The stability of sunspots

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Summary. An axisymmetric flux tube with a magnetic field in pressure equilibrium with an external, stratified plasma is stable if the radial component of the field decreases upwards at the interface. In the Sun, sunspots and pores with fluxes greater than about 10^{19} Mx are stable over the range where a vacuum field model is adequate. At greater depths an additional stabilizing mechanism is required.

1 Introduction

A sunspot can survive for several months, although the dynamical timescale (the time taken for an Alfvén wave to cross the spot) is only an hour. Sunspots must therefore be stable. They also tend to be round. Yet it is hard to demonstrate the stability of an axisymmetric magnetic flux tube in the solar photosphere (Parker 1975).

The field in such a flux tube spreads out towards the surface of the Sun and is concave towards the external, field-free gas. It is well known that a plasma contained by a magnetic field that is concave towards it is liable to interchange, or flute, instabilities. Parker (1975) and Piddington (1975) have therefore suggested that sunspots are intrinsically unstable. The problem is not quite so simple, for the field fans out owing to the vertical pressure gradient outside the flux tube. As the boundary becomes more inclined, the lighter magnetic region rests upon the denser gas outside: and such a configuration is stable to Rayleigh–Taylor interchanges. So the overall stability of the flux tube depends on the competing effects of curvature at its boundary and of the pressure stratification outside.

We shall show, using a simplified model, that a sunspot can be stable in and immediately below the photosphere. This stability can be related to the potential energy associated with the Wilson depression, as conjectured by Meyer et al. (1974) (though thermal effects are inessential). There is no need to invoke twisted fields (e.g. Piddington 1975), which have not been observed. Sunspots are round for the same reason that a bunch of tethered hydrogen balloons congregates into an approximately circular configuration.

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In Section 2 we consider the stability of a magnetic flux tube in equilibrium with an external, stratified field-free gas. Using the energy principle of Bernstein et al. (1958), we show that the configuration is stable to interchanges provided that the magnitude of the radial component of the field decreases upwards on the boundary of the flux tube. The corresponding conditions on the external pressure when there is a vacuum field inside the flux tube are investigated in Section 3. In Section 4 we demonstrate the existence of both stable and unstable vacuum field configurations.

A proper stability analysis requires a proper model of a sunspot, which in turn needs a theory of energy transport in a magnetic field. At the photosphere, however, the gas pressure is much less than the magnetic pressure in the umbra and the spot can be represented by a flux tube with a potential field in pressure equilibrium with the gas outside. This vacuum field model provides a reasonable approximation for magnetic pores though it cannot describe penumbral structure in a sunspot (Simon & Weiss 1970). At depths greater than a few hundred kilometres, however, the internal gas pressure cannot be ignored and the model becomes inadequate. In Section 5, we use this model to show that tubes with fluxes greater than about $10^{19}$ Mx are stable in the Sun. This result is related to observations of small-scale magnetic fields in Section 6 and to sunspot groups in Section 7.

Parker (1975) considered a sunspot model with the magnetic field concentrated into a shallow throat, below which it diverged. This configuration is unstable to interchanges at the throat, where the radial component of the field vanishes. It is reasonable to suppose that a perturbation which increases the local field strength thereby cools the gas and reduces the internal pressure (cf. Biermann 1941; Cowling 1976). Parker (1975) has suggested that interchanges may be stabilized by such a mechanism. But any configuration whose flute instability could be stabilized thus should be unstable to a radial symmetric contraction into another configuration for which any further contraction would increase the total internal pressure. Therefore it appears unlikely that the flute instability of a sunspot configuration can be stabilized by thermal effects.

We will show, however, that a model with a monotonically converging field can be stable throughout the region dominated by magnetic forces. Deeper down, where the magnetic pressure is less than the internal gas pressure, a second collar is required. This may be related to the formation of magnetic flux ropes by convection (Weiss 1964). A discussion of this problem requires a theory relating heat transport, the local magnetic field and convection.

### 2 The stability criterion

As an equilibrium model of a sunspot let us consider a flux rope with a magnetic field $B$ occupying the region $V_1$, and in equilibrium with the surrounding field-free plasma in the region $V_2$, separated from $V_1$ by the surface $S$, as shown in Fig. 1(a). We suppose that the gravitational acceleration $g$, acting in the negative $z$ direction, produces a pressure stratification $p(z)$ in the gas in $V_2$, with $dp/dz < 0$, and a corresponding density stratification $\rho(z)$. Let $p_1$ and $\rho_1$ be the pressure and density in $V_1$. Within the flux tube

$$\nabla \left( p_1 + \frac{B^2}{8\pi\mu} \right) = \rho_1 g + \frac{1}{4\pi\mu} (B \cdot \nabla) B,$$

while outside it

$$\nabla p = \rho g.$$

At the surface $S$ there is a current sheet and the magnetic field drops discontinuously to zero. Flux conservation implies that $S$ is made up of lines of force and continuity of normal
Bernstein et al. (1958) derived a stability criterion for a gravitationally layered plasma of constant density with no internal magnetic field in equilibrium with a vacuum magnetic field. We repeat their derivation below for the slightly more general equilibrium of the above sunspot model. We shall see there that the most dangerous perturbations to this equilibrium are driven by displacements of the surface $S$ and a local stability criterion is suggested by the following argument. Suppose that the surface $S$ is distorted as indicated in Fig. 1(b). After the perturbation the new equilibria in $V_1$ and $V_2$ will differ from the original ones. This difference however becomes arbitrarily small for such disturbances that allow a simple interchange of material and magnetic field on each side of $S$ leading to cancellation of first-order effects. Such distortions keep the total volumes of $V_1$ and $V_2$ unchanged and are constant along the fieldlines in $S$. In order that varying curvature and shear of the magnetic field do not interfere with the simple exchange the disturbances must be confined to a sufficiently close neighbourhood of $S$ by choosing the scale of the fluting sufficiently small. Let $n$ be the unit vector normal to $S$ pointing out of the field free plasma and let $\xi$ be the displacement of $S$. Then the configuration will be stable if there is a net restoring force on the displaced surface, i.e. if

$$|\xi \cdot n| n \cdot \nabla p < |\xi \cdot n| n \cdot \nabla \left( p_1 + \frac{B^2}{8\pi\mu} \right).$$

(2.4)
Hence the surface is stable if
\[ \mathbf{n} \cdot \left[ \nabla \left( p_1 + \frac{B^2}{8\pi\mu} \right) - \nabla p \right] > 0, \]  
(2.5)

where the gradients are evaluated on opposite sides of \( S \).

The same stability criterion can be rigorously derived from the energy principle of Bernstein et al. (1958). They considered an arbitrary adiabatic displacement \( \xi \), neglecting all dissipative terms, and obtained the condition that the second-order variation in the energy, \( \delta W \), must be positive for stability. This global stability criterion can be expressed in the form
\[ \delta W = \delta W_1 + \delta W_2 + \delta W_S > 0, \]  
(2.6)

where
\[ \delta W_1 = \frac{1}{2} \int_{V_1} \left[ \left( \frac{B^2}{4\pi\mu} + \rho \cdot \xi \cdot \nabla \right) \frac{1}{\rho} \cdot \nabla^2 + \frac{1}{\rho_1} \cdot \nabla \cdot \left( \rho_1 \cdot \xi \right) \right] \cdot \nabla p_1 \, d\tau, \]  
(2.7)

\[ \delta W_2 = \frac{1}{2} \int_{V_2} \left( p \cdot \nabla^2 \right) \left( \gamma - \frac{d \ln p}{d \ln \rho} \right) - \frac{1}{\rho} \cdot \frac{dp}{d\rho} \left[ \nabla \cdot \left( \rho \xi \right) \right] ^2 \, d\tau \]  
(2.8)

and
\[ \delta W_S = \frac{1}{2} \int_S \left( \mathbf{n} \cdot \xi \right)^2 \mathbf{n} \cdot \left[ \nabla \left( p_1 + \frac{B^2}{8\pi\mu} \right) - \nabla p \right] \, dS. \]  
(2.9)

Here the primes denote Eulerian perturbations; \( j \) is the electric current and \( \gamma \) is the adiabatic index. This decomposition of \( \delta W \) makes it possible to distinguish local instabilities in \( V_1 \) and \( V_2 \) from instabilities driven by distortions of the surface \( S \). The latter are the interchange instabilities with which we are concerned.

The integral over \( V_2 \) in (2.8) includes the effect of any superadiabatic stratification, which would render the plasma unstable to convection; if \( 0 < d \ln p / d \ln \rho < \gamma \) then \( \delta W_2 \) is always positive. The solar convection zone is of course unstably stratified but convective instability is irrelevant to our problem. We shall therefore ignore any instability that is driven by convection and assume that \( \delta W_2 > 0 \).

Similarly we assume that the plasma in \( V_1 \) is also stably stratified, so that along each line of force \( 0 < d \ln p_1 / d \ln \rho_1 < \gamma \). There remains the possibility that the magnetic field might be liable, for example, to kink instabilities. To eliminate this, we suppose that the flux rope is magnetohydrodynamically stable, so that the contribution from terms involving the magnetic field in (2.7) is always positive. Then \( \delta W_1 > 0 \) for all displacements \( \xi \).

With these assumptions any instability of the flux rope depends only on the surface integral in (2.9). A sufficient condition for stability of the field configuration is that the integrand should always be positive. If the integrand is somewhere negative, Bernstein et al. (1958) showed that it is always possible to consider incompressible interchanges with \( \nabla \cdot \xi = \xi \cdot \nabla \rho = 0 \), so that \( \delta W_2 = 0 \), and then to construct a localized displacement of \( S \) such that \( \delta W_S + \delta W_1 < 0 \). Hence the local criterion (2.5) is both necessary and sufficient for stability.

This criterion can be converted into a simple condition on the magnetic field at \( S \). Substitution of (2.1) and (2.2) into (2.5) gives
\[ \mathbf{n} \cdot \left[ \frac{1}{4\pi\mu} \left( \mathbf{B} \cdot \nabla \right) \mathbf{B} - \left( \rho - \rho_1 \right) \mathbf{g} \right] > 0 \]  
(2.10)

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(cf. equation (5.13) of Bernstein et al.). Moreover, the gradient

\[ \nabla \left( p_1 + \frac{B^2}{8\pi \mu} \right) - \nabla p = \frac{1}{4\pi \mu} (B \cdot \nabla) B - (\rho - \rho_1) g, \]  

(2.11)

from (2.1) and (2.2), and on the boundary \( S \) this vector must be normal to \( S \), from (2.3), and therefore parallel to \( n \). Gravity can be eliminated by taking the horizontal component of this vector. Let \( h \) be a horizontal vector pointing out of \( V_1 \) and into the field-free plasma. Then for stability

\[ h \cdot [(B \cdot \nabla) B] < 0, \]  

(2.12)

from (2.10) and (2.11): along any field line in \( S \) the magnitude of the component of \( B \) in any fixed outward horizontal direction must decrease upwards. If the flux rope is horizontally stratified so that \( p_1 = p_1(z) \) then, from (2.1), the stability condition (2.12) becomes \( h \cdot \nabla B^2 < 0 \) on \( S \).

The apparent stability of circular sunspots makes it natural to study axisymmetric meridional fields \((B_r, 0, B_z)\), referred to cylindrical polar coordinates \((r, \phi, z)\). Then, from (2.12), a necessary and sufficient condition for stability is that

\[ \frac{d}{dz} |B_{rs}| < 0, \]  

(2.13)

where \( B_{rs} \) is the value of \( B_r \) at the surface \( S \). As the pressure difference \((p - p_1)\) decreases, the field fans out. At the boundary \( B \) becomes more nearly horizontal, while \(|B|\) decreases: stability is decided by the balance between these competing effects.

It is sometimes convenient to express the stability criterion in terms of the external pressure \( p(z) \). Let \( \rho_c \) be the radius of curvature of a line of force on \( S \). Then, from (2.10),

\[ \frac{B^2}{4\pi \mu \rho_c} < (\rho - \rho_1) g \sin \chi, \]  

(2.14)

where \( \chi \) is the inclination of the surface to the vertical. Let

\[ \frac{\rho - \rho_1}{\rho} = \kappa \frac{p - p_1}{p} = \kappa \frac{B^2/8\pi \mu}{p}, \]  

(2.15)

with \( 0 < \kappa < 1 \) (if \( \rho_1 < \rho \) and \( T_1 < T \)). Then (2.14) becomes

\[ \frac{1}{\rho_c} < \frac{\kappa \sin \chi}{2H_p}, \]  

(2.16)

where \( H_p = -[d \ln p/dz]^{-1} \) is the pressure scale height in \( V_2 \). In particular, if \( \rho_1 < \rho \) and \( p_1 < p \) or if the temperature is continuous across \( S \) then \( \kappa = 1 \) and (2.16) reduces to

\[ \frac{1}{\rho_c} < \frac{\sin \chi}{2H_p}. \]  

(2.17)

This inequality shows explicitly that curvature produces instability and that the stabilizing effect of the pressure stratification depends on the inclination \( \chi \).

For a flux tube with radius \( R(z) \), the radius of curvature \( \rho_c = (1 + R'^2)^{3/2}/R'' \), where \( R' = dR/dz \) and \( R'' = d^2R/dz^2 \). Thus the stability criterion (2.17) can be written as

\[ \frac{R''}{R'(1 + R'^2)} < \frac{1}{2H_p}. \]  

(2.18)
3 The stability of an axisymmetric flux rope

We consider a circular flux tube with an axisymmetric meridional field $B$. Before investigating stability we have to set up an equilibrium model. At the photosphere the magnetic pressure is much greater than the gas pressure within a sunspot. We shall therefore neglect $p_1$. For this symmetry it then follows that the equilibrium field is current-free. This field can be described by a Stokes flux function $\psi(r, z)$ such that

$$B = \frac{1}{r} \left( -\frac{\partial \psi}{\partial z}, 0, \frac{\partial \psi}{\partial r} \right). \quad (3.1)$$

For a given pressure $p(z)$ and a fixed flux $F$ we should find the radius $R(z)$ of the surface $S$ and a flux function $\psi$ which, for a potential field, satisfies the equation

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (0 < r < R), \quad (3.2)$$

subject to the boundary conditions

$$2\pi \psi = F, \quad |\nabla \psi|^2 = 8\pi \mu r^2 \quad (r = R). \quad (3.3)$$

This is a free boundary problem and the radius $R(z)$ of the flux rope must be chosen to satisfy the boundary conditions (3.3). In principle, we could guess $\psi(z)$, solve (3.2) and then iterate to satisfy the pressure balance on the boundary, provided we also have conditions at some upper and lower boundary, e.g. at $z = \pm \infty$. We assume that $B$ behaves like a monopole field as $z \to \infty$; for a single sunspot flux returns sufficiently far from the axis for this to be a good approximation. Moreover, the observed stability of a sunspot is quite independent of what is going on around or above it; we shall see below in Section 5 that the stratification of the solar photosphere and chromosphere makes any flux rope locally very stable. The precise form of the upper boundary condition does not therefore affect the stability problem. The lower boundary condition is less straightforward; for the reasons mentioned in Section 1 we shall require the field to become vertical as $z \to -\infty$. In any case, this simple model does not apply at depths where the pressure gradient within a sunspot can no longer be ignored. If, however, the pressure inside the flux tube remains horizontally stratified ($p_1 = p_1(z)$) what is said in this section remains true if $p(z)$ is replaced by $p(z) - p_1(z)$ in equation (3.3). In particular the following discussion of stability can be performed in the same way.

Having obtained an equilibrium model we can easily discover whether $B_r$ decreases upwards on the boundary, and so determine its stability. For a given pressure distribution the only free parameter is the flux and we would like to find whether there is a critical flux $F_c$ that separates stable from unstable solutions. As we shall see, the exact value of $F_c$ is sensitive to the detailed pressure distribution immediately below the photosphere in the Sun, where $p(z)$ is not accurately known. However, we do not need to solve the full problem outlined above in order to show that a sunspot can be stable.

At the top of a sunspot, where $H_\theta \ll R$, the radius increases rapidly and the inclination $\chi$ approaches $\pi/2$. The field is approximately that of a monopole and

$$B_r \approx \frac{F}{2\pi R^2} \approx [8\pi \mu p(z)]^{1/2} \quad (3.4)$$

on the boundary. Hence

$$R(z) \approx \left( \frac{F^2}{32\pi^3 \mu} \right)^{1/4} p^{-1/4} \quad (3.5)$$
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which becomes infinite as \( p \to 0 \). From (3.4) we see that \( B_r \) decreases upwards and the configuration is therefore stable. From (3.5) the boundary becomes horizontal and the radius of curvature infinite as \( p \to 0 \). Hence the destabilizing effect is small and the criterion (2.17) is trivially satisfied.

At the bottom of the flux rope \( B_r < B_z \) and

\[
B_z \propto B \propto \frac{F}{\pi R^2} \propto [8\pi\mu p(z)]^{1/2}.
\]

(3.6)

Now \( R' = \tan \chi = B_r/B_z \) on \( \Omega \) and so

\[
B_r \propto R' B \propto -p^{-3/4} \frac{dp}{dz} = -4 \frac{d}{dz} (p^{1/4}).
\]

(3.7)

For marginal stability \( B_r \) is constant, so that \( p \propto (z_0 - z)^4 \) with some constant \( z_0 \). For stability the pressure must increase downwards more rapidly than \( (z_0 - z)^4 \). If \( p \) increases less rapidly the flux rope will be unstable at sufficiently great depths, where \( R' < 1 \), though it remains locally stable near the top, where \( R' \gg 1 \). Hence there is a transition from instability to stability. Near the marginal solution, for \( p \propto (z_0 - z)^4 \), we can investigate stability by an expansion for small \( R' \). This is carried out in Appendix A. Whether the next term in the expansion is stabilizing or destabilizing depends on the local values of the first four derivatives of \( p \).

Now the pressure stratification can be described by the pressure scale height \( H_p \) and the index \( m = - (dH_p/dz)^{-1} \). For sufficiently small flux tubes \( R \ll H_p \) and the field is nearly vertical. It follows from (3.7) that these small flux tubes are stable only if \( m > 4 \). We shall see in Section 5 that \( m < 4 \) in the solar photosphere. Small flux ropes are therefore unstable there, while large flux ropes are stable. So there must be a critical flux \( F_c \) separating stable from unstable flux tubes in the Sun.

To determine this critical flux it is necessary to consider the case with finite \( R' \). We have not attempted to calculate a full solution to the laborious free boundary problem posed by (3.2) and (3.3), partly because \( p(z) \) is not precisely known in the Sun. An alternative approach is to represent the magnetic field locally by exact solutions of equation (3.2), from which particular pressure distributions \( p(z) \) can be derived. This will be pursued in the next section, where we show that there exist stable solutions with \( R' \) of order unity even when \( m < 4 \). Then, in Section 5, we develop a crude model of a flux rope that can be used to estimate \( F_c \) in the Sun.

4 Examples of vacuum fields

In this section we construct equilibrium solutions by adopting a standard potential field over some range of \( z \) and taking the boundary \( R(z) \) to be some field line \( \psi = \) constant. The particular pressure distribution that would lead to this solution can then be found by evaluating \( |B| \) along the boundary. Exactly the same procedure can be used for two-dimensional plane fields; some results are summarized in Appendix B.

4.1 Bessel function model

The simplest separable solution of (3.2) is obtained by setting

\[
\psi = Ar J_1(kr) \exp (-kz),
\]

(4.1)

so that

\[
B_r = Ak J_1(kr) \exp (-kz), \quad B_z = Ak J_0(kr) \exp (-kz),
\]

(4.2)

where \( A, k \) are constants and \( 2\pi \psi(R, z) = F \). This solution corresponds to a flux rope like
that in Fig. 1(a) provided \(J_0(kr) > 0\). The corresponding pressure \(p(z)\) provides a fairly good representation of the solar photosphere (Simon & Weiss 1970) though it increases exponentially at great depths. At the boundary

\[
B_r = \frac{k\psi}{r} = \frac{kF}{2\pi R} \tag{4.3}
\]

and decreases upwards. Hence the field is everywhere stable. In this solution all lines of force are similar. Consider the flux tube defined by (4.1) with \(\psi\) a constant and suppose that the field becomes horizontal at \(z = z_0\), and that \(z_1 = z - z_0\). The normalized values of \(R, z_1, R', B^2, H_p\) and \(m\) are displayed in Table 1. This straightforward example shows that we can construct reasonable models of the magnetic field in a pore or sunspot that are not liable to interchange instabilities.

<table>
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<th>(kR)</th>
<th>(-kz_1)</th>
<th>(R')</th>
<th>((B/Ak)^2)</th>
<th>(kH_p)</th>
<th>(m)</th>
<th>((R_0/R))</th>
<th>((R'_0/R'))</th>
<th>((R''_0/R''))</th>
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### 4.2 DIPOLE AND MULTIPOLe FIELDS

In spherical polar coordinates \((s, \theta, \phi)\) axisymmetric separable solutions give multipole fields with

\[
\psi = \frac{A}{s^\nu} (1 - \cos^2 \theta) P'_\nu(\cos \theta) \quad (\nu > 1), \tag{4.4}
\]

where \(P_\nu(x)\) is the Legendre polynomial of degree \(\nu\). In particular, for the dipole solution, with \(\nu = 1\),

\[
\psi = \frac{A \sin^2 \theta}{s}, \quad B_r = \frac{3 \psi^3 \cos \theta}{A^2 \sin^3 \theta} \tag{4.5}
\]

and the field gives a model of a flux rope with \(B_z > 0\) for \(s > 0\), \(\cos \theta > 1/\sqrt{3}\). Hence \(B_r\) decreases monotonically with increasing \(\theta\) along a field line, and the configuration is stable.

For a monopole field occupying the cone \(|\theta| < \theta_0\), \(B_r\) varies as \(1/R^2\) and decreases upwards. For any \(\nu > 1\) (not necessarily an integer) we can construct multipole fields in the region \(|\theta| < \pi/\nu\). These provide a continuous sequence of model field configurations ranging from the dipole (\(\nu = 1\)) to the limit \(\nu = \infty\) which actually corresponds to the Bessel solution described above. It can then be proved that \(B_r\) decreases upwards for all these solutions in the region for which \(\theta > 0\).

In all these multipole solutions, \(B_z\) decreases to zero on the boundary at some finite value of \(z\), and the field lines return downwards. Below this level the field provides a reasonable model of a flux rope. The difference between these configurations and those satisfying the boundary conditions prescribed in Section 3 should be unimportant, for the reasons discussed there. All multipole fields have a singularity at some finite depth, where both \(B_r\) and \(B_z\) are infinite; in the Bessel solution this singularity is at \(z = -\infty\). These singularities are also irrelevant, for the vacuum field model cannot be applied at great depths in the Sun and,
in any case, we are most interested in instabilities produced near the surface, where the curvature is large.

4.3 FIELD OF A CURRENT LOOP

We can construct locally unstable solutions by making $B_r = 0$ on the plane $z = 0$. The simplest such field is that of a current loop of unit radius, shown in Fig. 2(a). Near the plane $z = 0$, $B_r$ increases from zero and any flux rope will be locally unstable. Far from the current loop the field is approximately that of a dipole and therefore stable. Near the current loop lines of force are almost circular and $B_r$ increases with the inclination. The surface separating the stable and unstable regions intersects the plane normally at $r = 1$ and meets the axis at $z = \sqrt{2/5}$. Analytical expressions for the field of a current loop involve elliptical integrals which can easily be evaluated (Martensen 1968, p. 36ff) to derive the shape of the separating surface.

It is possible to find an even simpler solution, corresponding to an azimuthal current distributed over the plane $\{r > 1, z = 0\}$ which shows a similar transition from stability to

![Diagram of a current loop](image)

Figure 2. (a) Field of a current loop (fieldlines from Fig. 10 of Martensen 1968). (b) Field of a perforated current sheet, given by equation (4.8). In the shaded regions flux tubes are unstable.
instability. We introduce oblate spheroidal coordinates \((u, v, \phi)\) such that
\[
    r = \cosh u \sin v, \quad z = \sinh u \cos v, \tag{4.6}
\]

Together with a scalar potential \(\Phi\) such that \(B = -\nabla \Phi\) and \(\nabla^2 \Phi = 0\). Now we restrict our attention to fields following the hyperbolas \(v = \text{constant}\), so that \(\Phi = \Phi(u)\) and

\[
    \frac{\partial}{\partial u} \left( \cosh u \sin v \frac{d\Phi}{du} \right) = 0. \tag{4.7}
\]

Hence \(d\Phi/du = -\text{sech} u\) and
\[
    B = \text{sech} u \left( \cosh^2 u - \sin^2 u \right)^{1/2} e_u, \tag{4.8}
\]

where \(e_u\) is the unit vector in the \(u\) direction (Parker 1975). This field is sketched in Fig. 2(b); at great distances from the origin it approximates to a monopole (rather than a dipole) field.

The radial field is zero in the plane \(u = 0\) (for \(v < \pi/2\)) and initially \(B_r\) increases upwards along each field line. From (4.8),
\[
    B_r = \frac{\tanh u \sin v}{(\cosh^2 u - \sin^2 v)^{1/2}}, \tag{4.9}
\]

and \(\partial B_r/\partial u = 0\) when
\[
    \cosh^2 u = \frac{1}{4} \left[ 3 + (9 - 8 \sin^2 v)^{1/2} \right]. \tag{4.10}
\]

This critical surface is shown in Fig. 2(b): it cuts the axis at \(z = 1/\sqrt{2}\) and intersects the plane \(z = 0\) at \(v = 1\). Any flux tube described by this field must be unstable near the throat, where the curvature is extreme and the pressure gradient itself (as well as the inclination \(\chi\)) is zero. Parker (1975) has shown directly that such a flux tube will be unstable to non-axisymmetric displacements in the plane \(z = 0\). However, the instability is confined to this anomalous region.

5 A crude model of a flux tube

A crude illustrative model of a cylindrical flux tube can be obtained by setting the flux
\[
    F = \pi R^2 \overline{B}, \tag{5.1}
\]

and assuming that the average field
\[
    \overline{B}(z) = B(R, z) = [8\pi \mu p(z)]^{1/2}. \tag{5.2}
\]

Then
\[
    R' = \frac{R}{4H_p}, \quad R'' = \frac{m + 4}{16m} \frac{R}{H_p^2} \tag{5.3}
\]

and, from (2.18), (5.1) and (5.2), the tube is stable if
\[
    F > F_c = 8\pi(8\pi \mu p)^{1/2} H_p^2 (4 - m)/m. \tag{5.4}
\]

At any level \(F_c\) provides an estimate of the minimum flux for which the rope is locally stable. This estimate is qualitatively adequate, though it cannot be justified by any consistent procedure (see Appendix A). Small flux tubes are unstable if \(m < 4\), as was shown in Section 3, but larger flux ropes, with \(R'\) finite, may be stable.

This crude model assumes that the average vertical field is equal to the value of \(|B|\) at the boundary. The variation of \(B_z\) with \(r\) is affected by two factors. First, the field increases
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more rapidly towards the axis when $R'$ is finite: this geometrical effect can be estimated by assuming that $|B|$ is constant over a spherical surface normal to the boundary. Secondly, curvature of the boundary enhances the field at the edge of the flux rope: the apparent reduction in total flux can be estimated by assuming that the curvature varies linearly across the spot. With these corrections, (5.2) becomes

$$\bar{B}(z) = B(R, z) \left[ 2 \left( 1 + R'^2 \right) - \left( 1 + R'^2 \right)^{1/2} \right] \left[ \frac{2\rho_c}{R} \left( 1 - \exp \left( \frac{R}{2\rho_c} \right) \right) \right].$$

(5.5)

The first factor is a monotonically increasing function of $R'$, varying from unit when $R' \approx 1$ (straight rope) to 2 when $R' \approx 1$ (monopole field). The second factor drops from unity to zero as the radius of curvature decreases. To second order in $R'$, equation (5.5) agrees with the expansion in Appendix A. This expression will be used in Section 6 to provide an estimate of the error in equation (5.2).

We can also investigate the accuracy of this crude model by comparing it with the exact Bessel function solution described above in Section 4. The pressure $p(z)$ corresponding to (4.2) can be inserted into (5.2) and $R(z)$ can then be found from (5.1). The ratios of the values of $R$, $R'$ and $R''$ derived from the crude model to those calculated from the exact solution are shown in the last three columns of Table 1. The crude model underestimates $R$, $R'$ and $R''$, and the error increases with decreasing $m$. Nevertheless, the agreement is sufficiently good to justify using this model to estimate the critical flux $F_c(z)$ in the solar atmosphere.

6 Sunspot models

The stability of a sunspot depends on the pressure stratification in the Sun. Near the surface of any star the atmosphere can be approximated by a plane layer with a pressure $p(z)$. We measure the height $z$ upwards from the reference level where the visual continuum optical depth $\tau_{0.5}$ at 0.5 $\mu$m is unity. It is convenient to introduce the pressure scale height

$$H_p = - \left[ d \ln p / dz \right]^{-1}$$

and the index

$$m = - \left[ dH_p / dz \right]^{-1}$$

(6.1)

(6.2)

as measures of the first and second derivatives of the pressure. Both $H_p$ and $m$ are functions of $z$; $n = m - 1$ is the index of the locally fitted polytrope. Simple equilibrium models of the magnetic field can be constructed by specifying $p$ and $H_p$ at a particular level but their stability is determined by $m$.

The variation of $m$ with $z$ is similar in all stars with outer convective zones. At the temperature minimum the scale height is also a minimum and $m$ is infinite; $m$ decreases rapidly downwards through the photosphere, reaching a value of 2.5, corresponding to a polytropic index $n = 3/2$, around $\tau_{0.5} = 1$, where the atmosphere becomes convectively unstable. Immediately below this level the superadiabatic gradient is high and $m$ falls to a minimum in the range $0.5 < m < 1.0$, around $z = -30$ km for the Sun. Then, as hydrogen becomes ionized, $m$ rises to a maximum of about 4.5 and finally settles down towards 2.5, the adiabatic value for a monatomic gas, deep in the convective zone. By then, however, variation of $g$ with radius has to be taken into account, so that $m$ remains greater than 2.5. This general behaviour of $m(z)$ is found in all models of the solar convective zone, though the precise values of $m$ vary from model to model. In Fig. 3(c) we show the variation of $m$ with $z$ for two theoretical models of the solar convection zone computed from mixing length theory with a hydrogen abundance $X = 0.7$, a metal abundance $Z = 0.02$ and a mixing length
$L = 1.1H_p$ and $L = 1.5H_p$ respectively. Both these models were fitted to the pressure and temperature of the Harvard–Smithsonian Reference Atmosphere (HSRA) (Gingerich et al. 1971) at $\tau_{0.5} = 0.79, z = +10.8 \text{ km}$. The fit is shown in Fig. 3(a). Data derived from the more recent atmospheric model of Vernazza, Avrett & Loeser (1976) (VAL) based solely on observations without use of an energy transport equation are also added in these diagrams.
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A good fit of theoretical models to the VAL data would require a mixing length of unrealistic magnitude which indicates that the temperature gradient in this model is inconsistent with the fact that the energy has to be transported almost exclusively by radiation near $\tau_{0.5} = 1$. The distribution of pressure with height for all these models is shown in Fig. 3(b). The differences between the models in these three diagrams indicate the observational and theoretical uncertainties in $p(z)$. We believe that for the deeper layers the uncertainty in the mixing length is larger than the observational uncertainties in the photospheric pressure and abundances. We choose the best fit $L = 1.1H_p$ as our standard model. For this model $m$ has a minimum value of 0.77 at $z = -33$ km.

Near the photosphere the internal pressure of a sunspot can be neglected. Therefore we shall restrict our discussion to the vacuum field model of a flux rope. This is adequate from the temperature minimum down to a depth of about 250 km below the photosphere, where the gas pressure corresponds to a magnetic field of about 3500 G (Simon & Weiss 1969). The pressure scale height increases from 100 to 300 km in this region. For $z < -250$ km the internal pressure of the gas within the flux rope can no longer be ignored. In any case, the treatment in this paper only explains the local stability of sunspots near the photosphere; a second lower collar is also needed to contain the flux at greater depths (Meyer et al. 1974).

In Section 3, we showed that a very small flux rope, with $R' \ll 1$, was unstable if $m < 4$. From Fig. 3 we see that $m < 4$ for $+30$ km $> z > -700$ km. Hence all sufficiently small flux tubes are unstable throughout a region extending at least 250 km below the photosphere. For these tubes, with almost vertical sides, the flute instability cannot be stabilized by the pressure gradient.

In larger flux ropes the field fans out towards the photosphere, so that $R' \gg 1$. Are they stabilized by this inclination? The stability criterion (2.17) can be written as

$$\rho_c > \frac{2H_p}{\sin \chi} = 2H_p \left( 1 + \frac{1}{R'^2} \right)^{1/2}.$$  

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Hence for $R' > 1$ the flux tube is locally stable if the radius of curvature $\rho_c \geq 3 H_p$. In a large spot we might expect $\rho_c$ to be comparable with $R$ and much greater than $H_p$; hence the spot should be stable. Consider, for example, a flux rope with a radius of $10^4\text{km}$ and, from (5.1), a flux of $10^{22}\text{mks}$. From (5.3), $\rho_c \approx 100 H_p$ and the rope is definitely stable. Provided $R'> 1$ the surface field is approximately that of a monopole and hence, as was shown in Section 3, the tube is stable.

A more elaborate discussion is needed in order to find the critical flux that separates stable from unstable flux tubes in the Sun. We can attempt to represent the fields by exact solutions of equation (3.2). For example, the Bessel function solution (4.1) has three arbitrary constants, which can be fixed by matching $p$ and $H_p$ at a particular level, and then specifying the flux (Simon & Weiss 1969). This provides a tolerable description of equilibrium configurations but cannot be used to investigate stability. Instead, we must match $p$, $H_p$ and $m$ at some level, so that the flux is uniquely determined. The corresponding flux for the dipole field (4.4) can similarly be found and any value of $F$ between these two fluxes can be represented by an appropriate multipole field.

We have shown that the flux ropes represented by these multipole and Bessel function solutions are all stable. However, the corresponding pressures $p(z)$ differ significantly from the solar stratification. Since $m$ decreases monotonically upwards along the boundary of the flux tube, the Bessel function solution cannot fully model the situation shown in Fig. 3. Nevertheless, it demonstrates the existence of stable configurations with $R'$ of order unity and $m < 4$.

To match the photospheric field we choose the level at which $m = 2.5$, where $z = +8.0\text{km}$, $(8\pi\mu p)^{1/2} = 1766\text{G}$, $H_p = 147\text{km}$. The Bessel function solution requires a flux of $3.9 \times 10^{19}\text{mks}$ and a radius of $862\text{km}$, while the corresponding dipole solution has a flux of $4.4 \times 10^{20}\text{mks}$ and a radius of $2481\text{km}$; in the latter case the central field of 6900 G is unrealistically high. If alternatively, we take the level at which $m = 1.3$, where $z = -15\text{km}$, $(8\pi\mu p)^{1/2} = 1896\text{G}$, $H_p = 158\text{km}$, then the Bessel function solution has $F = 1.04 \times 10^{20}\text{mks}$, $R = 1340\text{km}$.

A rough estimate of the critical flux can be obtained by constructing a family of flux tubes with radii $R(z) = \lambda R_B(z)$, where $\lambda$ is assumed to be constant for each flux tube and $R_B$ is the radius of the Bessel function solution, with flux $F_0$, matched to the standard atmosphere at $z = z_0$. From (2.18), a flux tube is locally stable if the parameter

$$Q(z) = \frac{2H_pR''}{R'(1 + R'^2)}$$

is less than unity. Let $Q(z_0) = Q_0$ for the Bessel function solution, with $R_B(z_0) = R_0$, $R_B'(z_0) = R'_0$. Then the similarity assumption implies that the critical flux tube has a radius $R_c = \lambda R_0$ at $z = z_0$, where

$$\lambda^2 = \frac{[Q_0(R_0^2 + 1) - 1]}{R_0^2},$$

and a flux $F_c \approx \lambda^2 F_0$. Matching at the photosphere with $m = 2.5$ gives $R_c \approx 386\text{km}$, $F_c \approx 7.9 \times 10^{18}\text{mks}$; matching at $m = 1.3$ gives $R_c \approx 678\text{km}$, $F_c \approx 2.4 \times 10^{19}\text{mks}$.

Alternatively, we may use the crude model of a cylindrical flux tube described in Section 5. This is not an exact solution of (3.2) and (3.3) but can be matched to any atmosphere. Given any $p(z)$, (5.1) and (5.2) yield $R(z)$ for any $F$, while the critical flux $F_c(z)$ can be obtained from (5.4). Fig. 4 shows $R(z)$ for tubes with fluxes between $10^{18}$ and $10^{20.5}\text{mks}$, matched by this crude procedure to our standard convective zone. The change from an almost vertical rope for small fluxes to a steeply inclined field at larger fluxes is clearly visible. Flux tube boundaries lying in the shaded region are locally unstable. This region
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Figure 4. Flux ropes in the Sun. The solid and broken lines show \( R(z) \) for the crude model of a flux rope, derived from (5.1) and (5.2) for \( L = 1 H_p \) and \( 1.5 H_p \) resp., for fluxes \( F = 10^{18}, 10^{18.5}, 10^{19}, 10^{19.5}, 10^{20}, 10^{20.5} \) mx. In the respective shaded region these flux ropes are unstable. The two thin lines are for the Bessel function solution (\( F = 10^{19.4} \) mx) and the dipole solution (\( F = 10^{20.4} \) mx) matched where \( m = 2.5 \). These solutions are always stable.

meets the axis where \( m = 4 \) and ropes with fluxes less than \( 10^{19} \) mx are unstable for the first 500 km below the photosphere. The region has its maximum extent around \( z = -30 \) km, where \( m \) is a minimum: at that level all ropes with fluxes less than \( 8 \times 10^{19} \) mx are unstable. When \( m = 2.5 \), \( R_c \approx 310 \) km and \( F_c \approx 5 \times 10^{18} \) mx; when \( m = 1.3 \), \( R_c \approx 650 \) km and \( F_c \approx 2.7 \times 10^{19} \) mx: these values are surprisingly close to those derived from the Bessel function solution. For comparison, the stable Bessel function and dipole solution, matched to \( p(z) \) where \( m = 2.5 \), are also shown in Fig. 4.

The error in Fig. 4 can be estimated by forming the ratio of \( \bar{B} \) from equation (5.5) to that from the cruder expansion (5.2) matched to the standard convection zone. For small fluxes (\( F \leq 10^{18} \) mx) the discrepancy is slight but the ratio drops to 0.7 around \( F \approx 10^{20} \) mx, owing to the curvature factor, and then rises gradually to 2 as the geometrical factor becomes important.

The results displayed in Fig. 4 are not exact but they are qualitatively correct. If \( p(z) \) were accurately known for the Sun, and (3.2) and (3.3) had been exactly solved, then a similar diagram would have been produced. In particular, the values of \( F_c \) are reliable to within a factor of 2. Hence tubes with fluxes greater than about \( 10^{20} \) mx should be stable throughout this sub-photospheric region, while ropes with less than about \( 10^{19} \) mx are unstable throughout most of this region. We might expect that a local instability confined to a narrow range in \( z \), small compared with \( (R_p \rho_c)^{1/2} \), would be stabilized by non-linear effects. Hence we estimate that the critical flux separating stable from unstable flux ropes is about \( 10^{19} \) mx.

7 Comparison with observations

So far we have only discussed the intrinsic stability of magnetic flux tubes. Any unstable stratification in the field-free region was explicitly ignored in Section 2. But pores and sunspots are embedded in a turbulent convective zone, where they are jostled and buffeted by individual granules. From Fig. 4 it is clear that the boundaries of tubes with fluxes less than \( 10^{19} \) mx are only slightly curved, so that relatively little energy can be released by interchanges; when \( F \approx 10^{19} \) mx the destabilizing effect of the magnetic field and, correspondingly, the stabilizing effect of the pressure stratification are much more powerful. Large flux tubes should therefore be able to withstand disturbances caused by external convection. On the other hand, a rope with a flux around \( 10^{19} \) mx, even if it satisfies the stability criteria derived above, may be torn apart by granular convection.
Recent observations have shown that more than 90 per cent of the magnetic flux outside pores and sunspots is concentrated into very small regions (Howard & Stenflo 1972; Frazier & Stenflo 1972). The properties of these small flux concentrations have been reviewed by Stenflo (1976). The field strength lies between 1500 and 2000 G (Frazier & Stenflo 1972; Stenflo 1973, 1975; Frazier 1974; Chapman 1974; Vrabec 1974) and the magnetic pressure therefore balances the external gas pressure. Mehltretter (1974) estimates that the typical flux is less than $5 \times 10^{17} \text{m}^2$, which corresponds to a radius of about 100 km. These small-scale magnetic fields are apparently associated with the filigree structure found by Dunn & Zirker (1973) and also with small-scale photospheric faculae (less than an arcsec in diameter) observed by Mehltretter (1974). These high-resolution observations provide an indirect means of relating magnetic fields to individual granules in the photosphere.

The filigree is made up of bright points and elongated crinkles, which lie in lanes between the granules (Dunn & Zirker 1973). The crinkles are moved about by the granules. Occasionally they form a ring around a few granules, which survives for about 10 min before breaking up into isolated crinkles or points. The bright points themselves appear or disappear in a few minutes and survive for 5–15 min. Small micropores, often ringed by crinkles, sometimes appear between the granules and last for 10–20 min (Mehltretter 1974). The magnetic knots observed near sunspots (Beckers & Schröter 1968; Harvey & Harvey 1973; Vrabec 1974) have fluxes of about $3 \times 10^{19} \text{m}^2$; these magnetic knots, like pores, survive for several hours while sunspots, with fluxes greater than about $10^{20} \text{m}^2$, have lifetimes ranging from a few days to several months. These periods should be compared with the lifetime, about 8 min, of an individual granule.

Apparently the lifetime of a magnetic feature increases with the flux (cf. Stenflo 1976). Sunspots evolve over a timescale of days and are affected by adjacent supergranules (Meyer et al. 1974). Pores have lifetimes characteristic of supergranular convection. We suggest that only ropes with fluxes greater than about $10^{19} \text{m}^2$ are stable to the disturbances produced by granules. Micropores, with fluxes between $10^{18}$ and $10^{19} \text{m}^2$ and lifetimes of 10–20 min, probably indicate the transition from stable to unstable configurations. Smaller flux concentrations are dominated by photospheric convection. They are formed by individual granules sweeping flux aside but, being unstable, spread irregularly through the lanes between the granules (cf. Fig. 7 of Dunn & Zirker 1973). Unless a flux approaching $10^{19} \text{m}^2$ is brought together the field is passively advected and magnetic structures dissolve after a period comparable with the granules’ lifetime but much greater than the dynamical timescale for a slender flux rope. Further observations are needed to confirm this distinction between stable and ephemeral configurations.

The gross stability of a large sunspot, determined by the surface integral in (2.6), should not be affected by small scale granulation. However, the spreading penumbral field may be liable to local interchanges driven by convection outside the flux rope. Indeed, it is likely that convective elements from below may penetrate through the penumbra to radiate their energy at the photosphere. Such penetration, starting at the edge of the sunspot and spreading inwards to form a bright filament, would be consistent with the observations of Muller (1973).

8 Sunspot groups

A circular sunspot with its axis vertical is stabilized by the component of the vertical pressure gradient acting normal to its boundary. Magnetic buoyancy tends to make a flux tube vertical but if the axis of a spot is inclined then the field on its inner side will be more nearly vertical, as shown in Fig. 5(a), so that the stability criterion (2.17) cannot be satisfied locally. Hence, if the flux rope below the photosphere is sufficiently inclined a sunspot will
be liable to interchange instabilities on its inner side and should decay, or be torn apart by supergranular convection. It follows that long-lived sunspots should be roughly circular in appearance.

![Diagram of an inclined flux tube](image)

**Figure 5.** (a) Instability of an inclined flux tube. (b) Sketches showing stages in the development of a sunspot group; the leading (westward) spot is to the right.

This suggests an explanation for the predominance of leading spots in groups. A sunspot group occupies a roughly oval region, with spots at the leading and trailing edges embracing pores and other smaller magnetic features within the region (see, e.g. Bray & Loughhead 1964). The asymmetry between leading and following spots must be related to differential rotation and various authors (e.g. Alfvén 1950; Ponomarenko 1970; Piddington 1975; Wilson 1975) have attempted to explain it.

A bipolar group is formed by a stitch of field rising from deep in the convective zone (Parker 1955; Schmidt 1968; Ponomarenko 1970). If we assume that the observed systematic difference in latitude between the leading and the following spots is caused by action of Coriolis forces on a rising and horizontally expanding blob of gas, it is also natural to assume that the rising stitch takes the form of an \( \Omega \) rather than a \( \cap \), as shown schematically in Fig. 5(b). The angular velocity apparently decreases with radius in the convective zone (Gilman 1974; Foukal 1972; Foukal & Jokipii 1975) and this retrograde motion tends to tilt the flux rope near the surface, making the leading side nearly vertical and the following side yet more steeply inclined. As a result, the leading spot is stabilized while the follower becomes more liable to decay. It may be significant that groups often show lines of pores trailing inwards from the spots.

9 Conclusion

Magnetic fields dominate the structure of a sunspot at and immediately below the photosphere. The vacuum field model is a reasonable approximation for this region and we have shown that a flux tube can be stable to adiabatic perturbations provided that the field converges sufficiently rapidly with depth. If the tube has a throat immediately below the photosphere, as suggested by Parker (1975), then it must be unstable. The survival of sunspots indicates that the fields are compressed underneath them.

It is conceivable that this stability might extend to depths as great as 2000 km if the fields there are stronger than those observed at the surface, as suggested by the sunspot models of...
Deinzer (1965) and Yun (1970). (Moreover, if granules produce fields of 1600 G then super-granules may be able to produce 10,000 G.) However, we expect the fields to become more nearly vertical, so that $|B_r|$ cannot increase indefinitely with depth. At some level, therefore the flux rope must be unstable. For fluxes less than about $10^{19}$ m$^2$s$^{-1}$ this level approaches the surface, and even for a sunspot the instability has been submerged but not removed.

At depths of several thousand kilometres the curvature is slight and the flute instability may be stabilized at finite amplitude. Nevertheless, a lower collar is needed to contain the field. At these depths the magnetic pressure is smaller than the thermal pressure in the flux rope and the thermal timescale is comparable with the dynamical timescale (Parker 1975). Any treatment of equilibrium or stability therefore requires a proper theory of convection and energy transport in a magnetic field. Such a calculation may confirm the existence of a collar. Alternatively, it may be necessary to introduce a twisted field at greater depth (Piddington 1975).

These conclusions are not restricted to the Sun. In any star with a hydrogen convection zone dynamo action may generate magnetic flux, which can be concentrated to form a stable spot. However, the likelihood of a flux rope remaining vertical depends on the magnitude of differential rotation. The estimated velocity of convection deep in the solar convective zone is about 0.1 km/s, which is comparable with the proper motions of sunspots and the differential rotation. In a rapidly rotating star, differential rotation will be faster and flux ropes may be so inclined that spots are generally unstable. When the Sun first reached the main sequence, before its rotation had been braked, sunspots may have been infrequent.

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References

Appendix A: Expansion for small flux tubes

In Section 3 we found a local stability criterion valid in the limit when $R' \to 0$ and the radius is given by (3.6). It is natural to extend this to finite $R'$ in the hope of demonstrating that finite inclinations stabilize the field. Unfortunately, though the algebra is complicated, the results are inconclusive. So we need only summarize them here.

Within the flux tube the field is described by a flux function $\psi$, which can be expanded as a Taylor series in $r$. At the boundary,

$$B(z) = [8\pi \mu p(z)]^{1/2}$$

and

$$\frac{1}{m} = \frac{1}{2} \left(1 - \frac{BB''}{B^2}\right)$$

where primes denote differentiation with respect to $z$. With the aid of equation (3.2) it can be shown that

$$F = \pi R^2 B \left[1 - \frac{1}{m} R^2 \right] + 0(R^6),$$

where

$$R'^2 = \frac{1}{4} \frac{B'}{B^2} R^2 + 0(R^4) = \frac{1}{4} \left(\frac{F'}{B^2} + 0(R^4)\right).$$

Since $\rho_c = (1 + R'^2)^{3/2}/R''$, the stability criterion (2.17) can be rewritten as

$$\frac{R''B}{R'B'} + 1 + R'^2 > 0.$$  \hfill (A5)

To zero order in $R'$, (A5) simply yields the condition that $m > 4$.

To proceed to higher order for a given $B(z)$ it is convenient to expand in terms of the small parameter $\epsilon$, such that

$$\epsilon^2 = \frac{\left(\frac{F'}{B^2}\right)}{\pi} B'^2 \ll 1,$$  \hfill (A6)

suggested by (A4). We also introduce the dimensionless quantities

$$\lambda = \frac{BB''}{B'^2} = 1 - \frac{2}{m}, \quad \mu = \frac{B^2 B''}{B^3}, \quad \nu = \frac{B^3 B'''}{B^4}.$$  \hfill (A7)
Then
\[ R = \left( \frac{F}{\pi B} \right)^{1/2} \left[ 1 + \frac{1}{16} e^2(1 - \lambda) + O(e^4) \right] \tag{A8} \]
and, after some algebra, (A5) becomes, to order \( e^2 \)
\[ \lambda - 1 \pm \frac{1}{2} e^2 [2
\begin{align*}
&= 2v - 11\mu + 32\lambda - 2\mu\lambda - 17] > 0. \tag{A9}
\end{align*}
This expression depends, through \( \nu \), on \( p'''' \) and is therefore sensitive to slight changes in the pressure stratification. We consider three special cases.

1 POLYTROPIC ATMOSPHERE

This is the case when \( m \) and \( \lambda \) are constant and for \( m > 0 \) (A9) reduces to
\[ \left( m - 4 \right) + \frac{e^2}{4m^2} \left[ 2m^3 - (m + 4)(3m + 8) \right] > 0. \tag{A10} \]
For \( 0 < m < 4 \), the field is unstable for all values of \( e \); for \( 4 < m < 4.51 \), it is stable if
\[ e^2 < \frac{4m^2(m - 4)}{(m + 4)(3m + 8) - 2m^3}, \tag{A11} \]
and for \( m > 4.51 \) the field is stable for all \( e \). As (A11) indicates, to this order finite inclination actually destabilizes the field when \( m = 4 \).

2 HYPERBOLIC FIELD

To show that finite inclination may stabilize the field, we consider the configuration described by oblate spheroidal coordinates in Section 4. For infinitesimal flux tubes, the transition from instability to stability occurs at \( r = 0, z^2 = \) when \( m = 4 \). For finite inclinations the transition occurs with
\[ \delta m = (m - 4) = -4e^2 = -16R'^2 < 0. \tag{A12} \]

3 SOLAR ATMOSPHERE

We would like to establish whether the Sun behaves like a polytropic atmosphere or like the hyperbolic field. Unfortunately, the stratification shown in Fig. 3 yields large values of \( |\nu| \) which are poorly determined, especially around the minimum in \( m \). However, it seems that, to this order, the field would be strongly stabilized for \( z > -50 \) km and destabilized below, where \( m \) varies more slowly.

Equation (3.2) is elliptic and \( R(z) \) cannot be locally determined. Hence stability depends on the global stratification and the expansion procedure is unlikely to be useful. We must either solve (3.2) and (3.3) exactly for a particular \( p(z) \) or else rely on the crude model described in Section 5. That model provides qualitatively adequate results, though (5.4) is an inconsistent approximation to (A5).

Appendix B: Two-dimensional fields

Consider a magnetic field \( \mathbf{B} = (B_x, B_z) = (-\partial \psi/\partial z, \partial \psi/\partial x) \) referred to plane Cartesian coordinates \( (x, z) \), where the flux function \( \psi(x, z) \) is harmonic in the region \( |x| < X(z) \) and
\[ \psi(x, z) = \frac{1}{2} F. \] From (2.12), the field is stable if
\[ \frac{d}{dx} |B_x| < 0 \] on the surface \( x = X. \)

The solution separable in \( x \) and \( z \) turns out to be marginal. Let
\[ \psi = A \sin kx \exp(-kz): \] then the horizontal field
\[ B_x = Ak \sin kx \exp(-kz) = k\psi \] which is constant along the boundary. The corresponding pressure,
\[ p(z) = A^2 k^2 \exp(-2kz)/(8\pi\mu), \] is independent of \( F \). This corresponds to an atmosphere with \( m \) infinite, so two-dimensional configurations are less stable than cylindrical flux tubes.

Nevertheless, two-dimensional multipole fields, with
\[ \psi = As^{-n} \sin n\theta \] in plane polar coordinates \( (s, \theta) \) have \( dB_k/d\theta < 0 \) on a field line throughout the region where \( B_z > 0 \), and they are therefore stable. The solution for two line currents at \( z = \pm d \) is
\[ \psi = A[\ln[(x + d)^2 + z^2] - \ln[(x - d)^2 + z^2]]. \] This shows a transition from instability to stability like that described in Section 4. Many other exact solutions are available in this geometry.