} \left[ r(l+1)Y_{l0} - \frac{\partial}{\partial \theta} \right]$$

and

$$E = -\left(\Omega \times r\right) \times B/c,$

so the internal electric field, $E = -(\Omega \times r) \times B/c$ because we assume the star to be perfectly conducting. In the crust the conductivity may be very small perpendicular to the magnetic field lines but will still be very large along the magnetic field lines. Electrons are free to migrate across the field lines beneath the crust, so all parts of the crust are in mutual electrical contact. We have not analyzed the multipolar induction problem for a dielectric sphere. The interested reader can find the problem of an unmagnetized dielectric sphere rotating in a uniform external magnetic field discussed in detail in Landau and Lifshitz (1960, p. 246). This is a similar problem to that of a dielectric sphere with an intrinsic magnetic dipole moment rotating in its own field.

The transverse component of the electric field, $E_o$, must be continuous at the surface $r = R$. This condition gives us the magnitude of the two multipoles that comprise the external electric field. The internal electric field can be rewritten (at $r = R$) as

$$E_o = -\left(\frac{4\pi}{2l + 1}\right) \frac{M_{l0}k}{r^{l+1}} \left[ \frac{r}{2l + 1} r(l+1)P_{l+1}(\cos \theta) - \frac{\partial}{\partial \theta} \frac{l(l+1)}{2l + 1} \left(P_{l+1}(\cos \theta) - P_{l-1}(\cos \theta)\right) \right].$$

(1)

The transverse component of the electric field, $E_o$, must be continuous at the surface $r = R$. This condition gives us the magnitude of the two multipoles that comprise the external electric field. The internal electric field can be rewritten (at $r = R$) as

$$E_o = \frac{4\pi}{2l + 1} \frac{M_{l0}k}{R^{l+1}} \left(\frac{I + 1}{2l + 1}\right) \frac{\partial}{\partial \theta} \left(P_{l+1} - P_{l-1}\right).$$

(3)

The multipole moments are thus

$$Q_{l-1,0} = \left(\frac{I + 1}{2l + 1}\right) \left(\frac{2l - 1}{2l + 1}\right) \frac{1}{2} \frac{M_{l0}kR^2}{I_{l+1} \Omega R^2/c},$$

(4)

and

$$Q_{l+1,0} = -\left(\frac{I + 1}{2l + 1}\right) \left(\frac{2l + 3}{2l - 1}\right) \frac{1}{2} \frac{M_{l0}kR^2}{I_{l+1} \Omega R^2/c},$$

(5)

where the $\theta$'s are what we shall call electric induction coefficients.

The method of matching the tangential electric fields does not address the question of the presence of a uniform surface charge; other arguments must be used to determine the possibility of a net charge on the star. In the aligned dipolar case, $I = 1$, the electric induction coefficient $Q_{0,-1}$ gives naively the net charge $Q = (4\pi)^{1/2} Q_{0,-1}$, which is the amount of charge required to ground the poles, and that conventionally assumed to be present on a real inductor. This net charge, by Gauss’s law, must be present inside the surface, but it could be canceled by an opposite surface charge.

The equation $E_o(r = R) = 0$ gives us the multipole moments for all $m \neq 0$. The $\phi$ component of the near fields of the electric multipoles must be canceled by the magnetic multipole’s electric induction field:

$$-\frac{M_{l0}k}{I(l+1)} \frac{\partial}{\partial \theta} Y_{lm} = Q_{l-1,m} \frac{1}{2l - 1 \sin \theta} Y_{l-1,m} + Q_{l+1,m} \frac{1}{2l + 3 \sin \theta} Y_{l+1,m}.$$

(6)
Using recurrence relations for the spherical harmonics, equation (6) becomes

\[ Y_{l-1,m} \left[ \frac{M l m k}{l} \left( \frac{l + 1}{2l + 1} \right) C_0 - \frac{Q_{l-1,m}}{2l - 1} \right] - Y_{l+1,m} \left( \frac{M l m k}{2l + 1} C_{+1} + \frac{Q_{l+1,m}}{2l + 3} \right) = 0, \tag{7} \]

where \( k = m \Omega/c \) and the \( C_k \) are the spherical harmonic recurrence coefficients,

\[ C_k \equiv \sqrt{\frac{(l + m + k)(l - m + k)}{[2(l + k) + 1][2(l + k) - 1]}}. \tag{8} \]

Since the coefficient of each spherical harmonic in equation (7) must be zero, the induced electric multipole moments are

\[ Q_{l-1,m} = i m \varphi_{-1} M_{lm} \Omega/c, \quad Q_{l+1,m} = i m \varphi_{+1} M_{lm} \Omega R^2/c, \tag{9} \]

where the electric induction coefficients are found from the expressions

\[ i m \varphi_{-1} = \frac{l + 1}{l - 2l + 1} C_0, \quad i m \varphi_{+1} = -\frac{2l + 3}{2l + 1} C_{+1}. \tag{10} \]

If we set \( m = 0 \) in equations (9) and (10), we find that the expressions for \( i m \varphi_{-1} \) and \( i m \varphi_{+1} \) reduce to the expressions for \( i 0 \varphi_{-1} \) and \( i 0 \varphi_{+1} \) in equations (4) and (5). Thus one set of electric induction coefficients suffices for all \(-l \leq m \leq l\).

Observe that the frequency of radiation emitted by an \( m \) multipole is at the \( m \)th harmonic of the rotational frequency \( \Omega \) of the star. Since different multipole radiation fields can be linearly superposed, there is no essential complication in considering the fields of a set of multipoles of different \( m \) values.

In Figure 1 we show the loci of \( E_r = 0 \) at the surface for different inclinations of a magnetic dipolar axis.

The electromotive field around a magnetic field line from foot to foot,

\[ \mathcal{E} = \int \mathbf{E} \cdot ds = \int \frac{\mathbf{E} \cdot \mathbf{B}}{\mathcal{B}} ds, \tag{11} \]

equals zero. We can see this by considering the parity of the spherical harmonics and equation (16.46) of Jackson (1975). The vector spherical harmonics have the same parity as the scalar spherical harmonics with the same value of \( l \), namely, \((-1)^l\), and their curls have the opposite. Thus all electric fields have parity \((-1)^l\) and all magnetic fields \((-1)^{l+1}\), where \( l \) is that of the inducing magnetic moment \( M_{lm} \). The fields \( \mathbf{E} \) and \( \mathbf{B} \) in equation (11) have opposite parity, so the parity of \( \mathcal{E} \) is odd. But \( \mathcal{E} \) must be invariant under inversion; thus \( \mathcal{E} = 0 \).

When \( l \geq 2, l > |m| \), we may ask what the ratio is of the radiated power from the multipole \( Q_{l-1,m} \) to that of \( M_{lm} \). We find (Jackson 1975) that

\[ \frac{P_g(l-1,m)}{P_m(l,m)} = \frac{l + 1}{l - 1} \left( \frac{2l + 1}{2l - 1} \right) \left( \frac{l^2 - m^2}{m^2} \right). \tag{12a} \]

It is remarkable that the induced electric moment \( Q_{l-1,m} \) can be the major radiator. For example,

\[ \frac{P_g(1,1)}{P_m(1,1)} = \frac{27}{5}, \quad \frac{P_g(2,1)}{P_m(2,1)} = \frac{80}{7}. \tag{12b} \]

Observe that the ratio (12a) does not depend on any parameters of the inductor.

III. OFF-CENTERED MAGNETIC MULTipoles

Pulsars have a high average space velocity (Helfand and Tademaru 1977; Gullaborn and Rankin 1978). The physical mechanism by which they acquire this velocity is not yet a matter of general agreement. In a well-reasoned and thoughtful paper, HT75 have proposed a novel mechanism, namely, that a magnetic dipole off-centered from the rotational axis will produce a radiation field containing a net linear momentum, thus rocketing the pulsar away.

We shall approach this problem using a different formalism, one in which the interference structure of the multipole radiation fields is most transparent (see also Peres 1962). Wherever there is overlap with the development in Jackson’s (1975) chapter 16, we follow his notation and conventions. In particular, we use the long-wavelength multipole coefficients \( a(l,m) \), even though the mechanism of HT75 has its chief application in a regime \([\Omega \lesssim (G\rho)^{1/3}]\), where this approximation is not valid for massive neutron stars and the particular formulae given by Jackson (1975) for the \( a(l,m) \) are inapplicable. Nevertheless, the generality of our result is not compromised. Campbell, Macek, and Morgan (1977) have shown that when the motions of the source are relativistic, one can calculate weighted multipole moments that, when inserted into the usual long-wavelength formulae for the multipole coefficients, yield a correct expansion of the electromagnetic field in multipole fields.
Fig. 1.—Curves describing the loci of points on which \( E_r = 0 \) for a dipole inductor for different values of the inclination angle \( \chi \) between the rotation axis (vertical) and the magnetic axis; \( \chi = (a) \frac{\pi}{2}, (b) \frac{3\pi}{2}, (c) \frac{5\pi}{2}, (d) \pi, (e) \frac{7\pi}{2}, (f) 3\pi \). The sign of the radial electric field at the surface is shown for all regions. The inductor is uncharged.
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Note that the multipole moments used in the preceding section may be regarded as either (a) static multipoles computed from real source densities as defined in Jackson (1975), chapter 4, or (b) radiation multipoles computed from complex Fourier amplitudes of the source density as defined in Jackson, chapter 16. In convention (a) the multipole moment corresponding to \(-m\) must be included in the total field: \(q_{l,m} = (-1)^m q_{l,m}^*\). In convention (b), which is used only at the end of this section, there is no moment corresponding to \(-m\). Misguided inclusion of radiation fields due to moments corresponding to \(-m\) (following the static convention) can lead to conclusions that are totally wrong, such as that the radiation field arising from electric and magnetic dipoles cannot contain net linear momentum relative to the radiator.

a) Linear Momentum in Multipole Radiation Fields

Given the electric and magnetic multipole radiation fields

\[
B_{lm}^{(E)} = \frac{(-i)^{l+1}}{kr} a_l(l, m) X_{lm} e^{i kr}, \quad B_{lm}^{(M)} = \frac{(-i)^{l+1}}{kr} a_m(l, m) (i \times X_{lm}) e^{i kr},
\]

we can construct integrals proportional to the linear momentum in the interference terms between any pair of multipole fields. Because the angular integrals (Teukolsky 1976) are

\[
\int d\Omega \cos \theta X_{lm} \cdot X_{l'm'} = \begin{cases} \frac{l(l+2)}{(l+1)^2} C_{l+1}, & l' = l + 1, m' = m \\ \frac{(l-1)(l+1)}{l^2} C_0, & l' = l - 1, m' = m \\ 0, & \text{otherwise}, \end{cases}
\]

and

\[
\int d\Omega \cos (i \times X_{lm}) \cdot X_{l'm'} = -\frac{im}{l(l+1)}, \quad l' = l, m' = m
\]

where

\[
C_l \equiv \left[ \frac{(l + |m| + k)(l - |m| + k)}{2(l + k + 1)(2l + k + 2)} \right]^{1/2},
\]

we need consider only the interference integrals

\[
\gamma_{lm}^{(EM)} \equiv \int d\Omega \cos \theta (B_{lm}^{(M)} \cdot B_{lm}^{(E)*} + B_{lm}^{(M)*} \cdot B_{lm}^{(E)}) = +2 \frac{I_{lm}^{(EM)}}{(kr)^2} \frac{m}{l(l+1)},
\]

\[
\gamma_{lm}^{(EE)} \equiv \int d\Omega \cos \theta (B_{lm}^{(E)} \cdot B_{l+1,m}^{(E)*} + B_{lm}^{(E)*} \cdot B_{l+1,m}^{(E)}) = -2 \frac{I_{lm}^{(EE)}}{(kr)^2} \left[ \frac{l(l+2)}{(l+1)^2} \right]^{1/2} C_{l+1},
\]

\[
\gamma_{lm}^{(MM)} \equiv \int d\Omega \cos \theta (B_{lm}^{(M)} \cdot B_{l+1,m}^{(M)*} + B_{lm}^{(M)*} \cdot B_{l+1,m}^{(M)}) = -2 \frac{I_{lm}^{(MM)}}{(kr)^2} \left[ \frac{l(l+2)}{(l+1)^2} \right]^{1/2} C_{l+1},
\]

where

\[
I_{lm}^{(EM)} = \text{Im} \left[ a_m(l, m) a_{m,*}(l + 1, m) \right],
\]

\[
I_{lm}^{(EE)} = \text{Im} \left[ a_m(l, m) a_{m,*}(l + 1, m) \right],
\]

\[
I_{lm}^{(MM)} = \text{Im} \left[ a_m(l, m) a_{m,*}(l, m) \right].
\]

These imaginary parts of the pairwise products of multipole coefficients depend only on the imaginary parts of the pairwise products of magnetic multipole moments,

\[
i_{lm} = \text{Im} \left( M_{lm} M_{l+1,m,*} \right),
\]

because the electric multipole moments are strictly determined by the magnetic multipole moments, the rotational speed, and the stellar radius. Armed with this knowledge, we can easily obtain explicit formulae for all the

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interference integrals (16) that result from a particular pair of neighboring multipole moments \(M_{lm}\), \(M_{l+1,m}\). We have, for the terms of \(\gamma_{lm}^{(MM)}\) and \(\gamma_{lm}^{(ME)}\) that involve only \(M_{lm}\) and \(M_{l+1,m}\),

\[
\gamma_{lm}^{(MM)} + \gamma_{lm}^{(ME)} = A \frac{(l+1)(l+2)}{l^2},
\]

\[
\gamma_{lm}^{(EE)} + \gamma_{l+1,m}^{(ME)} = A(kR)^2 \frac{2l+5}{2l+1} \frac{1}{m^2} C_2^{+},
\]

where

\[
A = -2 \frac{r_0}{k^2} \left[ \frac{4\pi k^{l+2}}{(2l+1)!} \right]^2 \frac{C_{l+1}}{2l+3}
\]

and we have used the electric induction coefficients given in § II.

Observe that there are no equations in \(-m\) corresponding to equations (13)-(18) because \(a(l, -m) = 0\) in the radiation multipole convention, when the radiation is produced by rigid rotation of multipole of fixed strength. The relation between the quantity \(\gamma\) and the linear momentum radiated is found as follows. The time-averaged energy density in harmonically varying vacuum fields is defined as (Jackson 1975, p. 242)

\[
u = (E \cdot E^* + B \cdot B^*)/8\pi,
\]

so that in the radiation zone \(\nu = (B \cdot B^*)/8\pi\). The time-averaged component of the linear momentum density along the axis \(z\) is \(p_z = \nu \cos \theta/\nu\). Thus, if \(\mathbf{B} = B_{lm}^{(M)} + B_{lm}^{(E)}\), we have

\[
\dot{P} = r^2 c \int (d\Omega/4\pi) p_z = (r^2/8\pi) \int d\Omega \cos \theta (B_{lm}^{(M)} \cdot B_{lm}^{(E)}) + B_{lm}^{(M)} \cdot B_{lm}^{(E)}.
\]

Whence

\[
\dot{P} = (r^2/8\pi)(\text{sum of } \gamma's).
\]

This rate of change of momentum in the radiation field is equal and opposite to the rate of change of the momentum of the rotating star; hence the star recoils. An immediate corollary of equations (9), (17), and (19) is that there is no recoil radiation unless the phases of at least two magnetic multipole moments differ by other than an integral multiple of \(\pi\).

b) Expansion of an Off-centered Multipole Moment in Centered Moments

A number of authors have investigated the representation of the external magnetic field of cosmic bodies by an off-centered dipole. Jupiter (Komesaroff 1976), Earth (Bernard et al. 1969), magnetic stars (Stift 1974), and pulsars (HT75) have all received such treatment. It has been found that the off-centered dipole field gives a representation of the magnetic field about as good as a general spherical harmonic fit with the multipoles as free parameters. The chief advantage of the off-centered dipole model is that, once the strength, distance from the center, and orientation of the dipole are chosen, all higher multipole moments are determined. It is easy to derive this dependence for an arbitrary off-centered multipole.

We give here a brief outline of the transformations necessary to obtain a centered representation for an off-centered solid harmonic. In this section we use static multipole moments derived from real source densities because these have nice transformation properties. We convert to radiation multipole coefficients at the end of the calculation. First, choose an off-centered multipole moment \(q_{lm}^{(c)}\) as coefficient of the solid harmonic \(Y_{lm}(\theta', \phi')/r_{l+1}\)

whose origin is at \(r_0 = r_0\), whose polar axis \(z'\) is positive outward along the line of centers, and whose \(x'\) axis is in the plane that we will call the offset plane defined by the displacement vector \(r_0\) and the rotation vector \(\Omega\), and is positive toward increasing \(\theta'\) at that point. This fixes the azimuthal phase of the off-centered multipole. Second, expand the off-centered solid harmonic in centered ones with the same axis (Hobson 1931, p. 140):

\[
Y_{lm}(\theta', \phi') = \sum_{k=0}^{\infty} r_0^k \left[ \frac{2l+1}{2l} \frac{l-m+k}{l-m} \frac{l+m+k}{l+m} \right]^{-1/2} Y_{l+1,k,m}(\theta', \phi').
\]

Thus the centered, but unrotated, multipole moments of this off-centered multipole field are

\[
q_{l+1, k, m}^{(c)} = r_0^k \left[ \frac{2l+1}{2l} \frac{l-m+k}{l-m} \frac{l+m+k}{l+m} \right]^{1/2} q_{lm}^{(c)} Y_{l+1,k,m}(\theta', \phi').
\]

Clearly, \(q_{lm}^{(c)} = q_{lm}^{(c)}\) for any multipole. Third, rotating the spherical harmonics \(Y_{l+1,k,m}(\theta', \phi')\) through the polar angle \(\beta\) so that their axis \(z'\) coincides with the axis of rotation \(z\) with the well-known expression,

\[
Y_{lm}(\theta, \phi') = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} Y_{lm}(\theta', \phi') r_{l+1}(\beta) \, d\beta,
\]

where
using the quantum mechanical polar rotation matrices \( r_{m'm}^{(k)}(\beta) \) (Messiah 1966), which are purely real, we observe that

\[
q_{l+k,m} = \sum_{m'=-l}^{l} q_{l+k,m}' r_{m'm}^{(l+k)}(\beta).
\]

Substituting equation (21) into equation (23), we obtain

\[
q_{l+k,m} = r_{0}^{k} \sum_{m'=-l}^{l} q_{m'm}^{*} G_{l}^{(k)}(\beta) r_{m'm}^{(l+k)}(\beta).
\]

This is the general formula for transforming a set of off-centered multipoles referred to the displacement axis \( z' \), \( q_{m'm}^{*} \) \((-l \leq m, l \leq l)\), to a set of centered multipoles referred to the rotational axis, \( q_{l+k,m} \) \([k = 0, 1, 2, \ldots, \infty, -l \leq m, l \leq l+k]\).

Although the coefficients \( G_{lm}^{(k)} \) and \( r_{mm}^{(l+k)} \) are real, in general, the phases of, say, \( q_{l+k,m} \) and \( q_{l+k+m} \) will not differ by an integral multiple of \( \pi \). This phase mixing can be seen clearly in the following way. Consider two sets of centered multipole moments referred to the displacement axis, \( \{ q_{l-1,m}' , -l \leq m \leq l \} \) and \( \{ q_{l+1,m}' , -l \leq m \leq l \} \). If these centered moments are derived from the same off-centered moment, the corresponding moments with the same \( m' \) have the same azimuthal phase by virtue of equation (21). These sets of moments when rotated to refer to the rotational axis will each retain the same \( l \) weight, \( l_{1} \) and \( l_{2} \), since the \( l \) weight of spherical harmonics is invariant under rotation (but not, of course, translation).

However, the rotated sets, \( \{ q_{l-1,m}' , -l \leq m \leq l \} \) and \( \{ q_{l+1,m}' , -l \leq m \leq l \} \), are no longer characterized by the moments with the same \( m \) having the same azimuthal phase. The rotation matrix \( r_{mm}^{(l+k)} \) forms the \( q_{l,m} \) with a different set of coefficients than the matrix \( r_{mm}^{(l+k)} \) supplies to form the \( q_{l,m} \). Since the linear combinations are different for the different \( l \) weights, the phases of the moments are mixed differently. This conclusion can, of course, be given a precise mathematical statement. A short calculation shows that the requirement that the phases be mixed identically is that

\[
(-1)^{m} r_{m'-m}^{(l)} = r_{mm}^{(l)}, \quad l = l_{1}, l_{2}.
\]

This relation is true for all \( m' \) only if \( m = 0 \), when the moments have no complex phase and in our problem cannot radiate.

c) Recoil Radiation from an Off-centered Dipole

It is cumbersome, and neither instructive nor useful, to develop a general formula for the recoil radiation from an arbitrary off-centered multipole. The only case of present interest is the off-centered dipole, explored recently by HT75. The translation coefficients \( G_{lm}^{(k)} \) for the dipole are

\[
G_{lm}^{(k)} = \frac{1}{k!} \left[ \frac{2k + 3 (k-m+1)! (k+m+1)!}{3 (1-m)! (1+m)!} \right]^{1/2}.
\]

The rotation matrices \( r_{mm}^{(l)} \) and \( r_{mm}^{(2)} \) can be found in standard texts (Rose 1957; Brink and Satchler 1962). We shall not consider the centered moments higher than the quadrupole.

In our decomposition of this problem we have four interference terms to consider in the production of recoil radiation. First, the magnetic dipole moments \( M_{11} \) and \( M_{21} \) will produce the interference integral \( Y_{11}^{(MM)} \). Second, an electric dipole moment \( Q_{11} \) having the same phase as \( M_{21} \) and an electric quadrupole moment \( Q_{21} \) having the same phase as \( M_{11} \) will produce the interference integral \( Y_{11}^{(EE)} \). Third, \( M_{11} \) and \( Q_{11} \) will produce \( Y_{11}^{(ME)} \); and fourth, \( M_{21} \) and \( Q_{21} \) will produce \( Y_{21}^{(ME)} \). These interference integrals are summed in pairs in equations (18a) and (18b). Implicit in equation (18c) is the assumption of slow rotation (long-wavelength limit), so that we must consider that the interference pair \( Y_{11}^{(EE)} \) and \( Y_{21}^{(ME)} \) are negligible compared to \( Y_{11}^{(MM)} \) and \( Y_{11}^{(ME)} \). From equations (8), (18a), (18c), and (19) we get

\[
P = -(8\pi/15\sqrt{5}) i_{11} k^{5}.
\]

The force exerted on the star is \( F = F_{z} = -\dot{P}_{z} \). It remains to determine the quantity \( i_{11} = \text{Im} (M_{11} M_{21}^{\ast}) \).

Let us revert to the convention \( q_{m} \) for the static multipoles and convert later to the radiation multipole \( M_{lm} \). We first establish a simple notation and give its correspondence to that of HT75. We take as given an arbitrary off-centered vector dipole \( q \), with a component \( q_{l} \) along the line of centers \( z' (q'z' = q_{l}) \) and a component \( q_{\perp} = q - z' q_{l} \) perpendicular to the line of centers (Fig. 2). The component \( q_{\perp} \) makes an angle \( \alpha \) with the inferior segment of the meridian defined by the intersection of the sphere with the \( z' - z' \) plane. In the cylindrical coordinates and notation of HT75 we have

\[
q_{l} = \mu_{z} \cos \beta + \mu_{r} \sin \beta,
\]

\[
q_{\perp} = [\mu^{2} + (\mu_{z} \sin \beta + \mu_{r} \cos \beta)^{2}]^{1/2} = \mu_{\phi}/\sin \alpha,
\]

\[
\tan \alpha = \mu_{\phi}/(-\mu_{z} \sin \beta + \mu_{r} \cos \beta).
\]
Fig. 2.—The vector dipole moment $\mathbf{M}$ is displaced a distance $r_0$ from the center of the ball. Its projection on the sphere makes the angle $\alpha$ with its meridian.

We are at liberty to choose the origin of azimuthal phase to be at $\alpha$, which gives us the real harmonic multipole moments (in a slight simplification),

$$q_{11}^* \equiv q_+ = -q_0(3/8\pi)^{1/2}, \quad q_{1, -1}^* \equiv q_- = +q_0(3/8\pi)^{1/2}, \quad q_{10}^* \equiv q_0 = q_0(3/4\pi)^{1/2}. \quad (28)$$

The centered moments are

$$q_{1, \pm 1} = q_0, \quad q_{10}^* = q_0, \quad q_{2, \pm 2} = 0, \quad q_{2, \pm 1}^* = 5^{1/2}r_0q_0, \quad q_{20}^* = 2(5/3)^{1/2}r_0q_0. \quad (29)$$

The important moments after rotation are

$$q_{11} = q_0(i \sin \alpha + \cos \beta \cos \alpha) - (q_0/2^{1/2}) \sin \beta,$$

$$q_{21} = -5^{1/2}r_0q_0[i \cos \beta \sin \alpha + (2 \cos^2 \beta - 1) \cos \alpha] - 2^{1/2}q_0 \sin \beta \cos \beta,$$

and

$$q_{1, -1} = -q_{11}^*, \quad q_{2, -1} = -q_{21}^*. \quad (30)$$

The connection to radiation moments is simply

$$M_{11} \leftrightarrow 2q_{11}, \quad M_{21} \leftrightarrow 2q_{21}, \quad 0 \leftrightarrow q_{1, -1}, \quad q_{2, -1}. \quad (31)$$

Hence the imaginary part of the moment product is

$$i_{11} \equiv \text{Im}(M_{11}M_{21}^*) = -4 \times 5^{1/2}r_0q_0 \sin \alpha (q_0 \cos \alpha(1 - 3 \cos^2 \beta) + (3/2^{1/2})q_0 \sin \beta \cos \beta). \quad (32)$$

The recoil force on the star is obtained by substituting the negative of equation (32) into equation (26). In the special case considered by HT75 the dipole lies in the rotational equatorial plane, $\beta = \pi/2$, so that

$$i_{11}(\beta = \pi/2) = -4 \times 5^{1/2}r_0q_0^2 \sin \alpha \cos \alpha = [(3 \times 5^{1/2})/2\pi]r_0\mu_0\mu_x, \quad (33)$$

and we get the following simple expression for the force on the star:

$$F = 2(4/5)k^2r_0\mu_0\mu_x. \quad (34)$$
This force is a factor 6 larger than that found by HT75. This difference is accounted for by their not having included the factor 2 in the conversion from static to radiation moments in their equations (A18)-(A20) and by their not having correctly treated the electrical polarization of the conducting star. We would not be surprised if our discussion had defects as well. Tademaru (1976) has also treated the case of the nonconducting star; but this case is not of astrophysical interest, so we will not discuss it.

We have analyzed this problem as a simple example of the application of our multipole induction formalism to an astrophysical problem, one which has already been approached by other means. But it does not seem to us that electromagnetic recoil radiation can play a great role in the generation of high pulsar space velocities. As HT75 point out, the mechanism requires that the newborn neutron stars be rotating close to the breakup velocity. Evidence is mounting, from the kinetic ages and slowing rates of pulsars, that neutron stars are formed with much smaller angular velocities. Further, the formalisms used in this paper and in HT75 are not valid for rapid rotators: one needs formulae that are complete to at least second order in $\Omega R/c$ (J. Mathews and W. J. Roberts, unpublished). For this reason it is not valid to compute, using this formalism, second-order corrections to the radiated power of the dipole produced by its being off-centered. The radiation from the electric dipole is second order in $\Omega r_0/c$, as is that of the magnetic quadrupole.

IV. A SLIGHT MODIFICATION TO THE DEUTSCH SOLUTION

Deutsch (1955) carried out a solution to third order in $q_s \equiv \Omega R/c$ of the Maxwell equations for a rotating oblique magnetic star with only a dipolar external magnetic moment. To do so self-consistently requires a very special assumption. The charge distribution convected with the material of the star produces a magnetic dipole moment and a magnetic octopole moment that are second order in $q_s$. Deutsch did not include these induced magnetic moments in his solution. This omission is equivalent to assuming that the intrinsic currents within the star rearrange themselves to cancel the induced currents. It is not clear to us that his assumption is plausible, whether one views the star as having been spun up or whether one considers it as being characterized by a steady-state dynamo. We make a different assumption. Like Deutsch, we admit the mathematical possibility that the nonrotating star has a vacuum dipolar internal magnetic field all the way to the center. With this as our starting assumption, our solution has no adjustable functions or parameters. We have developed elsewhere (Roberts 1976) the formalism for computing the electric and magnetic harmonic moments of such a star and give there all the relevant formulae for computing the induced multipoles of arbitrary intrinsic internal multipole fields. In particular, we show there that each intrinsic multipole can be treated independently of the others. We examine here only the perpendicular moment $M_{lm}$ with $l = m = 1$.

Deutsch’s solution is generally correct. A few minor errors should be pointed out: In equation (13) Deutsch gives a nonzero radial component for the magnetic wave field; the component he gives is actually that for the dipolar magnetic induction field, which is, of course, the leading term in the radial component of the magnetic field at large distances. The surface charge in his equation (22) has an extra factor $c^2$. His formulae for the near electric and magnetic fields are free of error, but his magnetic induction field of the electric quadrupole is too large by a factor 2.

a) Stellar Parameters

The radiating harmonic magnetic moment $M_{11}$ ($l = m = 1$) is related to the conventional Cartesian dipole moment $M \sin \chi$, where $\chi$ is the angle included between $\Omega$ and $M$, by

$$M_{11} = -(3/2\pi)^{1/2} M \sin \chi.$$  (35)

Because $M_{1,-1} \equiv 0$, $M_{11}$ is twice the magnetostatic dipole moment.

The intrinsic harmonic magnetic moment $M_{11}$ is modified by rotation to the effective external dipole moment,

$$M_{11}' = M_{12} + M_{12}^{(1)} = M_{12}(1 + q_s^2/6).$$  (36)

The circulation current of the quadrupolar charge distribution induced by the rotation of $M_{11}$ produces the external magnetic octopole moment ($kR \approx q_s$),

$$M_{31}^{(1)} = 14^{1/2}/15 M_{11} q_s^2 R^2.$$  (37)

Matching the vacuum electric induction fields of $M_{11}'$ and $M_{31}^{(1)}$ to the boundary condition $E_\theta(r = R) = 0$ yields the two electrostatic harmonic multipole moments,

$$Q_{21}' = Q_{21}^{(1)} + Q_{21}^{(2)} = -\frac{5^{1/2}}{3} M_{11}(1 + 67/70 q_s^2) kR^2,$$
$$Q_{41}^{(2)} = +\frac{1}{7} \left(\frac{6}{5}\right)^{1/2} M_{11} k^3 R^6.$$  (38)
These stellar multipole moments feed into the multipole radiation formalism developed in Jackson (1975) through his multipole coefficients \(a_M(l, m)\) and \(a_E(l, m)\), as used in his general vacuum solution to the Maxwell equations (16.46). In our problem the only nonzero multipole coefficients are

\[
a_M(1, 1) = i \frac{4\pi^2/3}{3} M_{11}(1 + q_e^2/6)k^3,
\]

\[
a_M(3, 1) = -i \frac{8\pi}{225} \left( \frac{2}{21} \right)^{1/2} M_{11} k^7 R^4,
\]

\[
a_E(2, 1) = \frac{2\pi}{3} \left( \frac{2}{15} \right)^{1/2} M_{11} \left( 1 + \frac{67}{70} q_e^2 \right) k^5 R^6,
\]

\[
a_E(4, 1) = -i \frac{\pi}{9!} \frac{2 \times 6^{1/2}}{7} M_{11} k^9 R^6.
\]

(39)

We shall ignore the magnetic induction field produced by the rotation of \(Q_{41}^{(2)}\); that is, we shall leave out of \(B\) the term with \(a_E(4, 1)\).

**b) Electromagnetic Field**

We can now write out the fields explicitly—first, the magnetic field:

\[
B = -\frac{i}{k} a_M(1, 1) \mathbf{V} \times (h_1 X_{11}) + a_E(2, 1) A_2 h_2 X_{21} - \frac{i}{k} a_M(3, 1) B_3 \mathbf{V} \times (h_3 X_{31})
\]

\[
\equiv B^{(1)} + B^{(2)} + B^{(3)}. 
\]

(40)

Because we impose no boundary conditions on \(B\), we must have \(B_1 = B_3 = 1\). The component from the rotating dipole is \((q = \Omega r/c)\)

\[
B^{(1)} = -\left( \frac{3\pi}{2} \right)^{1/2} M_{11} k^3 \left[ \frac{i}{q} 2h_1 \sin \theta + \hat{\theta} \left( h_1' + \frac{h_1}{q} \right) \right] e^{i\phi}.
\]

(41)

Rewriting equation (41) using the conventional dipole magnetic moment \([M = -(3/2\pi)^{1/2} M_{11}]\), we get

\[
B^{(1)} = M(1 + q_e^2/6)k^3 \left[ \frac{i}{q} 2h_1 \sin \theta + \hat{\theta} \left( h_1' - \frac{h_1}{q} \right) \right] e^{i\phi}.
\]

(42)

This component differs slightly from the corresponding component found by Deutsch (1955). The component induced by the rotating electric quadrupole is

\[
B^{(2)} = A_2 \left( \frac{3}{2} \right)^{1/2} M_{11} q_e^2 \left( 1 + \frac{67}{70} q_e^2 \right) k^3 h_2 \left[ \frac{i}{q} h_3 \sin \theta - \hat{\theta} \cos \theta \cos^2 \theta - \sin^2 \theta \right] e^{i\phi}.
\]

(43)

\(A_2 \sim 1\) is to be determined by matching \(E_\phi\) at the surface. Near the star \(B_2^{(2)} \approx B_2^{(1)} q_e^2/2\). Thus, to order \(q_e^2\), we find \(B_3\) increased by a factor \((1 + 2q_e^2/3)\). The component produced by the rotating magnetic octopole is

\[
B^{(3)} = \frac{e^{i\phi}}{225} \left( \frac{3}{2} \right)^{1/2} M_{11} k^7 R^4 \left[ \frac{i}{q} 12 h_3 \sin \theta (5 \cos^2 \theta - 1) + i\hat{\theta} \left( h_3' + \frac{h_3}{q} \right) \cos \theta (4 - 15 \sin^2 \theta)
\]

\[ - \hat{\theta} \left( h_3' + \frac{h_3}{q} \right) (5 \cos^2 \theta - 1) \right].
\]

(44)

The electric field is slightly more complicated, being derived from four multipole moments:

\[
E = a_M(1, 1) h_1 X_{11} + \frac{iA_2}{k} a_E(2, 1) \mathbf{V} \times (h_2 X_{21}) + a_M(3, 1) h_3 X_{31} + \frac{iA_4}{k} a_E(4, 1) \mathbf{V} \times (h_4 X_{31})
\]

\[= E^{(1)} + E^{(2)} + E^{(3)} + E^{(4)}.
\]

(45)
The components induced by the changing magnetic field are simple:

\[
E^{(1)} = i(3\pi)^{1/2} M_{11}(1 + q_s^2/6) \Omega^2 h_1(\theta + i\phi \cos \theta)e^{i\phi},
\]

\[
E^{(3)} = -\frac{i}{225}(3\pi)^{1/2} M_{11} \Omega^2 R^4 h_2[5 \cos^2 \theta - 1 + \phi \cos \theta(15 \cos^2 \theta - 11)]e^{i\phi}.
\]

The field produced by the rotating electric quadrupole is

\[
E^{(2)} = -\frac{A_s}{3}(3\pi)^{1/2} M_{11} \left(1 + \frac{67}{70} q_s^2 \right) \Omega^2 R^2 \left[\frac{R^2}{q} \sin \theta \cos \theta (7 \cos^2 \theta - 3) - i\theta \left(h_2 + \frac{h_s}{q}\right) (28 \cos^2 \theta - 27 \cos^2 \theta + 3) + \phi \left(h_2 + \frac{h_s}{q}\right) \cos \theta (7 \cos^2 \theta - 3) \right]e^{i\phi},
\]

and the electric 16-pole produces

\[
E^{(4)} = -\frac{A_s}{28\cdot 7!}(3\pi)^{1/2} M_{11} \Omega^2 R^6 \left[-i\frac{20h_s}{q} \sin \theta \cos \theta (7 \cos^2 \theta - 3) - i\theta \left(h_2 + \frac{h_s}{q}\right) (28 \cos^2 \theta - 27 \cos^2 \theta + 3) + \phi \left(h_2 + \frac{h_s}{q}\right) \cos \theta (7 \cos^2 \theta - 3) \right]e^{i\phi}.
\]

In setting \(E^{(1)} + E^{(2)} + E^{(3)} + E^{(4)} = 0\) at \(r = R\), we obtain two equations in \(A_2\) and \(A_4\). The coefficient of \(\cos \theta\) yields the equation

\[
A_2 \left[\frac{1}{3} \left(1 + \frac{16}{105} q_s^2 \right) q_s^2 \left(h_2 + \frac{h_s}{q}\right) \right] + A_4 \left[\frac{1}{20 \cdot 49} q_s^6 \left(h_2 + \frac{h_s}{q}\right) \right] = -2(1 + q_s^2/6)h_1.s - \frac{11}{225} q_s^4 h_3.s,
\]

and the coefficient of \(\cos^3 \theta\) gives

\[
A_4 = -\frac{28}{q_s^3} \left(h_2 + \frac{h_s}{q}\right) \sin \theta \cos \theta (7 \cos^2 \theta - 3).
\]

Substituting equation (51) into equation (50) gives

\[
A_2 = -\frac{16}{7!} \left(h_2 + \frac{h_s}{q}\right) \left(q_s^2 \left(1 + \frac{64q_s^2}{70}\right) - \frac{6}{q_s^3} \left(1 + q_s^2/6 \right) \left(h_2 + \frac{h_s}{q}\right) \right).
\]

In the limit \(q_s \to 0\), \(A_2\) reduces to

\[
A_2 \approx 1 - \frac{43}{7!} q_s^2
\]

and \(A_4 \approx 1\), to the accuracy of our calculation.

V. DISCUSSION

This paper develops a formalism for treating general multipole electromagnetic fields of neutron stars. We give the electric multipoles induced in a star by its rotation with an arbitrary magnetic multipole at its center. Then we show how to express a family of off-centered multipoles, having the same \(l\) weight, as an infinite array of centered multipoles of increasing \(l\) weight referred to the rotational axis. We then give general expressions for the linear momentum present in the superposition of arbitrary multipole fields. All the preceding results are then combined to compute the radiation rate of linear momentum by an off-centered dipole to zeroth order in \(\Omega R/c\). In § IV, not closely connected with §§ II and III, we rederive in a clear, consistent manner the general Deutsch solution to the vacuum electromagnetic field around a rotating magnetized star, and provide some minor additions and corrections.

Carl Caves and Peter Goldreich made useful remarks on the Fourier decomposition of source densities. Saul Teukolsky did the integrals (14) and (15) in a splendidly elegant manner. Gene Tademaru kept my interest in the off-centered dipole problem alive after some initial mistaken groping on my part. Kip Thorne and Peter Goldreich arranged a Visiting Associateship at Caltech, where this work was largely completed.
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