ANALYSIS OF BREMSSTRAHLUNG SOURCE SPECTRA IN TERMS OF INTEGRAL MOMENTS  
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ABSTRACT  
The problems of deriving temperature structure for thermal hard X-ray sources and electron spectra for nonthermal sources, from their bremsstrahlung continuum spectra, are briefly reviewed and the dangers of model-fitting are reiterated. A more satisfactory approach is developed in terms of the evaluation of integral moments of the distribution function. Using thermal analysis of the impulsive solar hard X-ray burst of 1970 March 1 as an illustration, the integral moment method is shown to give a rapid assessment of the real information content of a bremsstrahlung spectrum. In particular it is shown that due to limitations of bandwidth and, to a lesser extent, of spectral resolution, current hard X-ray spectrometry alone is incapable of distinguishing isothermal, multithermal, and nonthermal sources. Criteria are established for the spectrometer needed to define a thermal source distribution to within some specified accuracy.  
Subject headings: plasmas — Sun: flares — Sun: X-rays  
I. INTRODUCTION  
There are two types of collisional bremsstrahlung spectrum problem of interest in astrophysics. First, there is the problem of thermal bremsstrahlung from a nonisothermal source which can be described in terms of the differential emission measure $\xi(T)$ per unit temperature $T$ (see Craig and Brown 1976a for a precise definition). From such a source (or from the line-of-sight column filling the instrument field) the spectral energy flux is given by (Culhane 1969; Craig and Brown 1976a)  
$$J(\epsilon) = c_1 \int_0^\infty \frac{\xi(T)}{T^{1/2}} e^{-\epsilon/T} dT,$$  
where $c_1$ is a constant (depending on the source distance), $J(\epsilon)$ is the photon energy flux per unit photon energy $\epsilon$, and $T$ is in energy units. Second, there is the problem of nonthermal bremsstrahlung from a source of mean proton density $n_0$ containing an effective spectrum of energetic electrons $N(E)$ per unit electron energy $E$ [see Brown 1971 for precise definition of $n_0$ and $N(E)$]. In this case the observed spectrum is, for the Bethe-Heitler cross section (Brown 1971),  
$$\psi(\epsilon) = n_0 \int_\epsilon^\infty \frac{N(E) dE}{(E - \epsilon)^{7/2}},$$  
where $\psi(\epsilon) = -c_2 (2/3m_e)^{1/2} j(\epsilon)$, $c_2$ is a constant, and $m_e$ is the electron mass.  
The problem at hand is to determine $\xi(T)$ or $N(E)$ from $J(\epsilon)$ using (1) or (2), respectively. The first problem is important in determining the temperature structure of the plasma and hence in assessing the roles of energy input and transport mechanisms such as conduction and convection (see, e.g., Craig and McClymont 1976). The second should yield insight into the particle acceleration mechanism in terms of the spectrum produced (e.g., Brown and Hoyng 1975; Benz 1977). Both of these problems arise in the solar flare context and indeed may overlap in the energy range 1–50 keV (e.g., Brown 1971). (Indeed, hard X-ray spectra may be interpreted thermally.)  
For simplicity and convenience in data processing, a common procedure has been to fit an empirical model function for the source spectrum to the spectral data [e.g., the power-law $N(E) = aE^{-\gamma}$ for (2) (Kane and Anderson 1970; Hoyng, Brown, and van Beek 1976); and the form $\xi(T) = \xi_0 \ln (-T/T_0)$ for (1) (Chambe 1971; Acton et al. 1972)]. However, in several recent papers (Craig and Brown 1976a, b; Brown 1975, 1976; Underwood and McKenzie 1977; Craig 1977a, b) the dangers of such a procedure have been emphasized. The difficulty stems from the fact that the source mechanism itself imposes a high-frequency filter on the source function [$\xi(T)$ or $N(E)$] so that no details of it are visible in the data function, however high the spectral resolution is made; cf. Böhm (1963).  
This point was made long ago by Chandrasekhar and Münch (1950) in the physically quite different context of stellar rotation speeds which have a distribution to be derived from an equation entirely equivalent to (2). Chandrasekhar and Münch (1950), in emphasizing the arbitrariness of most model functions, propose that the best procedure is to derive a general mathematical representation of the source function which may be compared with any  
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sensible physical (or empirical) model when such becomes available. Specifically they show how their equivalent of (2) can be analyzed in terms of the integral moments of the source and data functions. In this paper such a moment analysis of the bremsstrahlung problems (1) and (2) is obtained, and it is suggested that such a procedure is a useful form for future publication of bremsstrahlung source data, since it brings out clearly the significance of the result.

II. MOMENT ANALYSIS OF THE INTEGRAL EQUATIONS

a) Thermal Bremsstrahlung

Consider the mth moment of the observed photon energies, weighted with respect to the photon number flux $I(e) = J(e)/e$, given by

$$\langle e^m \rangle = \int_0^\infty I(e)e^mde,$$

(3)

where, for the present, the unnormalized moment is used (cf. $\langle e^m \rangle$ below). Then if we insert $J(e)$ from (1) and reverse the order of integration, the integral over $e$ can be performed immediately to give

$$\langle e^m \rangle = c_1 \Gamma(m)\int_0^\infty \frac{\xi(T)}{T^{1/2}} T^m dT.$$

(4)

Thus

$$\langle T^m \rangle = \int_0^\infty \frac{\xi(T)}{T^{1/2}} T^m dT = \langle e^{m+1/2} \rangle / [c_1 \Gamma(m + \frac{1}{2})],$$

(5)

where $\langle T^m \rangle$ is the (unnormalized) mth moment of $T$ weighted with respect to $\xi(T)$. [In fact, the moments of $T$ and $e$ could be made of the same order in (5) by weighting $T$ with respect to $\xi(T)/T^{1/2}$ instead of $\xi(T)$, but this would be less physically interesting.]

Conversion to the normalized moments

$$\{e^m\} = \langle e^m \rangle / \langle e^0 \rangle, \quad \{T^m\} = \langle T^m \rangle / \langle T^0 \rangle$$

(6)

is readily done by application of (5) to the case $m = 0$, giving

$$\{T^m\} = \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m + \frac{1}{2})} \{e^{m+1/2}\}.$$  

(7)

Then the mean, mean square deviation $\{(T - \{T\})^2\} = \{T^2\} - \{T\}^2$, and skewness $\{(T - \{T\})^3\} = \{T^3\} - 3\{T\}\{T^2\} + 2\{T\}^3$ can be constructed, viz.,

$$\{T\} = 2\{e^{3/2}\}/\{e^{1/2}\},$$

$$\{(T - \{T\})^2\} = 4 \left[ \frac{1}{3} \{e^{3/2}\} - \frac{\{e^{3/2}\}^2}{\{e^{1/2}\}^2} \right],$$

$$\{(T - \{T\})^3\} = 8 \left[ \frac{1}{15} \{e^{3/2}\} - \frac{\{e^{3/2}\}^2}{\{e^{1/2}\}^3} + \frac{2\{e^{3/2}\}^3}{\{e^{1/2}\}^5} \right].$$

(8)

b) Nonthermal Bremsstrahlung

Reexpression of (2) in terms of the electron flow $F(E) = N(E)v(E)$ gives

$$J'(e) = -\frac{n_0}{c_2 e} \int_e^\infty \frac{F(E)dE}{[E(E - e)]^{1/2}}.$$  

(9)

Expressing $I(e)$ in terms of $J(e)$ in (3) and integrating once by parts, we obtain

$$\langle e^m \rangle = -\frac{1}{m} \int_0^\infty J'(e)e^mde.$$  

(10)

1 For typographic reasons, the notation $\langle x \rangle$ has been used throughout to denote the normalized average of quantity $x$. This has identical meaning to the usual $\bar{x}$ notation.
where we have assumed that $J(e)$ cuts off at some upper limit and that $J(e)$ does not approach $\infty$ as $e$ approaches 0 faster than $e^{-m}$ for any $m$ of interest (cf. § III). If we insert $J'(e)$ from (9) and reverse the order of integration, the integral over $e$ can again be performed immediately to give

$$\langle e^n \rangle = \frac{1}{c^2 m} B(m, \frac{1}{2}) \int_0^\infty F(E) E^{m-1} dE.$$  \hspace{1cm} (11)

Thus if the moments $\langle E^m \rangle$ of $E$ are those weighted with respect to the electron flux, i.e.,

$$\langle E^m \rangle = \int_0^\infty F(E) E^m dE,$$  \hspace{1cm} (12)

then they are related to the moments of $e$ by

$$\langle E^m \rangle = \frac{c^2}{B(m + 1, \frac{1}{2})} \langle e^{m+1} \rangle.$$  \hspace{1cm} (13)

or, in normalized form,

$$\{E\} = \frac{\langle E^m \rangle}{\langle E^0 \rangle} = \frac{(m + 1)}{B(m + 1, \frac{1}{2})} \{e\}.$$  \hspace{1cm} (14)

That is, the mean, mean square deviation, and skewness of the electron energy distribution are, similarly to (8):

$$\{E\} = \frac{1}{2} \{e\} / \{e\},$$

$$\{E - \langle E \rangle\}^2 = \frac{9}{4} \{e^2\} / \{e\}^2,$$

$$\{E - \langle E \rangle\}^3 = \frac{35}{4} \{e^3\} / \{e\}^3 + \frac{405}{4} \{e^2\}^2 / \{e\}^4 + \frac{54}{4} \{e^3\}^2 / \{e\}^5.$$  \hspace{1cm} (15)

The relationship of the $m$th moment of $E$ to the $(m + 1)$th moment of $e$ in (14) may be attributed to the occurrence of the derivative in (9).

### III. APPLICATION TO THE SOLAR HARD X-RAY BURST OF 1970 MARCH 1

In order to illustrate the application of moment analysis, and in particular to show how such analysis readily reveals the limited information content of typical spectral data, the formulæe of § II will now be applied to the solar hard X-ray burst peaking near UT 11:27 on 1970 March 1. This event is one of a selection of bursts (observed on OSO 5; cf. Frost et al. 1971), of simple temporal structure, recently analyzed by Crannell et al. (1978). This burst is particularly suitable here since it is intense enough to yield good photon count statistics and since Crannell et al. provide their "best fit" thermal spectrum parameters which are convenient for comparison with the results below.

In Table 1 are shown the spectral data obtained at burst peak by Crannell et al. (1978) in terms of the mean channel energy, channel width, and photon flux, in channels 2–8, corrected for photospheric albedo (Mätzler et al. 1978). Table 2 shows the moments of the photon distribution inferred according to the discrete approximation to (3), viz.,

$$\langle e^n \rangle = \sum_{j=1}^{\infty} e_j^n I_j \Delta e_j.$$  \hspace{1cm} (16)

<table>
<thead>
<tr>
<th>Channel</th>
<th>Mean Energy $\epsilon_j$(keV)</th>
<th>Differential Energy Flux $J(\epsilon_j)$(keV cm$^{-2}$ s$^{-1}$ keV$^{-1}$)</th>
<th>Photon Flux $I(\epsilon_j)\Delta \epsilon_j = J(\epsilon_j)\Delta e_j$ (cm$^{-2}$ s$^{-1}$)</th>
</tr>
</thead>
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<tr>
<td>2</td>
<td>38</td>
<td>50</td>
<td>36.8</td>
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<tr>
<td>4</td>
<td>94</td>
<td>14</td>
<td>4.20</td>
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<tr>
<td>5</td>
<td>122</td>
<td>8</td>
<td>1.85</td>
</tr>
<tr>
<td>6</td>
<td>150</td>
<td>3.6</td>
<td>0.67</td>
</tr>
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<td>0.34</td>
</tr>
<tr>
<td>8</td>
<td>206</td>
<td>1.6</td>
<td>0.22</td>
</tr>
</tbody>
</table>

* From Crannell et al. 1978 and Mätzler et al. 1978, with correction for albedo.
By means of equations (7) and (8), and (14) and (15), these yield the derived moments of the temperature distribution and electron energy distribution for thermal and nonthermal interpretations respectively, as shown in Table 3 for the first three moments.

Using a very simple fitting procedure, Crannell et al. (1978) claimed that this event was consistent with an isothermal source at around \( T_0 = 52 \) keV, but they did not discuss the relative goodness of fit of any other model such as a power-law particle spectrum or a distribution of temperature. The results of Table 3 permit an assessment of the realism and uniqueness of their conclusion, as follows.

First, the moment analysis gives a \( \{T\} \) approximately twice their "best-fit" value of \( T_0 \), a difference which might be attributed to the different weight given to each channel by their \textit{a priori} assumption of a spectral form. However, much more important is the fact that moment analysis leads to a negative variance for the \( T \) distribution, which is impossible for any real distribution function. In addition, the magnitude of the root variance and the skewness are comparable to \( \{T\} \) itself, suggesting a far from isothermal distribution. If we compare this with the nonthermal analysis, we see that the required electron distribution appears to be a broad distribution about \( \{E\} = 102 \) keV strongly skewed to smaller energies, similar to a power law \( (E^{-\alpha}) \). Also the variance here is satisfactorily positive. It is therefore tempting to conclude that a nonthermal (quasi power-law) distribution is a more satisfactory model for this burst than a thermal one since the latter gives rise to a meaningless variance. A closer inspection of the moment problem, however, reveals the true source of the discrepancy with the Crannell et al. result (and of the negative \( \{T\} \) variance) and proves that their spectral data (and all similar data) are quite inadequate for distinction between source models.

IV. DISCUSSION OF THE ERRORS IN SPECTRAL ANALYSIS

Typical X-ray continuum spectra \( I(\epsilon) \) are subject to three sources of error: (a) Truncation error due to the finite bandwidth of the instrument, i.e., \( I(\epsilon) \) is not observed over the full range of \( \epsilon \) [this is extremely important in deriving \( \xi(T) \) or \( N(E) \) since electrons at any \( T \) or \( E \) contribute significantly over a very wide range of \( \epsilon \), a fact that is at the root of the instability of eqs. (1) and (2)—cf. Craig and Brown (1976a)]; (b) discretization error due to the finite spectral resolution of the instrument within the observed spectral band; (c) statistical errors in the photon counting statistics in each instrument channel. In the following, an assessment is made of errors (a) and (b) (in terms of their effect on the spectral moments) but (c) is neglected. Inclusion of count statistics can only worsen the conclusion reached below (note that an increase in spectral resolution worsens the photon counting error in each band).

For simplicity, consider the results obtained by a moment analysis of a truly isothermal source (temperature \( T_0 \)) of total emission measure \( \xi_0 \) observed by an instrument spanning the continuum photon range \( \epsilon \) to \( \epsilon \) in \( n \) channels of equal width. (A similar analysis for other source types and instrumental characteristics will lead to the same conclusion.) The spectral moments in this case should be analytically [cf. eq. (4) with \( \xi(T) = \xi_0 \delta(T - T_0) \)]

\[
\langle \epsilon^n \rangle = c_1 \frac{\xi_0}{T_0^{1/2}} T_0^n \int_0^\infty x^{n-1} e^{-\xi} dx.
\]

If the instrument had perfect spectral resolution but a finite band, the observed spectral moments would, however, be

\[
\langle \epsilon^n \rangle' = c_1 \frac{\xi_0}{T_0^{1/2}} T_0^n \int_{\epsilon_{\text{low}}}^{\epsilon_{\text{high}}} x^{n-1} e^{-\xi} dx.
\]

### Table 3

| Derived Moments of the Temperature and Energy Distributions Required for the Burst of Table 1 |
|-------------------------------------------------|-------------------------------------------------|
| \( (T) = 120 \text{ keV} \) | \( (E) = 102 \text{ keV} \) |
| \( (T^2) = (79 \text{ keV})^2 \) | \( (E^2) = (131 \text{ keV})^2 \) |
| \( (T^3) = (63 \text{ keV})^3 \) | \( (E^3) = (150 \text{ keV})^3 \) |
| Variance = \(-90 \text{ keV}^2\) | Variance = \(82 \text{ keV}^2\) |
| Skewness = \(-64 \text{ keV}^3\) | Skewness = \(-39 \text{ keV}^3\) |
Comparison of (17) and (18) yields the relative truncation error $\Delta_T$ in the observed moments

$$\Delta_T = \frac{\langle e^m \rangle - \langle e^m \rangle_c}{\langle e^m \rangle_c} = \Delta_1 + \Delta_2,$$  \hspace{1cm} (19)

where

$$\Delta_1 = -\frac{1}{\Gamma(m)} \int_{e_i/T_0}^{e_f/T_0} x^{m-1} e^{-x} dx = -\left[1 - \exp\left(-\frac{e_i}{T_0}\right) \sum_{j=1}^{n-1} \frac{1}{j!} \left(\frac{e_i}{T_0}\right)^j\right] \approx -\frac{1}{\Gamma(m+1)} \left(\frac{e_i}{T_0}\right)^m,$$  \hspace{1cm} (20)

$$\Delta_2 = -\frac{1}{\Gamma(m)} \int_{e_f/T_0}^{\infty} x^{m-1} e^{-x} dx = -\exp\left(-\frac{e_f}{T_0}\right) \sum_{j=1}^{n-1} \frac{1}{j!} \left(\frac{e_f}{T_0}\right)^j \approx -\frac{1}{\Gamma(m)} \left(\frac{e_f}{T_0}\right)^{m-1} \exp\left(-\frac{e_f}{T_0}\right),$$  \hspace{1cm} (21)

where the approximations are valid in the limits $e_i/T_0 \ll 1$ and $e_f/T_0 \gg 1$, respectively (i.e., for small errors).

Figure 1 shows the exact result of (20) for $\Delta_1$ and $\Delta_2$ for a wide range of $x$ values. For the first four moments ($m = 1, 4$) the truncation errors are $\Delta_1 \approx -12\%$ for $m = 1$, $-14\%$ for $m = 2$, $-32\%$ for $m = 3$, and $-54\%$ for $m = 4$. Thus to keep truncation errors in the first two moments ($m = 1$, $4$) to within $10\%$ requires $e_i/T_0 \lesssim 0.1$, $e_f/T_0 \lesssim 3.5$—i.e., if the 1970 March event were indeed an isotermal source at $T_0 \approx 50$ keV, it would require a spectrometer spanning the range $5-175$ keV to derive $\langle e^m \rangle$. Consideration of equation (11) then shows that the mean, variance, and skewness of the temperature distribution are subject to truncation errors of order $100\%$, more in the Crannell et al. (1978) data. Thus the occurrence of a negative $T$ variance in Table 3 and the discrepancy between $\{T\}$ and the Crannell et al. $T_0$ value have no bearing whatever on the true nature of the source but arise because the spectrometer bandwidth involved is incapable of adequately defining the spectral or temperature distribution moments. (Note that a negative $T$ variance can occur, because the $T$ moments are derived from several $e$ moments with errors in them, rather than directly from a distribution function.)

Next the discretization error must be considered. Suppose the spectrometer used satisfied the criteria $e_i/T_0 \approx 0.1$, $e_f/T_0 \approx 3.5$ required for $\Delta_1 \lesssim 15\%$. Then the actual moments inferred from a spectrometer of $n$ equal width channels will be

$$\langle e^m \rangle_c \approx \frac{c_1 e_A}{T_0^{1/2}} T_0^m \left(\frac{x_A - x_d}{n}\right) \sum_{j=1}^{n} x_j^{m-1} \exp\left(-x_j\right),$$  \hspace{1cm} (22)

where $x_j = e_j/T_0$ and $e_j = e_A + (j - 1)(e_f - e_A)/n$. Thus the discretization error $\Delta_D = \langle e^m \rangle_c - \langle e^m \rangle$ will be

$$\Delta_D = \left(\frac{x_f - x_d}{n}\right) \sum_{j=1}^{n} x_j^{m-1} \exp\left(-x_j\right) \left/ \left(\int_{x_A}^{x_f} x^{m-1} e^{-x} dx\right)\right. - 1.$$  \hspace{1cm} (23)

Figure 2 shows the value of $\Delta_D$ from (23) for $m = 1, 2, 3$ for the above spectrometer bandwidth, with increasing numbers $n$ of independent spectral channels (cf. comments below), or increasing spectral resolution. (Note again that we are ignoring the increase of photon count errors arising from the decrease of flux per channel with increasing $n$.) It is at once clear that only a few channels ($\lesssim 5$) are needed for this spectrometer to keep the discretization errors well below the truncation errors and that for more than about 20 channels the discretization errors become negligible. This confirms the general conclusions of Craig and Brown (1976a) that increase of spectral resolution beyond a modest limit yields no new information due to the intrinsic smoothness of the measured function, and that the proper strategy is to maximize the bandwidth over which the spectral representation is obtained.

Since the physical quantities of interest, i.e., $\{T\}$, $\sigma(T)$, etc., depend on differences of powers of $e$, $\langle e^2 \rangle$, etc., it is clearly necessary to measure the moments of $J(e)$ to better than about $3\%$ if we are to prove isothermality of the source to within about $3\%$ [i.e., $\sigma(T) \lesssim 0.3\langle T \rangle$]. From Figures 1 and 2 it is clear that for $T_0 \approx 50$ keV this requires a spectrometer spanning about 2–250 keV in 5–10 channels. Thus the spectral resolution used by Crannell et al. (1978) should suffice but the instrument bandwidth should be increased, particularly to lower energies. The chief problem in doing so will be that of pulse pileup from the steep spectra involved. Such a bandwidth may be achieved by the next generation of spectrometers (Lin 1978). However, the very high spectral resolution obtainable with these (~100 channels) seems unnecessary. On the other hand, it should be borne in mind that the present analysis is only concerned with the purely numerical errors of trying to represent a function in a finite discretized band, and assumes that the $n$ bands are effectively decoupled. This is not satisfied by the Crannell et al. (1978) data since they estimate their channel spread at energy $e$ as $\Delta_e \approx 6(e\text{keV})^{1/2}$. Thus their lowest channels have $\Delta_e \approx 30$ keV $> e$, effectively reducing the number of independent channels. This problem should be overcome by the improved smearing characteristics of the next generation of instruments. Though results have only been presented here for linearly spaced channels, the method may readily be generalized to other spacings and utilized to optimize the channel division of the bandwidth in future spectrometers.
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Fig. 1.—Percentage truncation errors in the spectral moments \( \langle \epsilon^m \rangle \) (\( m = 1, 2, 3, 4 \)) of a bremsstrahlung source at temperature \( T_0 \), observed by a spectrometer of perfect spectral resolution but of finite spectral band \( \epsilon_A, \epsilon_B \). Dashed lines show error due to low-energy cutoff as a function of \( \epsilon_A/T_0 \) (lower abscissa) while solid curves show error due to upper cutoff as a function of \( \epsilon_B/T_0 \) (upper abscissa).

Fig. 2.—Percentage discretization errors in \( \langle \epsilon^m \rangle \) (\( m = 1, 2, 3 \)) for spectrometer as specified in Fig. 1 with \( \epsilon_A/T_0 \approx 0.1, \epsilon_B/T_0 = 3.5 \) but of finite spectral resolution, the \( n \) energy bands being uniformly spaced in \( \epsilon_A, \epsilon_B \). These errors are additional to truncation errors.

Conversely, the level of confidence for any existing data set can be usefully assessed by means of a moment analysis. (Note also that the moment method is not restricted to the exact form of eq. [1] used here but may be generalized to more complex \( \epsilon, T \) dependence such as adopted by Crannell et al. 1978). When such an assessment indicates inadequate spectral information, the best that can be done is to use all available a priori knowledge on the form of \( \epsilon(T) \) and to follow the usual model fitting procedures or preferably a regularization procedure (Craig 1977a). For instance, if there are independent observational and theoretical reasons for belief in an isothermal model (cf. Mätzler et al. 1978; Crannell et al. 1978; for example), then a temperature \( T_0 \) may be obtained by best-fitting which will be better defined than the \( \{T\} \pm \sigma(T) \) characterizing a moment analysis since the class of source functions has been restricted by a priori assumption. It is vital to remember, however, that the significance of such a "best fit" rests largely on the objectivity of the a priori "information" used, rather than on the spectral fit itself.

Finally some remarks are in order concerning the nonthermal problem (§ IIb). First, the choice between a thermal and a nonthermal spectral analysis is itself a matter of model selection and of use of independent data, since either type of source can fit the same spectrum (Brown 1971, 1974). Second, the Abel equation (2) is of a somewhat different character than (1) since \( \epsilon \) appears in the integral limit. This has the effect of making inversion of (2) somewhat more stable than (1) (Craig 1977b), particularly for the restricted class of source functions \( N(E) \) which decrease rapidly with increasing \( E \). Furthermore, only electrons of energy \( E > \epsilon_A \) contribute to a spectrometry in \( \langle \epsilon^m \rangle \). Thus Lin (1978) manages to reproduce some structure in model forms of \( N(E) \) for suitably smooth \( N(E) \) and suitably small errors in \( \langle \epsilon \rangle \). Though this success is somewhat dependent on assumed forms for \( N(E) \), the nonthermal case does possess a source of such a priori knowledge in the form of interplanetary electron spectra (Lin 1974). Nevertheless, the number of model parameters extractable is small regardless of the number of spectral points measured, sharp details in \( N(E) \) (high moments) being lost in \( \langle \epsilon \rangle \) through the filtering properties of (2). Third, if an a priori case exists for a monotonic decrease in \( N(E) \), the above type of moment analysis would best be modified since no peak exists within the spectrometer band (i.e., may be outside the range). A suitable procedure would be to write \( N(E) = E^{-\gamma}M(E) \) for some suitable index \( \gamma \), extract the power-law dependence, and conduct a moment analysis of the remaining factor \( M(E) \) to represent the superposed detailed structure.

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