1. Introduction

It is now generally agreed (see the recent reviews by Stenflo, 1976a, b) that the surface magnetic field of the Sun is made up of intense (1–2 kG), isolated flux tubes situated in the supergranular boundaries (Livingston and Harvey, 1969; Stenflo, 1973; Chapman, 1974). Over 90% of the total photospheric magnetic flux is found in this concentrated form (Howard and Stenflo, 1972; Frazier and Stenflo, 1972). Typically, the flux tubes are believed to have diameters at photospheric levels of the order of one or two hundred km. Below photospheric levels, theoretical considerations indicate that the tube narrows (Parker, 1976; Roberts, 1976b); the magnetic pressure, though increasing with depth, does so at a slower rate than the confining gas pressure. So, with increasing depth, the gas pressure in the tube (assumed to be in static equilibrium) tends to the external pressure.
The question of how such intense magnetic fields, compressed to magnetic pressures comparable with the gas pressure in the photosphere, are assembled has been considered by Parker (1976), who concluded that only cooling of the gas within the flux tube could produce the necessary high magnetic field intensities. The cooling, it was proposed, is achieved by the generation and subsequent emission of overstable Alfvén waves, in much the same way as Parker (1974a) suggested that sunspots are cooled. In support of this conjecture Roberts (1976a), in a detailed calculation that took account of the effect of the lateral flux tube boundaries on wave propagation, showed that cooling is indeed important in a wide range of magnetic structures, ranging from the sunspot down to the slender tube. However, it should be noted that the hypothesis of cooling in sunspots by overstable Alfvén waves has recently been criticised by Cowling (1976), chiefly on the grounds that such a mechanism is not efficient enough.

Whatever mechanism we eventually consider to be responsible for producing intense photospheric fields the fact remains that such fields do exist. So, in this paper, we consider the structure of small amplitude motions in an intense, slender flux tube regardless of the way in which it is formed. The atmosphere (both within and outside the tube) is taken to be non-isothermal, so that the characteristic functions describing the motions within the tube (namely the sound speed, the Alfvén speed, and the density scale-height) are height dependent.

The nature of wave propagation in such a stratified unbounded atmosphere has been discussed by Lighthill (1967), who considered only two of the three governing forces at any one time. In many solar situations the interaction of all three forces is important, and this is certainly the case in an intense magnetic flux tube (where the sound and Alfvén speeds at photospheric levels are about 10 km s\(^{-1}\)). The interaction of these forces poses a complex problem for the theoretician, and as a result various approximations (such as taking a uniform magnetic field and/or an isothermal atmosphere) have naturally been made (see, for example, Ferraro and Plumpton, 1958; McLellan and Winterberg, 1968). Consequently, such calculations are only of limited value in discussing wave propagation in flux tubes.

In addition to the interaction of the three governing forces it is necessary to consider the influence of the lateral boundaries. Assuming a uniform basic-state, the case of an incompressible fluid has been considered by Parker (1974b), a compressible fluid by Cram and Wilson (1975), and a Boussinesq fluid by Roberts (1976a). Recently, Defouw (1976) has discussed waves in flux tubes, assuming an isothermal atmosphere so that both the sound and Alfvén speeds are constant. His analysis is, in fact, a special case of our own, and will be discussed in Section 5.

At first sight it would appear that the inclusion of lateral boundaries, in addition to the three governing forces, would make an already complicated problem hopelessly intractible. However, for a slender tube this is not the case. In fact, it turns out that for an intense, slender tube such approximations as assuming a constant field or temperature (to make the problem tractible) are unnecessary; a more general treatment of the problem is possible, as we shall show in detail in this paper.
Instead of making such assumptions, we consider an \textit{expansion} of the complete equations noting that the field is intense and confined to a slender tube (for which the lateral dimensions are very much less than variations on a vertical scale). As a result, we derive a set of equations governing the \textit{zeroth-order state}, this essentially being a description of the behaviour on the line of axial symmetry. The assumption that the tube is \textit{slender} implies that the radius \( r_0(z) \) of the tube at the height \( z \) is very much less than variations on a vertical scale \( L_0 \).

The value of \( L_0(z) \) depends upon the precise conditions pertaining at the 'upper' and 'lower' ends of the flux tube, which we do not discuss. A lower estimate of \( L_0 \) is probably provided by taking \( L_0 = \Lambda_0(z) \), the temperature scale-height, but a value of \( L_0 = 4-10\Lambda_0(z) \), may be more reasonable. As we have noted, observations indicate that \( r_0 \approx 100 \) km (Stenflo, 1976b, gives 75 km) at photospheric levels and this is broadly comparable to \( \Lambda_0 \) at the observed level. As we descend into the tube \( r_0 \) falls whereas \( \Lambda_0 \) rises, and so the condition \( r_0 \ll \Lambda_0 \) is more readily met.

It is shown that the equation governing a linear perturbation of amplitude \( \hat{q}(z) \) and frequency \( \omega \) is of the general Helmholtz form (see Equations (30) and (33))

\[
\frac{d^2 \hat{q}}{dz^2} + f(z; \omega^2) \hat{q} = 0,
\]

for a cylindrical coordinate system (with the \( z \)-axis pointing upwards). This wave-like equation is derived under the assumption that the basic state of the flux tube is one of hydrostatic equilibrium, the magnetic pressure simply being balanced by the difference between the external (confining) pressure and the gas pressure within the tube (Parker, 1955).

The advantages of our expansion (in \( r_0/L_0 \)) procedure are clear: it allows a \textit{systematic} derivation of the governing equation, and this equation turns out to be amenable for the most part to an analytical investigation. Thus, we are able to retain magnetic fields, compressibility, and both density and thermal stratification, without the problem becoming virtually intractible, as would be the case, for example, with a wide structure not satisfying the conditions \( r_0/L_0 \ll 1 \). Such assumptions as an isothermal atmosphere or a Boussinesq fluid are thus seen to be unnecessary. Such a simple, systematic procedure does not appear to have been given before in this context, though the equations thus derived are in agreement with previous ones set up intuitively.

The validity of the expansion procedure is investigated (in Section 6) by considering the problem of waves in a uniform column of field embedded in a uniform background. Gravity effects are taken to be negligible. Such a problem is amenable to a complete analysis, taking into account lateral boundary conditions, without recourse to the expansion procedure. The two possible approaches are compared and it is found that good agreement exists.

Finally, we discuss the general structure of the expansion equations (in Section 7) and apply (in Section 8) the theory to the intense magnetic flux tubes of the solar photosphere.
2. The Governing Equations

The dynamical equations governing isentropic motions of a perfectly conducting, inviscid, ideal gas embedded in a magnetic field are taken to be:

\[ \frac{\partial \rho}{\partial t} + \text{div} \rho \mathbf{v} = 0, \quad (1) \]

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \rho \mathbf{g} + j \times \mathbf{B}, \quad (2) \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \text{curl} (\mathbf{v} \times \mathbf{B}), \quad (3) \]

\[ \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = \frac{\gamma p}{\rho} \left( \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \right), \quad (4) \]

\[ p = \frac{k}{m} \rho T, \quad (5) \]

\[ \text{div} \mathbf{B} = 0, \quad \mu j = \text{curl} \mathbf{B}. \quad (6) \]

In the above equations, \( p, \rho, T \) and \( \mathbf{v} \) are the pressure, density, temperature and velocity of the gas within the flux tube; \( \gamma \) is the (assumed constant) ratio of specific heats, \( k \) is Boltzmann's constant, \( m \) is the mean particle mass; \( \mathbf{B} \) is the magnetic (induction) field, \( j \) the current density, \( \mu \) the permeability and \( g \) the gravitational acceleration.

To discuss axisymmetric gas motions in an intense, vertical, untwisted flux tube we consider Equations (1)–(6) in a cylindrical coordinate system in which there is no azimuthal-dependence, so variables are functions of \( r, z \) and \( t \). We expand each of the variables in (1)–(6) in its Maclaurin series about \( r = 0 \), and retain only the zeroth-order \((r = 0)\) terms. Thus, we are assuming that the lateral scale for variations is very much less than the vertical scale. In terms of a scale \( L_0 \) we are thus assuming that, at each level \( z, r \ll L_0 \) over the cross-section of the tube. To describe small terms in our expansion it is convenient to introduce a small order parameter \( \varepsilon \), taken (for example) to be \( r_0(\hat{z})/L_0(\hat{z}) \) at a suitable depth \( \hat{z} \) where the condition \( \varepsilon \ll 1 \) is satisfied.

Thus, to zeroth-order, we consider a magnetic tube within which the field and flow are essentially axial:

\[ \mathbf{B} = (0, 0, B_z(z, t)) + O(\varepsilon), \quad \mathbf{v} = (0, 0, v_z(z, t)) + O(\varepsilon), \quad (7) \]

where \( B_z(z, t) = B_z(r = 0, z, t), v_z(z, t) = v_z(r = 0, z, t) \) are the vertical components of the magnetic and velocity fields, as measured on the axis of symmetry (the \( z \)-axis).
of the tube. With this procedure, Equations (1)–(4) give the zeroth-order, non-linear equations

\[
\frac{\partial p}{\partial t} + \rho \Delta + v \frac{\partial p}{\partial z} = 0 ,
\]

\[
\rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} \right) = - \frac{\partial p}{\partial z} - \rho g ,
\]

\[
\frac{\partial B}{\partial t} = - \Delta B + B \frac{\partial v}{\partial z} - v \frac{\partial B}{\partial z} ,
\]

\[
\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial z} = c^2 \left( \frac{\partial p}{\partial t} + v \frac{\partial p}{\partial z} \right) ,
\]

where \( c^2 = \gamma p/\rho \) and \( \Delta = (\text{div} \mathbf{v})_{r=0} \). In the above we have used the notation \( p = p(r = 0, z, t) \), etc. The details of the above expansion are given in Appendix A. The ideal gas equation determines the (zeroth-order) temperature \( T \):

\[
T = \left( \frac{\hat{m}}{k} \right) \frac{p}{\rho} .
\]

Equation (9) comes from the vertical component of the momentum equation. The radial component of the momentum equation shows that, to zeroth-order, the total (hydrodynamic + magnetic) pressure is constant across the tube. The radial component of the induction equation simply determines the radial field once the vertical field and velocity have been found (see Appendix A). It should be noted that the solenoidal condition (\( \text{div} \mathbf{B} = 0 \)) on the magnetic field in no way restricts the function \( B(z, t) \) occurring in (7). In fact, the solenoidal condition simply determines the radial component of the magnetic field, once \( B(z, t) \) is known. Finally, it may be noted that \( \Delta \) may be eliminated between (8) and (10) to give

\[
\frac{\partial}{\partial t} \frac{\rho}{B} + \frac{\partial}{\partial z} \frac{\rho v}{B} = 0 ,
\]

which is simply a re-statement of flux conservation.

### 3. Equilibrium of an Intense Flux Tube

Consider the equilibrium configuration \( B_0(z) \), \( p_0(z) \) and \( \rho_0(z) \) resulting from the above equations with \( v = 0 \), \( \partial / \partial t = 0 \). (Equilibrium variables inside the tube will be denoted by a subscript 0, and those outside by an e.) The flux tube is shown in Figure 1. Equation (9) gives

\[
\frac{dp_0}{dz} = - \rho_0 g ,
\]
which, when combined with the gas Equation (12), yields

\[ p_0(z) = p_0(0) \exp \left[ - \int_0^z \frac{dz'}{A_0(z')} \right]. \tag{15} \]

Here \( A_0(z) = k T_0(z)/\dot{m}g \) is the scale-height of the atmosphere within the flux tube, and \( z = 0 \) is an arbitrary reference level. We choose \( z = 0 \) to correspond to optical depth \( \tau_{5000} = 1 \) in the solar photosphere.

The external atmosphere is also in barometric equilibrium, so

\[ p_e(z) = p_e(0) \exp \left[ - \int_0^z \frac{dz'}{A_e(z')} \right], \tag{16} \]

where \( A_e(z) = k T_e(z)/\dot{m}g \) is the scale-height of the external atmosphere. To relate the interior of the tube to its environment we note (see Appendix A) that the total (hydrostatic + magnetic) pressure is, to zeroth order, constant across the tube, and so to balance across the sides of the slender tube, we must have

\[ p_e(z) = p_0(z) + \frac{1}{2\mu} B_0^2(z), \tag{17} \]

where \( B_0(z) \hat{z} \) is the zeroth-order magnetic field of the tube. It follows that

\[ \frac{d}{dz} \left( \frac{B_0^2(z)}{2\mu} \right) = g(\rho_0 - \rho_e), \tag{18} \]

a result first given by Parker (1955). If, further, the tube is in temperature balance...
with its surroundings, so that $T_0(z) = T_e(z)$, then (17) and (18) may be combined to give
\[
B_0^2(z) = B_0^2(0) \frac{p_e(z)}{p_0(z)} = B_0^2(0) \frac{p_0(z)}{\rho_0(0)},
\]
and so $B_0(z) \sim p_e^{1/2}(z)$. From conservation of magnetic flux it then follows that the radius $r_0(z)$ of the tube varies as $p_e^{-1/4}(z)$, and $r_0(z) = \frac{1}{\Lambda_0(z)} r_0(0)$. (A dash is used throughout to denote differentiation with respect to $z$.)

In order to illustrate as clearly as possible the structure of motions in an intense flux tube, and for the sake of simplicity, we will generally assume that $T_0(z) = T_e(z)$. (The equilibrium configuration for $T_0(z) \neq T_e(z)$ has been considered, in a slightly different context, in Roberts (1976a,b).) There is at present no conclusive observational evidence of either a temperature balance or imbalance. On theoretical grounds, it has been suggested elsewhere (see Parker, 1976; Roberts, 1976a) that a slight temperature difference may exist.

To characterise our problem it is convenient to introduce the sound speed $c_0(z)$ and the Alfvén speed $v_\Lambda(z)$, as given by
\[
c_0^2(z) = \frac{\gamma \rho_0(z)}{\rho_0(z)}, \quad v_\Lambda^2(z) = \frac{B_0^2(z)}{\mu \rho_0(z)}.
\]

In addition, the density scale-height $H_0(z)$ and the Brunt-Väisälä (buoyancy) frequency $N_0^2(z)$ naturally arise (though only $c_0^2$, $v_\Lambda^2$ and $N_0^2$ are required to characterise our problem):
\[
H_0^{-1}(z) = \frac{\rho_0(z)}{\rho_0(z)}, \quad N_0^2(z) = \frac{g}{H_0(z)} - \frac{g}{c_0^2(z)}.
\]

$N_0^2(z)$ may be positive or negative. (Under the sole influence of gravity, internal (or gravity) waves only exist if the stratification is stable, i.e. if $N_0^2 > 0$.)

It should be noted that for a flux tube in temperature balance with its surroundings, so that (19) holds, the sound speed and the Alfvén speed are related by
\[
\frac{c_0^2(z)}{v_\Lambda^2(z)} = \frac{c_0^2(0)}{v_\Lambda^2(0)},
\]
and so $c_0 \sim v_\Lambda$ at every level $z$.

It follows from the above description that the equilibrium state of the tube is completely characterised by the scale height $\Lambda_0(z)$. Thus, we have
\[
p_0(z) = p_0(0) e^{-n(z)}, \quad B_0(z) = B_0(0) e^{-1n(z)},
\]
\[
\rho_0(z) = \rho_0(0) \frac{\Lambda_0(0) e^{-n(z)}}{\Lambda_0(z)},
\]
where
\[
n(z) = \int_0^z \frac{dz'}{\Lambda_0(z')}.
\]
while the parameters are

\[ c_0^2(z) = \gamma g \Lambda_0(z), \quad v_\Lambda^2(z) = v_\Lambda(0) \frac{\Lambda_0(z)}{\Lambda_0(0)}, \]

\[ H_0^{-1}(z) = \frac{1 + \Lambda_0'(z)}{\Lambda_0(z)}, \quad N_0^2(z) = \frac{g}{\Lambda_0(z)} \left( \frac{(\gamma - 1)}{\gamma} + \Lambda_0(z) \right). \]  

(24)

Thus, in our treatment we regard \( \Lambda_0(z) (= \Lambda_\varepsilon(z)) \) as a prescribed function of \( z \) (though in the Sun, of course, \( \Lambda_\varepsilon(z) \) is determined by the overall energetics) that we may choose from a model atmosphere.

Finally, we note that two choices of profile \( \Lambda_0(z) \) that may be readily analysed are the exponential and linear forms, though it should be noted that the exponential profile \( (T_0(z) = T_0(0) e^{-2\nu z}, \nu > 0, z \ll 0) \) leads to unrealistic results in that \( \rho_0(z) \) (and \( \rho_\varepsilon(z) \)) tends to zero as \( z \to -\infty \), and this is independent of our flux tube model. The linear profile

\[ \Lambda_0(z) = \Lambda_0(0) \left( 1 - \frac{z}{\lambda \Lambda_0(0)} \right), \quad \lambda > 0, \]

gives the polytrope

\[ \rho_0(z) = \rho_0(0) \left( \frac{\Lambda_0(z)}{\Lambda_0(0)} \right)^\lambda, \quad \rho_0(z) = \rho_0(0) \left( \frac{\Lambda_0(z)}{\Lambda_0(0)} \right)^{\lambda - 1}. \]

So, for the density to monotonically increase with depth we require \( \lambda > 1 \).

4. Small Motions in an Intense Flux Tube

In order to discuss motions in an intense flux tube we linearise Equations (9), (11) and (13) about the equilibrium state described by (12) and (14). Thus the basic state is simply taken to be a gas in equilibrium under gravity, so Equation (14) holds. The resulting linear equations are found to be:

\[ \rho_0(z) \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} - \rho g, \]  

(25)

\[ \frac{\partial p}{\partial t} + \rho_0(z) v = c_0^2(z) \left( \frac{\partial p}{\partial t} + \rho_0(z) v \right), \]  

(26)

\[ B_0(z) \frac{\partial \rho}{\partial t} - \rho_0(z) \frac{\partial B}{\partial t} + (B_0(z) \rho_0(z) - B_0(z) \rho_0(z) v + \rho_0(z) B_0(z) \frac{\partial v}{\partial z} = 0. \]  

(27)

Note that, in contrast to the notation in Equations (1)–(13), here \( v, p, \rho \) and \( B \) refer to the perturbations.

For the above linearisation to be valid within the zeroth-order state is is necessary to assume that the characteristic measure of amplitude of the perturbations is very much less than \( r_0^2 / L_0 \ll r_0 \).
We relate the flux tube to its surroundings by assuming that the pressure perturbation in the exterior is negligible. Then, from the radial component of the momentum equation, for the flux tube surrounded by a quiescent fluid, we find that

\[ p + \frac{1}{\mu} B_0(z) B = 0. \] (28)

In fact, the perturbations in the exterior* may decline exponentially as we move away from the tube boundary (see results in Roberts (1976a)).

Equations (25)–(28) may now be combined to give, on assuming a time-dependence of the form \( e^{iot} \), with \( \omega^2 \) real, the second-order, ordinary differential equation

\[(\omega^2 - N_0^2(z))\ddot{p}'' + \alpha_0(z; \omega^2)\dot{p}' + \beta_0(z; \omega^2)\dot{p} = 0 \] (29)

for the (perturbation) pressure amplitude \( \dot{p}(z) \), defined by \( p = \dot{p}(z) e^{i\omega t} \). The coefficients \( \alpha_0, \beta_0 \) in (29) are given by

\[ \alpha_0(z; \omega^2) = (\omega^2 - N_0^2) \left( \frac{2c_0^2}{c_0^2} \frac{B_0'}{B_0} + \frac{1}{\Lambda_0} \right) + 2N_0N_0', \]

\[ \beta_0(z; \omega^2) = (\omega^2 - N_0^2) \left( \frac{1}{c_0^2} \frac{1}{\alpha^2} + \frac{g^2}{c_0^2 \alpha^2} \right) + 2\frac{gN_0N_0'}{c_0^2}. \]

Note that Equation (29) has a regular singularity at \( z = z_c \), where \( \omega^2 = N_0^2(z_c) \), provided that \( N_0(z_c)N_0'(z_c) \neq 0 \).

Introducing the perturbation quantity \( \dot{q}(z) \), defined by

\[ \dot{p}(z) = f_0^{1/2} \dot{q}(z), \]

with \( f_0(z) = \rho_0 B_0 |\omega^2 - N_0^2| \) assumed non-zero, allows us to write Equation (29) in the canonical form

\[ \dot{q}'' + f(z; \omega^2)\dot{q} = 0, \] (30)

where

\[ f(z; \omega^2) = \frac{\beta_0(z; \omega^2)}{(\omega^2 - N_0^2)} + \frac{f_0}{2f_0} - \frac{3}{4} \left( \frac{f_0}{f_0} \right)^2. \]

The solutions of Equation (30) only have a wave-like character for \( f(z; \omega^2) > 0 \). If \( f(z; \omega^2) < 0 \) the solutions of (30) are of exponential character (corresponding to motions exponentially damped out with depth). The equation \( f(z; \omega^2) = 0 \) may therefore be considered as defining a critical value of \( \omega^2 \), viz. \( \omega_p^2 \), giving the 'transition' frequency between vertically propagating pressure waves and vertically evanescent pressure modes. The critical frequency \( \omega_p \) is clearly a function of the

* The effect of a forced pressure variation on the tube boundary is considered in Appendix B.
various parameters (such as $\Lambda_0$, $c_0^2/v_0^2$, $c_0^2$, $v_\Lambda^2$) describing the structure of the flux tube, and so is also a function of height $z$. The nature of $\omega_\rho$ (and the corresponding transition frequency for the velocity field) as a function of $z$ will be considered in detail in the remainder of this paper. Of course, we may consider $f(z; \omega^2) = 0$ as defining a ‘transition’ height $z_{\text{trans}}$, for a given frequency $\omega$ – the functions $z_{\text{trans}}(\omega)$ and $\omega_\rho(z)$ are simply inverses of each other – but for comparison with observations the form $\omega_\rho(z)$ is more convenient.

Equations (29) and (30) apply for an arbitrary atmosphere, but with the present assumption of temperature balance between the flux tube and its surroundings, so that the structure of the tube is given by (23) and (24), we find that (29) reduces to

\[
(\omega^2 - N_0^2)\hat{p}'' + \left\{ (\omega^2 - N_0^2)\frac{3 + 2\Lambda_0}{2\Lambda_0} + 2N_0N_0' \right\}\hat{p}' +
\]
\[
+ \left\{ (\omega^2 - N_0^2)\frac{1}{\gamma\Lambda_0'(z)} \right\} \hat{p} = 0 ,
\]

with $H_0(z)$ and $N_0(z)$ given by (23), and $c_T^2(z) = c_0^2v_\Lambda^2/(c_0^2 + v_\Lambda^2)$. ($c_T(z)$ plays the role of a characteristic speed of propagation in the tube.)

Equations (25)–(28) may also be combined to give an equation for the velocity amplitude $\hat{v}(z)$, defined by $v = \hat{v}(z)e^{kz}$. We find that

\[
\hat{v}'' - \frac{1}{2\Lambda_0} \hat{v}' + \left( \frac{(\omega^2 - N_0^2)}{c_T^2} + (1 - \frac{1}{2}\gamma) \frac{N_0^2}{c_0^2} \right) \hat{v} = 0 ,
\]

where in deriving (31) we have used the fact that in the tube $c_0^2/v_\Lambda^2$ is a constant. The velocity and pressure amplitudes are related by

\[
\hat{v} = \frac{i\omega}{\rho_0 c_0^2 (\omega^2 - N_0^2)} (\hat{p}' + \gamma\hat{p}) ,
\]

(32a)

\[
\hat{p} = \frac{i\omega}{\rho_0 c_0^2 (\omega^2 - N_0^2)} \{(1 - \frac{1}{2}\gamma)\rho_0 g\hat{v} - \rho_0 c_0^2 \hat{v}'\} ,
\]

(32b)

for $\omega^2 \neq N_0^2$ (So $v$ is 90° out of phase with $p$). It may be noted in (31) that the equation for $\hat{v}$ does not, unlike that for $\hat{p}$, have a singularity at $\omega^2 = N_0^2$. Equation (31) is also algebraically simpler to investigate than (30).

We may put (31) into canonical form by writing

\[
\hat{Q} = \hat{v} e^{-\frac{1}{2}n(z)} ,
\]

and then $\hat{Q}(z)$ satisfies

\[
\hat{Q}'' + \left[ \frac{\omega^2 - N_0^2}{c_T^2} + (1 - \frac{1}{2}\gamma) \frac{N_0^2}{c_0^2} - \frac{1}{16\Lambda_0^2} - \frac{\Lambda_0'}{4\Lambda_0^2} \right] \hat{Q} = 0 .
\]

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Thus, oscillatory solutions (in $\hat{Q}$) occur for $\omega^2 > \omega_v^2$, where

$$\omega_v^2 = N_0^2 + \frac{c_T^2}{\Lambda_0^2} \left( \frac{3}{4} - \frac{1}{\gamma} \right) \left( \frac{3}{4} - \frac{1}{\gamma} + \Lambda_0' \right),$$

that is,

$$\omega_v^2 = \frac{c_T^2(z)}{\Lambda_0(z)} \left( \frac{3}{4} \Lambda_0'(z) + \frac{9}{16} - \frac{1}{2\gamma} + \frac{1}{\gamma} \left( \frac{\gamma-1}{\gamma} + \Lambda_0'(z) \right) \frac{c_T^2}{v_A^2} \right).$$

Equation (34) provides the generalization of Defouw's (1976) result, obtained for an isothermal atmosphere (for which $c_T$ and $\Lambda_0$ are constants), to the case of an arbitrary atmosphere. Note that if in (34) the right-hand side of the equation is negative, then oscillatory solutions occur for all $\omega^2 > 0$, and so we effectively have $\omega_v^2 = 0$.

We may rewrite Equation (33) in the form

$$\hat{Q}'' + \frac{1}{c_T^2(z)} (\omega^2 - \omega_v^2(z)) \hat{Q} = 0.\quad (33)'$$

We may estimate the group velocity $c_g^2$ of a propagating wave (with $\omega^2 > \omega_v^2$) and its phase speed $c_{ph}$ if we regard the coefficients of (33)' as essentially constant. (This is the so-called local analysis, valid if the functions of $z$ in question vary only slightly over a wavelength, and gives a local dispersion relation.) Thus, the local wavenumber $k$, for $\hat{Q} \sim e^{ikz}$, is given by

$$k^2 c_T^2 = \omega^2 - \omega_v^2,$$

which is analogous to the dispersion relation for an electromagnetic wave in a plasma (with $\omega_v > 0$ playing the role of the plasma frequency). So

$$c_g = \frac{\partial \omega}{\partial k} = \frac{c_T^2}{c_{ph}} \left( 1 - \frac{\omega_v^2}{\omega^2} \right)^{1/2} c_T,$$

where

$$c_{ph} = \frac{\omega}{k} = \left( \frac{\omega^2}{\omega^2 - \omega_v^2} \right)^{1/2} c_T.$$

Thus, $c_g < c_T < \min (c_0, v_A)$. As a numerical illustration, with $c_0 = v_A$ (typical for an intense flux tube in the photosphere), we see that $c_g < (1/\sqrt{2})c_0 = 5.7$ km s$^{-1}$. More precise estimates of $c_g$ (and $c_{ph}$) may be made once $\omega$ is known (perhaps from observations) at a definite height.

Before discussing (in Section 6) the general structure of the above equations, we consider in the next section several special cases which may help to clarify the nature of the tube equations.
5. Special Cases

In this section we discuss several special cases that are commonly treated in the literature. These cases have the virtue that they are amenable to an analytical treatment which, for the most part, is devoid of the complexity of the general case. Thus we are able to illustrate several features of the general equation in a fairly concise manner. However, it should be borne in mind that special cases can be misleading (as we point out in more detail in Section 8) and so should only be used with due caution as a guide to the general case. Also, only one of the special cases treated has a singular point, where $\omega^2 = N_0'(z)$, in the governing pressure equation, and so are only partly representative of the complete problem. Nevertheless a study of the following special cases is considered to be of value. For convenience, we will assume in this section that the mean particle mass $\bar{m}$ is a constant (so, for example, a constant temperature means a constant scale height $\Lambda_0$).

5.1. Isothermal Atmosphere

Consider the special case of a flux tube with isothermal atmosphere. Then $\Lambda_0$, $c_0$ and $H_0$ are constants, and (from (22)) so also is $v_\Lambda$, whereas the magnetic field declines exponentially ($B_0(z) = B_0(0) \exp (-z/2\Lambda_0)$) with height. Equation (29) reduces to

$$\ddot{\theta} + \frac{3}{2\Lambda_0} \dot{\theta} + \left[ \left( \omega^2 - \frac{g}{H_0} \right) \left( \frac{1}{c_0^2} + \frac{1}{v_\Lambda^2} \right) + \frac{g^2}{c_0^2 v_\Lambda^2} \frac{3g}{2\Lambda_0 c_0^2} \right] \dot{\theta} = 0, \quad \text{(35)}$$

giving an equation with constant coefficients. Since $N_0' = 0$ no singularities occur in the isothermal case. This case has recently been considered by Defouw (1976). Thus, solving (35) we find that the pressure $p(z, t)$ is given by

$$p \sim e^{-3z/4\Lambda_0} e^{i(\omega t \pm k z)}, \quad \text{(36)}$$

where

$$k^2 = \omega^2 \left( \frac{c_0^2 + v_\Lambda^2}{c_0^2 v_\Lambda^2} \right) - \frac{1}{\Lambda_0^2} \left[ \frac{9}{16} - \frac{1}{2\gamma} + \left( \frac{\gamma - 1}{\gamma^2} \right) \frac{c_0^2}{v_\Lambda^2} \right]. \quad \text{(37)}$$

The velocity perturbation is readily found from (36) and (32b), or directly from (31), with the result that

$$v \sim e^{-3z/4\Lambda_0} e^{i(\omega t \pm k z)}. \quad \text{(38)}$$

Wave propagation (i.e., $k^2 > 0$) occurs only for frequencies greater than a critical value, given by setting $k = 0$ in Equation (37). So for an isothermal atmosphere the transition frequencies $\omega_p$ and $\omega_v$, given by (37) with $k = 0$, are identical:

$$\omega_p^2 = \omega_v^2 = \frac{c_T^2}{\Lambda_0^2} \left[ \frac{9}{16} - \frac{1}{2\gamma} + \left( \frac{\gamma - 1}{\gamma^2} \right) \frac{c_0^2}{v_\Lambda^2} \right], \quad \text{(39)}$$

where $c_T^2 = c_0^2 v_\Lambda^2 / (c_0^2 + v_\Lambda^2)$. Equation (39) has been obtained by Defouw (1976). Its generalization to a non-isothermal atmosphere was given earlier (Equation (34)).
If, further, we suppose that gravity is absent, then
\[ v, p \sim e^{i(\omega t \pm (\omega/c_T)z)}. \]

In the limit \( v_A \to \infty \) we find \( c_T = c_0 \), and the above gives ordinary sound waves. Thus, wave propagation is at a phase speed \( c_T < \min(c_0, v_A) \), and is therefore in general both subsonic and sub-Alfvénic. This may be contrasted with the solution of the equations for a uniform medium (of infinite extent) in the absence of gravity, as given (for example) by Lighthill (1960). There one finds that the wave modes separate out into an Alfvén mode and the fast and slow magnetoacoustic modes. For propagation in the direction of the applied magnetic field,\(^*\) the magnetoacoustic modes give phase speeds \( c_0 \) and \( v_A \). Thus, the effect of confining the region of magnetic field to an ideally slender tube is to reduce the speed of propagation of the wave to a value below either of the possible magnetoacoustic speeds \( c_0, v_A \) in an infinite, uniform medium. We consider this matter further in Section 6.

5.2. Absence of Gravity

Other special cases of Equation (29) are of interest. For example, in the absence of gravity, with the temperature within the tube an arbitrary function of \( z \), Equation (29) reduces to
\[ \hat{\beta}'' + \frac{T_0'(z)}{T_0(z)} \hat{\beta}' + \frac{\omega^2}{c_T^2(z)} \hat{\beta} = 0, \tag{40} \]
where \( c_T^2(z) = c_T^2(0) T_0(z)/T_0(0) \). If we suppose that the temperature profile is linear, so that
\[ T_0(z) = T_0(0) \left( 1 - \frac{z}{\lambda} \right), \quad \lambda > 0, \]
then Equation (40) becomes
\[ \left( 1 - \frac{z}{\lambda} \right) \hat{\beta}'' - \frac{1}{\lambda} \hat{\beta}' + \frac{\omega^2}{c_T^2(0)} \hat{\beta} = 0, \]
with solutions
\[ \hat{\beta} \sim J_0 \left( \frac{2 \omega \lambda}{c_T(0)} \left( 1 - \frac{z}{\lambda} \right)^{1/2} \right), \quad Y_0 \left( \frac{2 \omega \lambda}{c_T(0)} \left( 1 - \frac{z}{\lambda} \right)^{1/2} \right) \]
in terms of Bessel functions \( J_0, Y_0 \) of the first and second kind. Bessel functions also occur in the case of an exponential atmosphere: for \( T_0(z) = T_0(0) e^{-2\nu z}, \nu > 0 \), we

\(^*\) For propagation at an angle \( \theta \) to the applied (uniform) magnetic field, the phase speed of the slow magnetoacoustic mode is \( \sqrt{\frac{1}{2} (c_0^2 + v_A^2) - \frac{1}{4} (c_0^2 + v_A^2)^2 - c_0^2 v_A^2 \cos^2 \theta} \) \( 1/2 \), which for \( \theta \approx \pi/2 \) reduces to \( c_T \cos \theta \ll c_T \). In fact, \( c_T \) is the speed at which the cusp tail of the slow mode propagates along the applied field.
find solutions

\[ \hat{\rho} \sim e^{\nu z} J_1 \left( \frac{\omega}{\nu c_T(0)} e^{\nu z} \right), \quad e^{\nu z} Y_1 \left( \frac{\omega}{\nu c_T(0)} e^{\nu z} \right). \]

It is of interest to determine the range of \( \omega \) over which oscillatory solutions may occur. Equation (40) gives

\[ \hat{q}'' + \left\{ \frac{\omega^2}{c_T^2} - \left( \frac{T_0'}{2T_0} \right)' - \left( \frac{T_0'}{2T_0} \right)^2 \right\} \hat{q} = 0, \quad (40)' \]

where \( \hat{q} = T_0^{1/2}(z) \hat{\rho} \). Thus, oscillatory solutions in \( \hat{q} \) are possible for

\[ \omega^2 \geq \left\{ \left( \frac{T_0'}{2T_0} \right)' + \left( \frac{T_0'}{2T_0} \right)^2 \right\} \frac{1}{c_T^2}. \]

For the linear profile this reduces to

\[ \omega^2 \geq -\frac{1}{4} c_T^2(z) \left( \frac{T_0'(0)}{T_0(z)} \right)^2, \]

which is satisfied for all \( z \) (and \( \omega^2 > 0 \)). Thus, oscillatory-type solutions for \( \hat{q} \) exist at all depths for the linear profile, and so \( \omega_\rho^2 = 0 \). On the other hand, for the exponential profile Equation (40)' reduces to

\[ \hat{q}'' + \left( \frac{\omega^2}{c_T(z)^2} - \nu^2 \right) \hat{q} = 0, \]

which has the oscillatory-type solutions for \( \omega^2 > \omega_\rho^2(z) \equiv \nu^2 c_T(z) \). So \( \omega_\rho^2 \) is here a monotonically increasing function of depth.

Turning now to the velocity equation we see that in the absence of gravity Equation (31) reduces to

\[ \hat{v}'' + \frac{\omega^2}{c_T(z)^2} \hat{v} = 0. \quad (41) \]

Thus, whatever the temperature profile, the velocity is always oscillatory in form, i.e., \( \omega_v^2 = 0 \). We may solve (41) for the special cases of a linear profile and an exponential profile. For the linear profile

\[ \hat{v} \sim \left( 1 - \frac{z}{\lambda} \right)^{1/2} \right. J_1 \left( \frac{2\omega\lambda}{c_T(0)} \left( 1 - \frac{z}{\lambda} \right)^{1/2} \right), \]

\[ \left( 1 - \frac{z}{\lambda} \right)^{1/2} Y_1 \left( \frac{2\omega\lambda}{c_T(0)} \left( 1 - \frac{z}{\lambda} \right)^{1/2} \right), \]

while for the exponential profile

\[ \hat{v} \sim J_0 \left( \frac{\omega e^{\nu z}}{\nu c_T(0)} \right), \quad Y_0 \left( \frac{\omega e^{\nu z}}{\nu c_T(0)} \right). \]
5.3. Constant density

For incompressible motions in a medium of uniform density \( \rho_0 \), the governing Equation (29) for the pressure becomes

\[
\hat{\rho}'' - \frac{v_A'(z)}{v_A(z)} \hat{\rho}' + \frac{\omega^2}{v_A(z)} \hat{\rho} = 0.
\]  

(42)

With an exponential Alfvén speed, \( v_A(z) = v_A(0) e^{-\lambda z} \), \( \lambda > 0 \), we find that (42) has solutions \( \hat{\rho} \sim e^{-\lambda z} \frac{\sin(\omega z/\lambda v_A(0))}{\cos(\omega z/\lambda v_A(0))} \).

As before, we may determine \( \omega_p \) by writing (42) in canonical form:

\[
\hat{q}'' + \left( \frac{\omega^2}{v_A} + \frac{v_A'^2}{2v_A} - 3 \left( \frac{v_A'}{2v_A} \right)^2 \right) \hat{q} = 0,
\]

where \( \hat{q} = \hat{\rho}/v_A^{1/2} \). Thus, for the exponential atmosphere we find

\[
\omega_p = \frac{3}{2} \lambda v_A(z) = \frac{3}{2} \lambda v_A(0) e^{-\lambda z},
\]

and so \( \omega_p \) is again a monotonically increasing function of depth.

The velocity amplitude \( \delta \) is readily found on noting that (32b) reduces to \( \delta = (i/\omega \rho_0) \hat{\rho}' \).

5.4. General case of a linear profile

Consider now the velocity Equation (31) for the special case of a linear temperature profile. Suppose then that \( \Lambda_0(z) \) is a non-zero constant, and put then that \( \Lambda_0(z) \) is a non-zero constant, and put

\[
x = \frac{2\omega \Lambda_0^{1/2}(0)}{\Lambda_0' c_T(0)} \Lambda_0^{1/2}(z), \quad x > 0.
\]

Then (31) becomes

\[
\frac{d^2 \delta}{dx^2} \left( 1 + \frac{1}{\Lambda_0'} \right) \frac{1}{x} \frac{d \delta}{dx} + \left( 1 - \frac{\left( s^2 - \left( 1 + \frac{1}{2\Lambda_0'} \right)^2 \right)}{x^2} \right) \delta = 0,
\]

(43)

where

\[
s^2 = \frac{4}{\gamma(\Lambda_0')^2} \left( \frac{\gamma - 1}{\gamma} + \Lambda_0 \right) \left( \frac{1}{2} \gamma + \frac{c_s^2}{v_A^2} \right) + \left( 1 + \frac{1}{2\Lambda_0} \right)^2.
\]

(44)

Note that \( s^2 \) may be negative, but is positive for \( N_0^2 > 0 \). Solving (43) (see Abramowitz and Stegun, 1967, p. 362) we find

\[
\delta \sim x^{\frac{1+(1/2\Lambda_0^2)}{\gamma}} J_0(x), \quad x^{\frac{1+(1/2\Lambda_0^2)}{\gamma}} Y_0(x).
\]

(45)

The pressure \( \hat{\rho} \) may be found from (45) and (32a).

The nature of these solutions and their bearing on the solar atmosphere is considered in Section 8.
6. On the Validity of the Expansion Procedure

The expansion procedure that we have extensively described clearly leads to a considerable gain in simplicity, despite the complexity introduced by considering a stratified atmosphere. The question that naturally arises is: how valid are the solutions thus obtained? In this section we solve a special case of the full problem, taking proper account of the lateral boundary conditions, and without recourse to the approximate treatment given by the expansion procedure and the boundary condition (28). Excellent agreement is found between the two approaches, thereby lending strong support to the use of the expansion procedure in the complete problem (for which a fuller treatment is not possible).

To begin with recall that in the special case of an isothermal, compressible atmosphere, with gravity negligible, the dispersion relation (given by (36), with \( \Lambda_0 \to \infty \)) is simply

\[
\omega^2 = k^2 c_T^2. \tag{46}
\]

Now consider this situation afresh, without using the expansion procedure. The basic-state inside the tube of constant cross-section is one of uniform pressure \( p_0 \), density \( \rho_0 \), and magnetic field \( B_0 \), so that \( c_0^2, v_A^2 \) and \( c_T^2 \) are constant. Outside the tube (where \( B_0 = 0 \)) the uniform pressure and density are \( p_e, \rho_e \). This is similar to the situation discussed by Lighthill (1960), except Lighthill did not consider the important effect of lateral boundaries and his treatment is in Cartesian coordinates. The governing equations are (1), (2) with \( g = 0 \), (3), (4) and (6).

Consider axisymmetric \( (\partial / \partial \theta = 0) \) linear perturbations from the basic-state described above, and write \( \mathbf{v} = (v_r, 0, v_z) \), \( \mathbf{b} = (b_r, 0, b_z) \) for the perturbation velocity and magnetic field in cylindrical polar coordinates \( (r, \theta, z) \). Then, as in Lighthill (1960), we find that

\[
\frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} - (c_0^2 + v_A^2) \nabla^2 \right) \Delta + c_0^2 v_A^2 \frac{\partial^2}{\partial z^2} (\nabla^2 \Delta) = 0, \tag{47}
\]

where

\[
\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad \text{and} \quad \Delta = \text{div} \, \mathbf{v}.
\]

With

\[
\Delta = R(r) e^{kz + ilkz}
\]

Equation (47) reduces to a form of Bessel’s equation, with solution

\[
R(r) = A_0 I_0(m_0 r), \quad 0 \leq r < r_0,
\]

where

\[
m_0^2 = (k^2 c_0^2 - \omega^2)(k^2 v_A^2 - \omega^2)/[(k^2 c_T^2 - \omega^2)(c_0^2 + v_A^2)],
\]
$A_0$ is an arbitrary constant, and $I_0$ is the modified Bessel function of zero order. In writing this solution we have assumed that $m_0^2 > 0$, and taken the solution for which $R(r)$ is finite at the origin.

Having determined $R(r)$ it is a straightforward matter to obtain $v_r, v_z, p$, etc., the result being

$$
v_r = \frac{A_0(\omega^2 - k^2 c_0^2)}{\omega^2 m_0} I_1(m_0 r) e^{i\omega t + ikz},
$$

$$
v_z = -i\frac{A_0 k c_0^2}{\omega^2} I_0(m_0 r) e^{i\omega t + ikz},
$$

$$
p = \frac{iA_0 \rho_0 c_0^2}{\omega} I_0(m_0 r) e^{i\omega t + ikz}, \quad (0 \leq r < r_0)
$$

(48)

$$
b_r = \frac{A_0(\omega^2 - k^2 c_0^2)}{\omega^3 m_0} k B_0 I_1(m_0 r) e^{i\omega t + ikz},
$$

$$
b_z = \frac{iA_0}{\omega} (\omega^2 - k^2 c_0^2) B_0 I_0(m_0 r) e^{i\omega t + ikz},
$$

where $I_1$ is the modified Bessel function of order one.

A similar treatment holds in the region outside the tube, where the sound speed is $c_\epsilon$ and $v_A = 0$, with the result that in the exterior we find that

$$
v_r^{(e)} = \frac{m_\epsilon c_\epsilon^2}{\omega^2} A_\epsilon K_1(m_\epsilon r) e^{i\omega t + ikz},
$$

$$
v_z^{(e)} = -\frac{ik c_\epsilon^2}{\omega^2} A_\epsilon K_0(m_\epsilon r) e^{i\omega t + ikz}, \quad (r > r_0)
$$

(49)

$$
p^{(e)} = \frac{i\rho_0 c_\epsilon^2}{\omega} A_\epsilon K_0(m_\epsilon r) e^{i\omega t + ikz},
$$

where $A_\epsilon$ is an arbitrary constant, $K_0$ and $K_1$ are modified Bessel functions of the second kind, and

$$
m_\epsilon^2 = (k^2 c_\epsilon^2 - \omega^2)/c_\epsilon^2.
$$

In (49) we have assumed $m_\epsilon^2 > 0$ and that perturbations vanish as $r \to \infty$. Note that $c_\epsilon^2$ and $c_0^2$ are related through the pressure balance condition (17):

$$
c_\epsilon^2 = (\rho_0/\rho_\epsilon)(c_0^2 + \gamma \nu_k^2).
$$

The solutions given by (48) and (49) must be suitably joined at the boundary $r = r_0$. The appropriate boundary conditions are:

$$
\text{across } r = r_0, \quad v_r \text{ and } p + \frac{B_0}{\mu} b_z \text{ are continuous.}
$$

(50)
These boundary conditions may be justified on physical grounds or from a consideration of the governing equations. In the latter method we consider the governing perturbation equations for a basic-state in which \( p_0, \rho_0 \) and \( B_0 \) are functions of \( r \), and then integrate these equations across the boundary \( r = r_0 \), bearing in mind that for the problem under discussion \( p_0, \rho_0 \) and \( B_0 \) are step-functions (and as such have unbounded derivatives at \( r_0 \)). A similar situation is discussed in Chandrasekhar (1961).

Applying the two boundary conditions in (50) gives the dispersion relation

\[
\rho_0 \left( \frac{(k^2c_T^2 - \omega^2)(c_0^2 + v_A^2)}{(k^2c_0^2 - \omega^2)} \right)^{1/2} I_0(m_0r_0) = \frac{1}{I_1(m_0r_0)} = \rho_e \omega^2 \left( \frac{c_e^2}{k^2c_T^2 - \omega^2} \right)^{1/2} K_0(m_0r_0) K_1(m_0r_0). \tag{51}
\]

Now we are interested in the circumstances where the tube is slender, so that \( k^2r_0^2 \ll 1 \). Then, noting that

\[
I_0(m_0r_0) \sim 1, \quad I_1(m_0r_0) \sim \frac{1}{2}m_0r_0 \quad \text{as} \quad m_0r_0 \to 0,
\]

\[
K_0(m_0r_0) \sim -\log (m_0r_0), \quad K_1(m_0r_0) \sim 1/m_0r_0 \quad \text{as} \quad m_0r_0 \to 0,
\]

we see that (51) reduces to

\[
\rho_0(k^2c_T^2 - \omega^2)(c_0^2 + v_A^2) = \frac{1}{2} \rho_e(k^2c_0^2 - \omega^2) \omega^2 r_0^2 \log (m_0r_0)^2, \tag{51}'
\]

valid for \( k^2r_0^2 \ll 1 \). An examination of (51)' reveals that

\[
\omega^2 = k^2c_T^2 + \frac{1}{4} \left( \frac{\rho_e}{\rho_0} \right) \frac{c_T^2(c_0^2 - c_T^2)}{(c_0^2 + v_A^2)} k^4 r_0^2 \log (k^2 r_0^2) + O(k^2 r_0^2), \tag{52}
\]

so that for small \( k^2r_0^2 \) we have

\[
\omega^2 \approx k^2c_T^2,
\]

in agreement with the result (Equation (46)) found by the expansion procedure, coupled with the simplifying boundary condition (28). Thus, at least for this special case, it appears that the procedure is fully vindicated. This, we feel, lends substantial support for the use of the expansion procedure in the general problem, considered in the following section.

It should be noted that the conclusion of the above calculation, namely that \( \omega^2 \approx k^2c_T^2 \), seems at first sight to be violated by the requirement that \( m_0r_0^2 \ll 1 \), in that \( m_0^2 \to \infty \) as \( \omega^2 \to k^2c_T^2 \). However, a closer examination, using (52), reveals that \( m_0^2r_0^2 \) does tend to zero in the limit of small radius, even though \( m_0^2 \to \infty \). Thus there is no contradiction.

Finally, returning to the general dispersion relation (51), we may note the reduction in the limiting case of incompressibility (i.e., \( c_0^2 \to \infty \)). Thus, as \( c_0^2 \to \infty \), we find

\[
\omega^2 = k^2v_A^2 \left[ 1 + \frac{\rho_e K_0(kr_0)I_1(kr_0)}{\rho_0 K_1(kr_0)I_0(kr_0)} \right],
\]
and so for \( kr_0 < 1 \) this reduces to \( \omega^2 \approx k^2 v_\Lambda^2 \), with an error of order \( k^2 r_0^2 \log_e (k^2 r_0^2) \) reducing \( \omega^2 \) below the value \( k^2 v_\Lambda^2 \).

7. The General Structure of Motions

Returning now to the more general case of a non-isothermal atmosphere in the presence of gravity we note that the structure of the governing Equation (29) is markedly different from the special cases discussed in Section 5. For example, instead of constant coefficients, as found for the isothermal atmosphere, we now have functions of \( z \), and further (as remarked earlier) Equation (29) for the pressure is singular at \( \omega^2 = N_0^2(z) \). Such singular points (‘critical levels’) in wave propagation problems have been discussed in fluid mechanics (e.g. Lin, 1966; Booker and Bretherton, 1967), magnetohydrodynamics (e.g. Acheson, 1973; McKenzie, 1973; Rudraiah and Venkatachalappa, 1972), and in the solar physics context (e.g. Adam, 1974, 1977; Thomas, 1976). However, in the context of magnetic flux tubes no discussion has yet been given; it turns out, though, that the critical level is here only of secondary interest.

We may consider the behaviour of \( \hat{p} \) near the regular singularity \( z = z_c \), where \( \omega^2 = N_0^2(z_c) \), by using the method of Frobenius (see, for example, Whittaker and Watson, 1962); this gives \( \hat{p} \) as a linear combination of the two fundamental solutions \( \hat{p}_1 \) and \( \hat{p}_2 \), where

\[
\hat{p}_1 = \frac{\hat{p}_1(0)}{A_0(0)} (z - z_c)^2 [1 + C_1(z - z_c) + O(z - z_c)^2],
\]

and

\[
\hat{p}_2 = A \hat{p}_1(z) \log_e \left( \frac{z - z_c}{A_0(0)} \right) + C_2 + C_3(z - z_c) + O(z - z_c)^2.
\]

The constants \( A, C_1, C_2 \) and \( C_3 \) may be calculated by following the Frobenius procedure. It turns out that \( A \) is zero, and so the second solution \( \hat{p}_2 \) has no logarithm term. Thus, we find

\[
\hat{p}_1 \sim (z - z_c)^2, \quad \hat{p}_2 \sim 1 - \frac{8}{c_0(z_c)}(z - z_c),
\]

showing that \( \hat{p} \) is well-behaved at the singularity \( z = z_c \). The behaviour in the velocity amplitude \( \hat{v} \), near \( z = z_c \), may be found by using (32b); thus

\[
\hat{v}_1 \sim \text{constant}, \quad \hat{v}_2 \sim (z - z_c),
\]

and so \( \hat{v} \) is well-behaved at \( z = z_c \).

The behaviour of the perturbations in the vicinity of the critical level, as described by (53) and (54), is quite different from that found in the problems studied by Booker and Bretherton (1967), Acheson (1972) and Adam (1977). These authors found an exponential change in amplitude on crossing the critical
level. The level may act as a valve, allowing propagation of waves across the level in a preferred direction (Acheson, 1972). In our problem, both the pressure and velocity change continuously on crossing the critical level and so the valve-effect is not present. It appears that the critical level $z = z_c$ in our flux tube is only of secondary interest.

The special case of a slowly varying medium may be considered by using the WKB method (see Budden, 1961; Ginzburg, 1960). Then, if the function $f(z; \omega^2)$, given in (30), is such that

$$\left| \frac{3}{4} \left( \frac{d}{dz} f^{1/2} \right)^2 - \frac{1}{2f^{3/2}} \frac{d^2(f^{1/2})}{dz^2} \right| \ll 1,$$

the WKB solutions of the pressure Equation (30) are

$$\hat{q} \sim f^{-1/4} \exp \left( \pm i \int_0^z f^{1/2} \, dz \right).$$

Note that the WKB solutions fail, in particular, at points where $f = 0$, the so called 'reflection level'. As we have already noted, in the vicinity of such a zero of $f$, i.e. in the 'transition region', the nature of $\hat{q}$ changes from oscillatory ($f > 0$) to exponential ($f < 0$); in place of the WKB solutions we have Airy's equation (with the Airy functions as solutions). Thus, near a zero $z = z_0$ of $f$ we have $f \approx (z - z_0)f'(z_0; \omega^2)$, and so (30) becomes

$$\hat{q}'' + (z - z_0)f'(z_0; \omega^2)\hat{q} = 0,$$

which is essentially Airy's equation. A detailed explanation of how the solutions of Airy's equation are joined to the WKB solutions may be found in, for example, Murray (1974).

We turn now to a consideration of the behaviour of the solution deep down in the tube (as $z \to -\infty$). The nature of the point $z = -\infty$ depends in detail upon the profile $\Lambda_0(z)$. To illustrate this behaviour it is convenient to consider Equation (30) for two special cases: the linear profile and the exponential profile. For the exponential profile, $\Lambda_0(z) = \Lambda_0(0) e^{-2\nu z}$, $\nu > 0$, we find (after some algebra) that

$$f(z; \omega^2) \sim -\nu^2 + (\omega^2 + 2g\nu)/c_I(z) + O\left(\frac{\Lambda_0^2(0)}{\Lambda_0^2(z)}\right), \quad (55)$$

as $z \to -\infty$. With this asymptotic form for $f(z; \omega^2)$ we may consider the asymptotic form of the solution for $\hat{q}$ and thence $\hat{p}$, with the result that for the exponential profile

$$\hat{p} \sim \exp \left( \nu z \pm \nu z - \frac{3 e^{2\nu z}}{8\nu \Lambda_0(0)} \right), \quad (56)$$

as $z \to -\infty$. In a similar fashion we find that for the linear profile,

$$\Lambda_0(z) = \Lambda_0(0) \left( 1 - \frac{z}{\lambda \Lambda_0(0)} \right), \quad \lambda > 0,$$
the function \( f(z; \omega^2) \) behaves like

\[
f(z; \omega^2) \sim \frac{\omega^2}{c_T(z)} + \frac{1}{\Lambda_0} \left[ -\frac{9}{16} + \frac{3}{2\gamma} + \frac{1}{4\lambda^2} + \frac{(c_0^2 - c_T^2)}{\gamma^2 c_T^2} + \frac{(1-\lambda)c_0^2}{\lambda \gamma c_T^2} \right] + O\left(\frac{\Lambda_0^3}{\Lambda_0}\right),
\]

as \( z \to -\infty \). For the asymptotic behaviour of \( \hat{p} \) we find (in the case of a linear profile) that

\[
\hat{p} \sim \left(1 - \frac{z}{\lambda \Lambda_0(0)}\right)^{(3\lambda - 1)/4} \sin \left[ \frac{2\omega \Lambda_0(0)}{c_T(0)} \left(1 - \frac{z}{\lambda \Lambda_0(0)}\right)^{1/2} \right],
\]

as \( z \to -\infty \).

The frequency \( \omega_p \) may also be determined asymptotically for the linear and exponential profiles. For the exponential profile, using (55), we find that

\[
\omega_p^2 \sim \nu^2 c_T^2(z) - 2g\nu,
\]

as \( z \to -\infty \), whereas for the linear profile Equation (57) gives

\[
\omega_p^2 \sim \frac{c_T^2(z)}{\Lambda_0^2(z)} \left[ \frac{9}{16} - \frac{3}{2\gamma} + \frac{1}{4\lambda^2} + \frac{(c_T^2 - c_0^2)}{\gamma^2 c_T^2} + \frac{(\lambda - 1)c_0^2}{\lambda \gamma c_T^2} \right],
\]

as \( z \to -\infty \). (It may be noted that these results are in agreement with those obtained earlier for the special case of no gravity.)

An analysis along the same lines as the above (e.g. use of the WKB method) may be given for the velocity equation and for \( \omega_v \). However, the form of \( \omega_v \) may be more readily determined than \( \omega_p \), under fairly general conditions, and so such a discussion is unnecessary.

### 8. Intense Magnetic Fields on the Sun

We now consider the application of the above analysis to the Sun. We are thus supposing that the assumption of a basic, static state, as described in detail in Section 3, is a reasonable one. Any motions, then, are considered to be adequately described by the linear analysis of Sections 4, 5 and 7. One consequence of our model is that one can no longer assume (as is commonly done) that \( c_0(z) \gg v_A(z) \) (or vice versa) as we descend into the tube, simply because (according to (22)) the ratio of the sound speed to the Alfvén speed is determined by their ratio at \( z = 0 \) (i.e. at observed levels), where they are found to be broadly comparable.

Having specified the detailed structure of the flux tube, through (23) and (24), under the assumption that the tube is in temperature balance with its surroundings, it remains to prescribe the temperature profile (or, more precisely, \( \Lambda_e(z) \) since we allow the mean particle mass \( \bar{m} \) to vary with \( z \)) in these surroundings. To do this we make use of Spruit's model of the convection zone (Spruit, 1974), taking \( z = 0 \) to
correspond to the optical depth* $\tau_{5000} = 1$. For above $\tau_{5000} = 1$ we use the HSRA
(Gingerich et al., 1971).

Pressure variations in the tube are described by Equation (29) and (30), where
$f_0(z)$ may be taken (on using (23)) to be

$$f_0(z) = \frac{1}{\Lambda_0(z)} (\omega^2 - N_0^2) e^{-\frac{3}{2}n(z)}.$$

The transition from solutions of (30) with exponential character (i.e. motions
decaying with depth) to those with sinusoidal character (i.e. pressure waves) takes
place, as we have noted earlier, at a critical frequency, the ‘transition’ frequency $\omega_p$,
given by

$$f(z; \omega_p^2) = 0,$$

(60)

where $f(z; \omega^2)$ is given following Equation (30). Equation (60) gives a cubic
behaviour in $\omega_p^2$ and so has either three real roots or one real root and a complex
conjugate pair. We may display (60) in the form (for $\omega^2 = \omega_p^2$)

$$f(z; \omega^2) = \frac{1}{c_T(z)(\omega^2 - N_0^2)} \left\{ (\omega^2 - N_0^2)^3 + A(\omega^2 - N_0^2)^2 + B(\omega^2 - N_0^2) + C \right\} = 0,$$

(61)

where, for the flux tube described by (23) and (24), the coefficients of the cubic are

$$A(z) = \left\{ \left( \frac{3}{2} \gamma - 1 \right) \frac{g^2}{c_0^4} + \frac{m''}{2m} - \frac{3}{4} \left( \frac{m'}{m} \right)^2 \right\} c_T^2,$$

$$B(z) = \left\{ \left( \frac{g}{c_0^2} + \frac{m'}{2m} \right)(N_0^2)' - \frac{1}{2} (N_0^2)' \right\} c_T^2,$$

$$C(z) = -\frac{3}{4} c_T^2 [(N_0^2)'^2],$$

where $m(z) = (1/\Lambda_0) e^{-3n/2}$. It follows from the general theory of cubic equations
that (61) has a single real root if

$$18ABC - 4A^3C + A^2B^2 - 4B^3 - 27C^2 < 0.$$

The behaviour of the positive root, $\omega_p^2(z)$, of (60) as a function of depth is given
in Figure 2, where we have for purposes of illustration taken $\Lambda_0(0) = 152$ km, $\gamma = \frac{5}{3}$
and $c_0 = v_A$. Figure 2 should not be confused with the dispersion diagrams usually
given in the literature; these generally refer to variation with horizontal
wave number, whereas here we focus attention on the variation with depth.

In our calculation of $\omega_p^2$ we have faithfully modelled both the behaviour of the
scale height $\Lambda_0(z)$ and its (depth) gradient $-\Lambda_0(z)$. Spruit’s model gives $-\Lambda'_0$
sharply peaked at about 40 km below $\tau_{5000} = 1$ (i.e. $z = 0$). Before the peak $-\Lambda'_0$ is
almost linear, after the peak approximately gaussian at least down to a depth of

* Note that the $\tau$-scale refers to the surrounding photosphere, not to the tube’s interior.
around 1000 km. (The temperature gradient in Spruit’s data is conveniently sketched in Parker (1977).)

The function \( \omega_p^2(z) \) has a peak at a depth of about 9.2 km, where \( \omega_p^2 \) takes its maximum value (in units of \( g/\Lambda_0(0) = 0.0018 \text{ s}^{-2} \)) of 5.72, giving an oscillation with a period of 62 s. Wave-like pressure variations may occur for frequencies above \( \omega_p \), evanescent variations below \( \omega_p \). Thus, at the peak, 9.2 km below \( \tau_{5000} = 1 \), the maximum allowable period for a pressure wave is 62 s. It should be noted that periods of around 60 s are not usually associated with photospheric oscillations. However, such periods have been considered by Stein (see Stein and Leibacher, 1974) in relation to the generation of sound waves by the Lighthill mechanism. It has been suggested that such short period waves may account for microturbulence in the photosphere and low chromosphere.

As we descend into the tube, Figure 2 shows that the maximum (allowable) pressure period rises: by about one scale height down this period has risen to its maximum (corresponding to the minimum in \( \omega_p^2 \)) value, a period of around 330 s. Further, it is clear from Figure 2 that pressure waves inside the tube may persist to quite moderate depths. Deeper still in the tube, \( \omega_p \) rises as propagation gives way to evanescence. For several special cases (considered in detail in Section 5) this rise is exponential. The general behaviour of \( \omega_p^2 \) for \( |z| \gg \Lambda_0(0) \) has been determined in Section 7 (see, in particular, (58) and (59)), and is used to complete Figure 2 by the dashed curve.

The behaviour depicted in Figure 2 for a model of the convection zone may be contrasted with the results for an isothermal atmosphere. The critical frequency is then given by Equation (39), namely

\[
\omega_p = \left( \frac{67g}{160\Lambda_0} \right)^{1/2} = 0.0275 \text{ s}^{-1},
\]
(with a corresponding period of 229 s), where (for purposes of illustration) we have set $\gamma = \frac{5}{3}, c_0 = v_\Lambda$ and $\Lambda_0 = 152$ km. (It may be noted that the general cubic given by (61) is degenerate for the isothermal case, giving only the one relevant root (namely that given by (39))). These frequency estimates ($\omega_p (z = 0) = 0.0275$ s$^{-1}, 0.0976$ s$^{-1}$ for the isothermal and convection models, respectively) show clearly that the use of an isothermal model may well lead to results that are markedly different from those found for a more realistic temperature profile. Thus, only with a proper degree of caution should one use isothermal models to investigate the behaviour of waves in the non-isothermal atmosphere of the Sun.

We may also compare $\omega_p$ with the critical frequencies found in a compressible, non-magnetic, stratified atmosphere by Moore and Spiegel (1964). They found two critical frequencies (squared), namely $\gamma g/4\Lambda_0$ and $N_0^2$, both of which, at $z = 0$, are an order of magnitude smaller than $\omega_p^2$. In fact, only the (squared) frequency $\gamma g/4\Lambda_0$ arises for the case of vertical propagation, which is essentially the case considered here.

It should be noted that the role of the Brunt-Väisälä frequency, $N_0$, in the problem of waves in a compressible atmosphere and its role in propagation in a flux tube is, in one respect markedly different. In the tube, the presence of $N_0^2$ gives rise to a singularity (where $\omega^2 = N_0^2$) in the governing pressure equation, whereas in the open atmosphere $N_0^2$ only arises in the ‘oscillatory’ part, viz. $f(z)$ of (30), of the governing equations (see Bray and Loughhead, 1974, p. 258).

Turning now to a consideration of motions in the tube, as governed by the velocity Equation (31) and its canonical form (33), we see that oscillatory solutions occur for $\omega^2 > \omega_v^2$, with $\omega_v$ given by (34). For purposes of illustration we again take $c_0^2 = v_\Lambda^2$ and $\gamma = \frac{5}{3}$, and then (34) reduces to

$$\omega_v^2 = \frac{9g}{8\Lambda_0} (\Lambda_0^2 + \frac{67}{180}).$$  \hspace{1cm} (62)

Thus, for $\Lambda_0' < -\frac{67}{180}$ waves occur for an arbitrary temperature profile, and $\omega_v^2$ is essentially taken to be zero.

The form of $\omega_v$ and its corresponding period $\tau_v = 2\pi/\omega_v$, for Spruit’s model of the solar convection zone combined with the HSRA, are readily determined: $\tau_v$ is sketched in Figure 3. Below $z = 0$ we see that $\tau_v$ descends from infinity at about a depth of 110 km, falling rapidly to a minimum value of 438 s (with corresponding maximum in frequency of 0.0143 s$^{-1}$) at about 370 km below $\tau_{5000} = 1$. Deeper in the tube $\tau_v$ rises gradually, reaching 443 s at a depth of 500 km. At a depth of 1000 km $\tau_v$ has risen to only 491 s. Above $z = 0$ we see that $\tau_v$ descends from infinity at about 12 km above $\tau_{5000} = 1$, and then continues to fall to a local minimum of 195 s (with corresponding $\omega_v$ at a local maximum of 0.032 s$^{-1}$) at a height of 410 km. By a height of 500 km, $\tau_v$ has risen slightly to 199 s; at the temperature minimum (in HRSA at 4170 K, some 557 km above $\tau_{5000} = 1$), $\tau_v = 186$ s and $\omega_v = 0.034$ s$^{-1}$. Much above this height is going beyond the range of validity of our theory, and is therefore not shown in Figure 3. However, it is of some interest to
consider the trend in $\tau_v$ as we rise higher in the tube. We find that $\tau_v$ has a second minimum, at about a height of 600 km, where $\tau_v = 159$ s (and $\omega_v = 0.040$ s$^{-1}$). Higher still, $\tau_v$ rises steadily reaching a period of 200 s (with $\omega_v = 0.031$ s$^{-1}$) at a height of 1000 km. (The curious bump, at around a height of 600 km, in the projected part of the curve, appears to be due to the rapid change in temperature gradient in the HSRA model, and not to any aspect of our theory.)

We may readily contrast Figure 3 with the curves pertaining to the various special cases discussed earlier. For example, considering the region $z \gg 0$ and assuming an isothermal atmosphere, with $\Lambda_0 = 126$ km (the mean of $\Lambda_0 (z = 0)$ and $\Lambda_0 (z = 500$ km) in HSRA), we find the straight line $\tau_v = 210$ s. While for a linear temperature profile, with $\Lambda'_0 = -0.1$, we find a gently sloping curve, falling from $\tau_v \approx 270$ s at $z = 0$ to $\tau_v \approx 220$ s at a height of 500 km. Other special cases may easily be investigated.

Consider now the exact solution, obtained in Section 5, for the case of a linear temperature profile. Such an approximation is probably reasonable for the region $z = 0$ to just short of the temperature minimum (but not, of course, for the region embracing the rapid change in $\Lambda'_0$ that occurs just below $z = 0$). With $c_0 = v_A$ and $\gamma = \frac{3}{5}$, $s^2$ (as given by (44)) reduces to

$$s^2 = \left(1 + \frac{3}{2\Lambda'_0} \right)^2 + \frac{22}{25(\Lambda'_0)^2}(2 + 5\Lambda'_0).$$

A reasonable approximation is $\Lambda'_0 = -\frac{1}{10}$; if in fact we take $\Lambda'_0 = -0.1178$, then $s$ turns out (conveniently) to be an integer (and thus giving tabulated Bessel functions): $s = 10$ and

$$\hat{v} \sim x^{-3.24} J_{10}(x), \quad x^{-3.24} Y_{10}(x),$$

where $x \approx 35.79 \omega A_0^{1/2}(z)$, for $\Lambda_0$ in km and $\omega$ in s$^{-1}$. Thus, for the range $z = 0$ to the temperature minimum $\Lambda_0(z)$ declines, and so $x \leq 35.79 \omega A_0^{1/2}(0)$ at $z = 0$, with
\(A_0(0) = 152 \text{ km},\) we find \(x \approx 441 \omega.\) Thus, for example, taking \(\omega = 0.02 \text{ s}^{-1}\) would give \(x \leq 8.8\) over the range \(z\) of interest. A glance at a sketch of \(J_{10}\) and \(Y_{10}\) for \(x \leq 8.8\) (see Abramowitz and Stegun, 1967, p. 359) shows that we are in the evanescent region, where the oscillatory solutions have given way to a solution \(J_{10} \to 0\) and a solution \(Y_{10} \to -\infty.\) Note that from (62) we have, for \(A' = -0.1178,\) \(\omega_e(0) = 0.022 \text{ s}^{-1}\) indicating (since \(\omega < \omega_e\)) non-oscillatory solutions. On the other hand, if we take \(\omega = 0.03 \text{ s}^{-1}\) (so now \(\omega > \omega_e\) and oscillatory solutions are to be expected) we find \(x = 13.2\) at \(z = 0,\) and this is the range of oscillatory \(J_{10}\) and \(Y_{10}\).

Observations of the detailed structure of individual flux tubes are still somewhat in the preliminary stages, especially in regard to oscillations associated with such magnetic tubes. But it is interesting to note that line-of-sight velocity oscillations of periods in the 5 to 6 min range have been recently recorded (Giovanelli and Brown, 1977; Giovanelli et al., 1978). These spectrograph observations refer to levels of line formation above the level \(z = 0.\) It is proposed to extend these observations so as to cover oscillations deeper in the tube (J. W. Harvey, 1977, private communication) and therefore covering the complete range of height and depth discussed here. We await this further development with interest.

In comparing observations with the results presented here, it should be borne in mind that the origin \(z = 0\) is taken from the level \(\tau_{5000} = 1\) in the surrounding photosphere and not inside the tube. The tube is less dense than the surrounding material, being partially evacuated by the magnetic field. In fact, from the pressure balance condition (Equation (17)), the assumption of temperature balance (i.e., \(T_{\tau}(z) = T_0(z)\)), and for illustration taking \(c_0 = v_A,\) we have

\[
(1 + \frac{1}{2} \gamma)p_0(0) = p_e(0),
\]

so

\[
p_0(0) = \frac{6}{11}p_e(0), \quad \rho_0(0) = \frac{5}{11}\rho_e(0),
\]

for \(\gamma = \frac{5}{3}.\) Thus, at \(\tau_{5000} = 1\) the density of the material in the tube is some 54% of that outside the tube. According to Spruit’s model of the convection zone, densities inside the tube comparable to those in the normal photosphere at \(\tau_{5000} = 1\) (where \(\rho_e(0) \approx 3.19 \times 10^{-7} \text{ gm cm}^{-3}\)), are only met at a depth of some 300 km.

The critical level occurs at \(z = z_c,\) where \(\omega^2 = N_0^2(z_c).\) We briefly consider its likely position in the flux tube of the Sun. According to Spruit’s (1974) model, the square of the Brunt-Väisälä frequency is negative from about \(z = 0\) to about \(z = -100 \text{ km},\) and so for \(\omega^2 > 0\) the critical level cannot occur in this part of the flux tube. Below about \(z = -100 \text{ km},\) \(N_0^2\) becomes positive and the possibility of \(\omega^2 = N_0^2\) arises. For example, with \(\omega^2 = 10^{-4} \text{ s}^{-2}\) we find (for \(N_0^2(z_c) = \omega^2\)) the critical level \(z = z_c\) at about one scale-height below \(z = 0;\) at this depth, \(\omega_p^2 \approx 4 \times 10^{-4} \text{ s}^{-2}.\) Of course, above \(z = 0 N_0^2\) again becomes positive and a critical level will occur. However, since both pressure and velocity are well-behaved on crossing a critical level its importance in the flux tube appears to be slight.

Finally, some comments on the assumptions made in this paper may be of value. To begin with, we have assumed that the basic state of the tube is a static one,
whereas recent observations (e.g. Harvey, 1976; Stenflo, 1976b) indicate strong
downdrafts ($\approx$0.5–2.0 km s$^{-1}$ at photospheric levels) within the tube. (Parker (1977)
has recently considered such a basic (steady flow) state, and has suggested that such
downdrafts may well contribute to field intensification.) However, it may well be (as
suggested by Galloway et al. (1977)) that the strong downflow is a transient effect
taking place in the formation of the intense magnetic tube but becoming negligible
once the tube has been formed – achieving its final equilibrium state of (essentially)
static balance with its surroundings.

The theory developed and applied in this paper depends upon the assumption
that the expansion equations represent the behaviour of motions in the tube, at
least approximately. We have considered a simplified problem in some detail in
Section 6 and found that good agreement does exist for sufficiently long waves,
i.e., for sufficiently small wavenumbers. In general terms we have required
$r_0(z) \ll L_0(z)$, for suitable vertical scale $L_0$. Now the flux tube fans out with height,
expanding from (say) $r_0 \approx 100$ km at $\tau_{5000} = 1$ to $r_0 \approx 350$ km at $\tau_{5000} = 10^{-4}$. So if
we take, as a conservative estimate $L_0 = \Lambda_0(z)$ then the condition $r_0 \ll L_0$ is violated
(strongly at $\tau_{5000} = 10^{-4}$, but only weakly at $\tau_{5000} = 1$) and so our results for the
region $z > 0$ are at best only rough approximations. Going deeper into the tube
(where $z < 0$), $r_0$ falls and $\Lambda_0$ rapidly rises, and so the condition $r_0 \ll \Lambda_0$ is then
readily met. Of course, if we take $L_0$ to be several – perhaps ten – scale-heights
(observations suggest wavelengths in excess of 1500 km; and the condition that the
tube doesn’t fan out too rapidly implies $|r'_0| \ll 1$, i.e., $r_0 \ll 4\Lambda_0$), then the condition
$r_0 \ll L_0$ is likely to be met over the complete range of $z$ presented in our figures.
It would appear, then, that our results are likely to be a reasonable guide to the
behaviour in the tube over the ranges of height and depth presented here, but
should be viewed with the above caution in mind.

The role of dissipative processes (viscous, thermal and electrical) have also been
ignored, again for the sake of simplicity. The role played by thermal dissipation,
such as by heat conduction along the tube and from its sides is clearly of interest
(and, in fact, currently under investigation), and may relate to the possibility of
cooling (and subsequent field intensification) by overstable Alfvén waves, in-
vestigated for a uniform field (with lateral boundaries) by Roberts (1976a).

Acknowledgements

We would like to thank our colleagues Drs. J. Adam, A. D. D. Craik and E. R. Priest
for helpful suggestions and criticisms of a draft of this paper. A. R. Webb would like
to thank the University of St. Andrews for financial support.

Appendix A: Derivation of the Expansion Equations

We consider the expansion of Equations (1)–(6) of the text in terms of a small
parameter $\varepsilon$. For example, we may take $\varepsilon = r_0(\hat{z})/L_0(\hat{z})$, where $\hat{z}$ is some suitable
reference level chosen so that the condition $\varepsilon \ll 1$ is satisfied at (and below) that level for suitable vertical scale $L_0$.

Stretch the radial coordinate $r$ by writing $r = \varepsilon R$, for $R$ assumed to be of order one; then

$$\frac{\partial}{\partial r} = \varepsilon^{-1} \frac{\partial}{\partial R},$$

e etc.

Set

$$\mathbf{B} = \mathbf{B}(r, z, t) = (0, 0, B(z, t)) + \varepsilon R(b_1, 0, b_2) + O(\varepsilon^2), \quad (A1)$$

$$\mathbf{v} = \mathbf{v}(r, z, t) = (0, 0, \nu(z, t)) + \varepsilon R(\nu_1, 0, \nu_2) + O(\varepsilon^2), \quad (A2)$$

$$p = p(r, z, t) = p(z, t) + \varepsilon R p_1 + O(\varepsilon^2), \quad (A3)$$

$$\rho = \rho(r, z, t) = \rho(z, t) + \varepsilon R \rho_1 + O(\varepsilon^2), \quad (A4)$$

where $b_1$, $b_2$, $\nu_1$, $\nu_2$, $p_1$ and $\rho_1$ are functions of $z$ and $t$ only. So, to zeroth-order (i.e. as $\varepsilon \to 0$), $\mathbf{B}$, $\mathbf{v}$, $p$ and $\rho$ assume axial values.

Consider the equation of continuity (Equation (1) of main text): we have

$$\frac{\partial \rho(z, t)}{\partial t} + \left[ \rho(z, t) + O(\varepsilon) \right] \left[ \varepsilon^{-1} \frac{\partial}{\partial R} (\varepsilon R v_1) + (\varepsilon R)^{-1} (\varepsilon R v_1) \right] +$$

$$+ \varepsilon R v_1 + O(\varepsilon) \varepsilon^{-1} \frac{\partial}{\partial R} (\rho(z, t) + \varepsilon R \rho_1 + O(\varepsilon^2)) +$$

$$+ \nu(z, t) \frac{\partial \rho(z, t)}{\partial z} + 0(\varepsilon) = 0.$$

Thus, to zeroth-order, we have

$$\frac{\partial \rho(z, t)}{\partial t} + \left( 2 \nu_1(z, t) + \frac{\partial \nu(z, t)}{\partial z} \right) \rho(z, t) + \nu(z, t) \frac{\partial \rho(z, t)}{\partial z} = 0. \quad (A5)$$

In a similar fashion, we may show that the $z$-component of the momentum equation gives (to zeroth-order)

$$\rho(z, t) \left( \frac{\partial \nu(z, t)}{\partial t} + \nu(z, t) \frac{\partial \nu(z, t)}{\partial z} \right) = - \frac{\partial \rho(z, t)}{\partial z} - \rho(z, t) g. \quad (A6)$$

It may be noted that the magnetic field is absent in (A6). This is because the current density $j$ is in the azimuthal direction and of magnitude

$$j = \frac{1}{\mu} \frac{\partial b_1}{\partial z} - \frac{1}{\mu} \frac{\varepsilon^{-1} \partial}{\partial R} (\varepsilon R b_2) = - \frac{1}{\mu} b_2 + O(\varepsilon),$$

and so the $\mathbf{j} \times \mathbf{B}$ force is radial and (to zeroth-order) of magnitude $-(b_2/\mu) B(z, t)$.  

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In fact, the radial component of the momentum equation gives

$$p_1 + \frac{1}{\mu} B(z, t)b_1 = 0 . \quad (A7)$$

Thus, to within order $\epsilon$, the total (gas plus magnetic) pressure on the axis of the tube is equal to the total pressure on the boundary ($r = r_0(z)$).

The induction equation, on expansion, yields

$$\frac{\partial b_1}{\partial t} + \Delta b_1 + v \frac{\partial b_1}{\partial z} - B \frac{\partial v}{\partial z} = O(\epsilon) \quad (A8)$$

and

$$\frac{\partial B}{\partial t} + \Delta B + v \frac{\partial B}{\partial z} - B \frac{\partial v}{\partial z} = O(\epsilon) , \quad (A9)$$

where

$$\Delta = 2v_1 + \frac{\partial v}{\partial z} + O(\epsilon) .$$

Only (A9) is needed to determine $v$ and $B$; (A8) simply determines $b_1$, etc., in terms of $v$ and $B$.

The energy equation, on expansion, readily gives

$$\left( \frac{\partial}{\partial t} + v(z, t) \frac{\partial}{\partial z} \right) p(z, t) - \frac{\gamma p(z, t)}{\rho(z, t)} \left( \frac{\partial}{\partial t} + v(z, t) \frac{\partial}{\partial z} \right) \rho(z, t) = O(\epsilon) , \quad (A10)$$

whilst the ideal gas relation is (to zeroth-order) simply

$$p(z, t) = \frac{k}{m} \rho(z, t) T(z, t) , \quad (A11)$$

and so determines the zeroth-order temperature perturbation.

Finally, it should be noted that the solenoidal condition on $B$ gives

$$2b_1(z, t) + \frac{\partial B(z, t)}{\partial z} + O(\epsilon) = 0 , \quad (A12)$$

which thus determines $b_1$ once $B(z, t)$ is known. Equation (A12) makes it clear that the zeroth-order field $(0, 0, B(z, t))$ is in general a function of $z$, and not independent of $z$ as is sometimes assumed.

Appendix B: The Effect of Forced External Pressure Variations

Consider the radial pressure balance. In equilibrium we have the balance condition (Equation (17)):

$$p_0(z) + \frac{1}{2\mu} B_0^2(z) = p_e(z) .$$
In the presence of motions we have the linear pressure balance condition

$$p(z, t) + \frac{1}{\mu} B_0(z) B(z, t) = \delta p_e(z, t),$$  \hspace{1cm} (B1)

where $\delta p_e(z, t)$ is the linear perturbation of external pressure evaluated on the undisturbed tube boundary (i.e. on $r = r_0(z)$). Note that in (B1) we have made use of the condition that the total internal pressure is, to zeroth-order, constant across the tube (see (A7)). In general, $\delta p_e$ is determined by motions outside the tube. To illustrate its effect, let us suppose that $\delta p_e(z, t) = G(z) e^{i\omega t}$, then (B1) becomes

$$\dot{p}(z) + \frac{1}{\mu} B_0(z) \dot{B}(z) = G(z),$$  \hspace{1cm} (B2)

where $p(z, t) = \dot{p}(z) e^{i\omega t}$ and $B(z, t) = \dot{B}(z) e^{i\omega t}$. We may regard the pressure $G(z)$ as specified. (For example, we may take $G(z) = G_0(0) e^{a z^2 / A_0(0)}$, $z < 0$ and $a > 0$, which gives a declining pressure.)

The effect of the pressure term $\delta p_e$ is readily investigated, with the result that Equation (29), governing the amplitude $\hat{p}(z)$ of pressure perturbations inside the tube, becomes

$$(\omega^2 - N_0^2) \hat{p}'' + \alpha_0(z; \omega^2) \hat{p}' + \beta_0(z; \omega^2) \hat{p} = \frac{1}{v_A^2} (\omega^2 - N_0^2)^2 G(z).$$  \hspace{1cm} (B3)

Thus, the effect of the manipulative pressure $\delta p_e$ is to introduce a forcing term into the equation for $\hat{p}$. We may note that this forcing term is negligible for large Alfvén speed. For non-zero forcing term the solution given in the text corresponds to the complementary function of (B3).

As a simple illustration of the solution of (B3), suppose $G(z) = G_0(0) e^{a z^2 / A_0(0)}$, $z < 0$, and that the basic state is isothermal ($A_0 = A_0(0)$). Then, from (36), we find that the complementary function is

$$\hat{p} \sim e^{-(3z/4A_0) \pm ikz},$$  \hspace{1cm} (B4)

where $k$ is given in (37). The particular integral is $A_0 G(z)$, where

$$A_0 = \frac{(\omega^2 - N_0^2) \Lambda_0^2}{v_A^2 \left[ \frac{94}{25} + \frac{6A_0}{5g} \left( \omega^2 - \frac{g}{A_0} \right) \right]},$$

for $c_0^2 = v_A^2$. With $\gamma = \frac{5}{3}$, $A_0 = 152$ km and $\omega = \omega_p = 0.0275$ s$^{-1}$ we find $A_0 = 0.0038$.

References


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