Radiative equilibrium of highly nongray atmospheres. I. General analysis

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The radiative equilibrium of nongray atmospheres whose nongrayness is very small-scale in frequency ν, although it may be as strong as desired, is discussed. It is shown that one can use a statistical approach to describe the very small-scale dependence of the intensity on ν, having simultaneously preserved the information on the large-scale variations of the intensity with ν. A new concept introduced in this paper—the partial radiation intensity—lies at the basis of this description. It is shown that if the opacity probability distribution function does not vary over the spectrum and with depth, then the temperature profile \( T^4 = T^4(\tau) \) satisfies a homogeneous integral equation having a symmetrical difference kernel which generalizes Milne’s equation.

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INTRODUCTION

Milne’s problem on the calculation of the temperature distribution in a plane-parallel gray atmosphere in local thermodynamic equilibrium (LTE) serves as the principal standard in the theory of stellar atmospheres. It is assumed that the atmosphere is in radiative equilibrium and the radiation field is produced by a source lying at infinitely deep layers. This problem reduces to the well-known Milne linear integral equation for \( T^4 = T^4(\tau) \)

\[
T^4(\tau) = \frac{1}{8\pi} \int E_1(|\tau - \tau'|) T^4(\tau') d\tau',
\]

where \( \tau \) is the optical depth and \( E_1(\tau) \) is the exponential integral function. Various methods — analytical, numerical, and approximate — have been developed for its solutions; the solution has been derived in explicit form, studied in detail, and tabulated (for example, see refs. 1–4).

The radiative equilibrium of a class of nongray LTE atmospheres is investigated in this paper for which it is possible to reduce the calculation of their thermal structure to an integral equation for \( T^4(\tau) \), which generalizes Milne’s equation,

\[
T^4(\tau) = \frac{1}{8\pi} \int K_1(|\tau - \tau'|) T^4(\tau') d\tau'.
\]

Here \( \tau \) is some mean optical depth. The kernel function \( K_1(\tau) \) is normalized to unity on the interval \((0, \infty)\) and can be represented as a superposition of exponential functions:
\[ K(\nu) = \int e^{-\alpha(\nu)} G(\nu) d\nu, \]

wherein the form of \( G(\nu) \) is defined by a specific model of the absorption coefficient's frequency dependence. The class of atmospheres being investigated includes the picket-fence model and its generalizations which have been discussed in the literature.

The principal assumptions about the nature of the non-grayness reduce to the following. In the first place, it is assumed that it is very small scale in the frequency \( \nu \), so that the absorption coefficient \( k_\nu \) varies rapidly in a random or regular way; however, the mean value of \( k_\nu \) over a large enough frequency interval is constant across the spectrum. The amplitude of the fluctuations in the opacity can be as large as desired; however, their statistical properties should remain constant as one goes from one part of the spectrum to another. The exact meaning of these statements will be explained below. In the second place, it is assumed that the frequency dependence of \( k_\nu \) is the same at all depths, i.e., it is assumed that the variables \( \nu \) and \( \tau \) are separated in \( k_\nu \).

We note that Eq. (2) for the temperature profile in a non-gray atmosphere agrees in form with the thoroughly studied equation for the source function in the problem of the conservative scattering of photons of a spectral line, which occurs with complete redistribution in the frequency. Of course, the physical meaning of the solutions in these two problems is completely different.

Numerous lines and absorption bands make a large, and often dominant, contribution to the opacity in the atmospheres of late-type stars. On account of this situation, the maximum values of the absorption coefficient are larger by several orders of magnitude than the minimum values, and one should expect qualitative differences in the temperature profile from the gray case. The class of models introduced in this paper permit the investigation of these differences by analytic means. This investigation will comprise the content of a second article.

**MODEL FOR THE ABSORPTION COEFFICIENT**

In the calculation of the radiative equilibrium of non-gray LTE atmospheres everything is determined essentially by the nature of the wavelength dependence of two functions — the Planck intensity \( B_\nu(\tau) \) and the absorption coefficient \( k_\nu \). A characteristic frequency interval, \( \Delta \nu_1 \) and \( \Delta \nu_2 \), in which the corresponding function varies significantly, is associated with each of these functions. Let us assume that \( \Delta \nu_2 \ll \Delta \nu_1 \), i.e., that the absorption coefficient varies rapidly compared to \( B_\nu \) as a function of the frequency. Such a situation is typical for atmospheres having temperatures, roughly speaking, below \( 8000^\circ \)K, in which the spectrum is chopped up by numerous lines and molecular bands.

Let \( (\nu, \nu + \Delta \nu) \) be an arbitrary section of the spectrum of width \( \Delta \nu \) such that \( \Delta \nu_1 \gg \Delta \nu \gg \Delta \nu_2 \). We will assume that the average value of \( k_\nu \) over such a section — let us call it \( k \) — does not depend on \( \nu \), so that

\[
\frac{1}{\Delta \nu} \int_\nu^{\nu+\Delta \nu} k_\nu d\nu', \quad \Delta \nu_1 \gg \Delta \nu \gg \Delta \nu_2.
\]

Thus the atmosphere is considered to be gray on the average. The opacity at any specific frequency can differ as much as desired from the mean value. In other words, the variations in \( k_\nu \) are considered to be very small-scale in \( \nu \), but no restrictions are placed on their amplitude.

Let us set \( k_\nu = k_\nu(\nu) \), so that \( \alpha_\nu \) is the frequency profile of the absorption coefficient, and we assume that \( \alpha_\nu \) does not depend on depth (i.e., on density and temperature). One can consider the function \( \alpha_\nu \) to be, within the limits of the averaging interval \( (\nu, \nu + \Delta \nu) \), the application of some random process. Let \( p(\alpha) d\alpha \) be a probability distribution that a value \( \alpha_\nu \) for an arbitrary frequency selected at random from the averaging interval lies between \( \alpha \) and \( \alpha + d\alpha \). Thus \( p(\alpha) d\alpha \) is the fraction of the spectral section located near frequency \( \nu \) of any width \( \Delta \nu \) such that \( \Delta \nu_1 \gg \Delta \nu \gg \Delta \nu_2 \) in which \( \alpha_\nu \) lies between \( \alpha \) and \( \alpha + d\alpha \).

The normalization of \( p(\alpha) \) is as follows:

\[
\int p(\alpha) d\alpha = 1.
\]

The form of the function \( p(\alpha) \), which is often called the (differential) opacity probability distribution function (OPDF) or the opacity distribution function (ODF), generally varies along the spectrum. We will assume — and this is the last assumption of our model — that \( p(\alpha) \) does not depend on \( \nu \) (of course, it also does not depend on \( \tau \)), since we are assuming that \( \alpha_\nu \) does not vary with depth. The form of the dependence of \( k_\nu \) on \( \nu \) which is compatible with this assumption can be most varied: regular, for example, periodic (molecular band), irregular (random overlapping of lines), and the regions of periodic variation can be replaced with sections of irregular fluctuations, and so on. The point is that the frequency independence of the double-point distribution functions which determine the correlations between the values of \( \alpha_\nu \) at different \( \nu \) does not follow from the constancy along the spectrum of the single-point distribution function \( p(\alpha) \).

Substituting \( k_\nu = \int k_\nu(\nu) d\nu \) into Eq. (4), we obtain

\[
\frac{1}{\Delta \nu} \int \alpha_\nu d\nu' = 1.
\]

According to the meaning of the function \( p(\alpha) \), the contribution to this interval made by sections in which \( \alpha_\nu \) lies between \( \alpha \) and \( \alpha + d\alpha \) is equal to \( \alpha p(\alpha) d\alpha \). Integrating this expression over \( \alpha \), we find in place of Eq. (6)

\[
\int p(\alpha) d\alpha = 1.
\]

The units in which the opacity is measured are specified by this condition.

The harmonic (Rosseland) mean

\[
\alpha_R = \left( \int p(\alpha) \frac{d\alpha}{\alpha} \right)^{-1}
\]

also plays an important role. It is easy to see that \( \alpha_R \leq 1 \). It is sufficient to use the Cauchy inequality

\[
\left( \int g f d\alpha \right)^2 \leq \int g^2 d\alpha \int f d\alpha,
\]

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setting $g(\alpha) = \alpha p(\alpha)$ and $f(\alpha) = p(\alpha) / \alpha$ in it and taking Eq. (7) into account.

We set

$$G(z) = \int_a^b x^p p(x) dx.$$  

This is the function $G(x)$ which determines the kernel function $K_1(\tau)$ of the principal integral of the equation for $T^H(\tau)$ [see Eq. (3)]. According to Eqs. (9) and (3), under the assumptions made above concerning the absorption coefficient, $K_1(\tau)$ and this means $T^H(\tau)$, are completely determined by the function $p(\alpha)$. The details of the absorption coefficient's frequency dependence, which is not affected by the form of $p(\alpha)$, are not reflected in the depth distribution of the temperature.

**PARTICULAR CASES, PARAMETRIZED MODEL**

It is obvious for the gray case that

$$p(\alpha) = \delta(\alpha - 1),$$  

and therefore, according to Eq. (9), $G(x) = 1$, $|x| \leq 1$ and $G(x) = 0$, $|x| > 1$, so that Eq. (3) gives $K_1(\tau) = E_1(\tau)$, as it should.

The simplest nontrivial case is Chandrasekhar’s picket-fence model. An absorption coefficient for which

$$p(\alpha) = p_1 \delta(\alpha - \alpha_1) + p_2 \delta(\alpha - \alpha_2),$$  

where $p_1$ and $q$ are constants, is so named. By virtue of Eqs. (5) and (7), these constants are subject to the conditions

$$p_1 + p_2 = 1, \quad p_1 \alpha_1 + p_2 \alpha_2 = 1,$$

so that Chandrasekhar’s picket-fence model is a two-parameter model. This was historically the first stochastic model of the absorption coefficient. No specific dependence of the absorption coefficient on frequency is assumed here. In addition, $\alpha_p$ is considered to be the application of some random process, the only restriction on which is that this single-point distribution function is of the form of Eq. (11) (so that this random process is not necessarily time-independent).

Figure 1 illustrates several specific $\alpha_p$, which are different applications of the same Chandrasekhar picket fence (for which $p_1 = p_2 = 1/2, \alpha_1 = \alpha_2/3 = 1/3$).

An obvious generalization of the preceding model introduced by E. Hopf is an absorption coefficient for which

$$p(\alpha) = \sum_{i=1}^l p_i \delta(\alpha - \alpha_i),$$

whereby

$$\sum_{i=1}^l p_i = 1, \quad \sum_{i=1}^l \alpha_i p_i = 1.$$  

It is assumed in V. A. Ambartsumyan’s model that the absorption coefficient is some periodic function of rather short period, the explicit form of which is not specified. The distribution of the absorption coefficient values within the limits of a single period then defines the form of the function $p(\alpha)$, which remains arbitrary in this model.

And so, stringent restrictions are imposed on the form of $p(\alpha)$ and relatively mild ones on the absorption coefficient's frequency dependence in picket-fence models; on the other hand, in V. A. Ambartsumyan's model $p(\alpha)$ is arbitrary but rigorous restrictions are imposed on the form of the frequency dependence of $\alpha_p$. A synthesis of the strong points of both these groups of models—an arbitrary $p(\alpha)$, as V. A. Ambartsumyan does, and an arbitrary frequency dependence of the absorption coefficient—compatible with this $p(\alpha)$, as S. Chandrasekhar does—will lead to that general model of the absorption coefficient which was formulated in the preceding section of this article. This general model was first introduced, insofar as we are aware, by J. Steward, I. Kusce, and N. McCormick. Later it was rediscovered by A. Arking and B. Grossman. A lot of attention was given in their paper to finding $p(\alpha)$ for idealized models of molecular bands both of a regular sort (the model of Elsasser et al.) and with a random arrangement of the lines within a band.

The opacity probability distribution function for actual stellar atmospheres varies both along the spectrum (at scales of the order of $\Delta \lambda$) and with depth. It was suggested in a paper of S. E. Strom and R. L. Kurucz to use such functions in computer calculations of model atmospheres. After this, there appeared a significant number of articles devoted to determining these functions for conditions characteristic of stellar atmospheres (refs. 13-15 and others).

In the case of an analytical investigation, we must restrict ourselves to consideration of a model in which $p(\alpha)$, does not vary either along the spectrum or with depth. Moreover, it is useful in studies of the qualitative nature of the effects caused by large departures from grayness to introduce a family of functions $p(\alpha)$ which depend on a small number of parameters with whose help one would be able to approximate rather well the actual opacity probability distribution functions. It is easy to understand that it is necessary to have available no more than three parameters. Two parameters, just as in the picket-fence model, are necessary to indicate what fraction of the spectrum is occupied by lines and what their average strength is relative to the continuum. The third parameter describes, roughly speaking, the shape of a typical line.
However, all this should not be taken too literally: in the case of a statistical description, the division into continuum and line spectrum does not arise, and the concept of an individual line loses its meaning.

One can suggest the following expression

$$p(\alpha) = A_0(\alpha - \alpha_0) + B_0(\alpha - \alpha_0)^2(\alpha - \alpha_0),$$

(15)

as a convenient parametrized model, where $\alpha_0$ and $\alpha_0'$ are the minimum (continuum) and maximum value of the absorption coefficient, respectively, and $\phi(x)$ is the unit jump function: $\phi(x) = 0, x < 0, \phi(x) = 1, x \geq 0$. The quantity $A$ is in fact the fraction of the spectrum occupied by the continuum, and the parameter $\gamma (0 \leq \gamma \leq 1)$ describes the relative contribution to the energy transfer made by photons having different mean free paths (i.e., the shape of a typical line, if one is thinking in traditional terms). The smaller $\gamma$ is, the greater the role played in the energy balance by photons having large mean free paths. Since $p(\alpha)$ should satisfy the normalization relation (5) and the condition (7), Eq. (15) contains three free parameters.

We note that the behavior of $p(\alpha)$ for large values of $\alpha$ affects the temperature distribution only in the outer layers, and the thermal structure of the inner regions of the atmosphere is insensitive to it. Therefore, a discrepancy between the model and actual functions $p(\alpha)$ at large values of $\alpha$ does not play a significant role. Good agreement between them for small values of $\alpha$ is far more important. It ensures taking proper account of the contribution of photons with large mean free paths, which determine the principal features of the temperature distribution.

STANDARD FORM OF THE BASIC EQUATIONS

In order to find the depth dependence of the temperature $T = T(\tau)$ in a plane-parallel, seminfinite LTE atmosphere in radiative equilibrium, it is sufficient to solve simultaneously the equation of radiative equilibrium

$$\int_0^\infty B_\nu(T(\tau))T(\tau)\,d\nu = \frac{1}{2} \int_0^\infty T(\tau)\,d\nu \int_0^\infty I(\tau, \mu)\,d\mu$$

(16)

and the radiative transfer equation

$$\alpha(z) I(\tau, \mu) = \alpha(z) I(\tau, \mu) - B_\nu(T(\tau))$$

(17)

with the boundary condition $I(0, \mu) = 0, \mu \leq 0$. Here $\tau$ is the optical depth: $d\tau = -kd\nu$, where $\tau$ is the height in the atmosphere and $k$ is the frequency mean value of the absorption coefficient defined by Eq. (4); $\alpha$ is the cosine of the angle between the direction of the radiation's propagation and the outer normal to the atmosphere; $I(\nu)$ is the radiation intensity; and $B_\nu$ is the Planck function:

$$B_\nu(T) = \frac{2hv^3}{c^2} \left( \exp \left( \frac{hv}{kT} \right) - 1 \right)^{-1}.$$

The assumptions about the absorption coefficient which were made above do not in fact figure in Eqs. (16) and (17). If one were to use these assumption, one could reduce the nonlinear problem of determining $T(\tau)$ from Eqs. (16) and (17) to the linear integral Eq. (2) for $T^4(\tau)$. It works out well to do this, because many details of the frequency dependence of $\alpha(\nu)$, which are completely taken into account by Eqs. (16) and (17), are insignificant for calculating the temperature distribution.

The main idea in replacing Eqs. (16) and (17) by the simpler ones consists of the rejection of superfluous information, insofar as finding $T(\tau)$ is concerned, about the exact positions of the lines and of a transition from a complete description of the very small-scale frequency structure of the radiation field to a less informative statistical description. The concept of partial radiation intensity, which is introduced in the next section, lies at the basis of this description.

PARTIAL RADIATION INTENSITY

We will start from the guiding considerations. Photons of similar frequencies, which have practically the identical energy, play a significantly different role in the heat transfer process if their mean free paths differ significantly. When the absorption coefficient fluctuates rapidly, this is usually the situation. In this case, the use of the frequency as the parameter by which photons are classified is not adequate for the physics of the problem: nearness of frequencies does not guarantee the nearness of mean free paths important to the transfer process. Meanwhile, it is not possible to reject completely the use of frequency, since equality of mean free paths alone does not make photons equally valuable for heat transfer — if their frequencies differ greatly, the energy being transported is significantly different. A compromise is attained in the fact that the frequency is retained to describe the smooth variations occurring at a scale of the order of $\Delta \nu_1$, while a supplementary classification of photons by their mean free paths is introduced for a statistical description of the radiation field's very small-scale structure in $\nu$.

We will formalize these concepts. Let us consider a section of the spectrum $(\nu, \nu + \Delta \nu)$ such that $\Delta \nu \ll \Delta \nu_1$. One can neglect the energy difference of photons within these limits, assuming it to be equal to $h\nu$ for all the photons. Let us in addition of $\Delta \nu \gg \Delta \nu_1$, and let us denote by $p(\alpha) I(\nu, \mu, \alpha) d\alpha d\nu$ the contribution to the integral

$$\int I(\nu, \mu, \alpha) d\alpha d\nu,$$

made by those photons of energy $h\nu$ whose inverse mean free paths (measured in units of $k$) lie between $\alpha$ and $\alpha + d\alpha$. We will call $I(\nu, \mu, \alpha)$ the partial radiation intensity. It is obvious that

$$\int I(\nu, \mu, \alpha) d\alpha d\nu = \Delta \nu \int p(\alpha) I(\nu, \mu, \alpha) d\alpha, \quad \Delta \nu \gg \Delta \nu_1.$$ (18)

The appropriateness of introducing the factor $p(\alpha)$ in the definition of the partial intensity will become clear from the following.

It may seem that describing the radiation field with the help of the partial radiation intensity only complicates matters, since the number of variables on which $I(\nu, \mu, \alpha)$ depends is greater than for the usual intensity $I(\nu, \mu)$. This impression is erroneous. Since $I(\nu, \mu)$ oscillates rapidly with a change in $\nu$ in order to characterize the radiation field in the spectral interval $(\nu, \nu + \Delta \nu)$, it is necessary to know the intensity at an enormous number of
frequencies in this interval. In describing the field with the help of the partial intensity it is necessary to know $I(\nu, \mu, \alpha)$ for a single $\nu$, which characterizes the interval $(\nu, \nu + \Delta \nu)$ as a whole, and for several values of $\alpha$ in all, since the dependence on $\alpha$ is smooth.

**AN ALTERNATIVE FORM OF THE BASIC EQUATIONS**

The equation of transfer for the partial intensity is of the same form as the usual equation of transfer:

$$\frac{dI_\nu(\nu, \mu, \alpha)}{d\nu} = \alpha[I_\nu(\nu, \mu, \alpha) - B(\nu)]$$

(19)

It is derived from consideration of the balance of the energy transported by photons having inverse mean free paths ($\alpha + \Delta \alpha$) in the spectral interval $(\nu, \nu + \Delta \nu)$ with the condition that $\Delta \nu_1 \gg \Delta \nu \gg \Delta \nu_2$. Here and everywhere below the mean free paths are taken to be dimensionless, expressed in units of $\kappa^{-1}$. Let us turn our attention to the absence of the factor $p(\alpha)$ in front of $I_\nu(\nu, \mu, \alpha)$ in Eq. (19). This circumstance is explained as follows. The energy emitted under LTE conditions by a unit volume per second per unit solid angle in the spectral range $(\nu, \nu + \Delta \nu)$ is equal, obviously, to

$$B(\nu) = \frac{\omega}{\kappa} \int p(\alpha) d\alpha.$$

Let us separate out from it that part which is emitted in energy in the form of photons having inverse mean free paths ($\alpha + \Delta \alpha$). If we take $\Delta \nu_2 \gg \Delta \nu_1$, then the mean range $\Delta \nu$ in which the value of $\Delta \nu$ is confined between $\alpha$ and $\alpha + \Delta \alpha$ is equal to $p(\alpha) d\alpha$. Therefore, the energy emitted by 1 cm$^3$ in 1 sec per unit solid angle in the form of photons having inverse mean free paths ($\alpha + \Delta \alpha$) in the spectral range $(\nu, \nu + \Delta \nu)$ of width $\Delta \nu$ such that $\Delta \nu_1 \gg \Delta \nu \gg \Delta \nu_2$ is $B(\nu) \kappa p(\alpha) d\alpha$. The factor $p(\alpha)$ is cancelled out in the derivation of the transfer equation introduced here by the same factor in terms containing $I_\nu(\nu, \mu, \alpha)$ (which is why it was introduced in the definition of the partial intensity).

Let us now derive the radiative equilibrium equation written with the use of $I_\nu(\nu, \mu, \alpha)$. In accordance just with what has been stated in the discussion of the derivation of the equation for $I_\nu(\nu, \mu, \alpha)$, one can write the energy lost by a unit volume per second in the frequency interval $(\nu, \nu + \Delta \nu)$ as $4\pi B(\nu) \Delta \nu \kappa \int p(\alpha) d\alpha$. Integrating this expression over all large-scale frequencies $\nu$ (energies) and taking Eq. (7) into account, we find that the total energy loss rate per unit volume due to radiation is equal to

$$\kappa 4\pi \int B(\nu) d\nu.\quad \text{Incidentally, since one can obviously}
$$

**REPRESENT THIS QUANTITY AS**

$$4\pi \int B(\nu) d\nu,$$

we see that

$$\kappa = \int B(\nu) d\nu / \int B(\nu) d\nu.$$

Thus if $p(\alpha)$ satisfies the condition (7), then $\kappa$ is the Planck mean.

According to the definition of the partial intensity, the rate of energy increase per unit volume due to absorption of radiation is equal to

$$\alpha 2\pi \int \int p(\alpha) I_\nu(\nu, \mu, \alpha) d\alpha d\nu.$$

Therefore, one can write the radiative equilibrium condition with $E_\nu$ also in the form

$$S(\nu) = \frac{1}{2} \int \int p(\alpha) I_\nu(\nu, \mu, \alpha) d\alpha d\nu,$$

(20)

where we have denoted

$$S(\nu) = \frac{\sigma}{\kappa} T(\nu) = \int B(\nu) d\nu,$$

(21)

and $\sigma$ is the Stefan–Boltzmann constant. This equation preserves its form when $\sigma_\nu$ varies with $\nu$ in such a way that $\sigma_\nu$ remains unchanged. The form of the equation of transfer for one partial intensity changes significantly when $\sigma_\nu = \sigma_\nu(\nu )$. Travelling through space in free flight, a photon maintains its frequency; however, its mean free path $1/\sigma_\nu$ changes, generally speaking, due to the variation of $\sigma_\nu$ with depth. This circumstance leads to the appearance of additional terms in the equation for $I_\nu(\nu, \mu, \alpha)$.

The equation of transfer Eq. (19) with the boundary condition $I_\nu(0, \mu, \alpha) = 0$, $\mu < 0$, together with the radiative equilibrium equation in the form of Eq. (20), give a set of conditions alternative to Eqs. (16) and (17) which are sufficient to obtain $T = T(\nu)$. Since our special assumptions about the absorption coefficient were used in the derivation of Eqs. (19) and (20), they have a narrower range of applicability than the general Eqs. (16) and (17), but in return they are simpler than them.

**REDUCTION TO A LINEAR INTEGRAL EQUATION**

We denote by $I(\nu, \mu, \alpha)$ the partial radiation intensity integrated over the spectrum

$$I(\nu, \mu, \alpha) = \int I_\nu(\nu, \mu, \alpha) d\nu.$$

(22)

Eqs. (19) and (20) give

$$\frac{dI(\nu, \mu, \alpha)}{d\nu} = \alpha(I(\nu, \mu, \alpha) - S(\nu)),$$

(23)

$$S(\nu) = \frac{1}{2} \int \int p(\alpha) I(\nu, \mu, \alpha) d\alpha d\nu.$$

(24)

Introducing the formal solution of Eq. (23) with the boundary condition $I(0, \mu, \alpha) = 0$, $\mu < 0$, into Eq. (24), we obtain the desired equation for $S(\nu)$, i.e., actually, for $T(\nu)$:

$$S(\nu) = \frac{1}{2} \int K_\nu(|\tau - \tau'|) S(\tau') d\tau'$$

(25)

with the kernel function

$$K_\nu(\tau) = \int E(\nu, \mu, \alpha) p(\alpha) \alpha^2 d\alpha.$$

(26)

Substituting here the explicit expression for the exponen-
tial integral function $E_1$ and changing the order of the integration, we find for $K_1(\tau)$ the form Eq. (3) cited earlier with $G(z)$ specified by Eq. (9). We have from Eqs. (26), (5), and (7)

$$\int K_1(\tau) d\tau = 1, \quad \int K_1(\tau) \tau d\tau = -\frac{1}{\mu}.$$  \(27\)

We note that $I(\tau, \mu, \alpha)$ does not depend on the variables $\mu$ and $\alpha$ separately but only on their combination $z = \mu/\alpha$, which represents the projection of a photon's mean free path onto the $\tau$ axis. This circumstance follows directly from Eq. (23) for $I(\tau, \mu, \alpha)$. Having designated $I(\tau, z) = I(\tau, \mu, \alpha)$, $z = \mu/\alpha$, we can represent Eq. (24) as

$$S(\tau) = \frac{1}{2} \int I(\tau, z) G(z) dz.$$  \(28\)

The multiplicative constant which sets the accuracy to which the solution of Eq. (25) is determined is fixed, as always, by specifying the radiative flux $\pi F$ or the effective temperature $T_0$:

$$\pi F = \alpha_\nu T_0^4 = 2\pi \int \mu d\mu \int p(\alpha) I(\tau, \mu, \alpha) d\alpha,$$  \(29\)

which can also be rewritten in the form

$$F = 2 \int I(\tau, z) G(z) z dz$$  \(30\)

or

$$F = 2 \int S(\tau') K_1(\tau - \tau') d\tau' - 2 \int S(\tau') K_1(\tau - \tau') d\tau',$$  \(31\)

where

$$K_1(\tau) = \int K_1(t) dt.$$  \(32\)

Equations of this class coincide with the equations describing the conservative scattering of photons of a spectral line in a semi-infinite atmosphere with a source at infinity, which occurs with total redistribution over frequency (for example, see refs. 16 or 5, Ch. VI). The far-reaching formal similarity of the problems of radiative heat transfer in LTE media having a rapidly-varying absorption coefficient and of multiple scattering of photons in a line has a simple physical basis: In both cases complete redistribution of photons with respect to mean free paths occurs during the process of interacting with matter.

The partial intensity integrated over the spectrum $I(\tau, \mu, \alpha)$ was introduced already in papers of the 1930’s.\(^4\leftarrow6\) Then the integral equation (2) for $T^4(\tau)$ was derived,\(^6\) which was subsequently rediscovered in ref. 17. Simplifications associated with the fact that $I(\tau, \mu, \alpha)$ does not depend on $\mu$ and $\alpha$ separately but only on $z = \mu/\alpha$ were pointed out in ref. 9. However, this fact was not employed in numerous papers which subsequently appeared (for example, see refs. 18–20) on the application of the case method\(^21\) to the solution of radiative heat transfer problems for the picket-fence model, which hindered their solution and made it un-

justifiably cumbersome. As far as the energy dependence of the partial intensity $I_p(\tau, \mu, \alpha)$ is concerned, this concept is new to the best of our knowledge.

We note that $I_p(\tau, \mu, \alpha)$ also depends in fact on $z$ and not on $\mu$ and $\alpha$. If we set $I_p(\tau, z) = I_p(\tau, \mu, \alpha)$, $z = \mu/\alpha$, then Eq. (19) takes the form

$$z \frac{dI_p(z)}{d\tau} = I(z) - B_\nu (T(\tau)).$$  \(33\)

One should not confuse the function $I_p(\tau, z)$ with the usual radiation intensity, which is in fact denoted just the same $I_p(\tau, \mu)$.

After Eq. (25) is solved, i.e., the temperature field in the atmosphere is found, it is easy to calculate the radiation field in it. If there is detailed information about the frequency dependence of the absorption coefficient, i.e., $\alpha_\nu$ is known, the radiation intensity is obtained as the solution of the transfer Eq. (17) with the known $T(\tau)$. Incidentally, the less detailed description of the radiation field given by the partial intensity may turn out to be sufficient. When $T(\tau)$ is known, it is easy to find the partial intensity from Eq. (19) or Eq. (33). We emphasize that one can do this when $p(\alpha)$ is known and there is no more detailed information about $\alpha_\nu$. In other words, having available only some statistical information on the absorption coefficient, it is possible to calculate not only the temperature field but also the statistical properties of the radiation field in the atmosphere.

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