NON-LTE TRANSFER—II
TWO-LEVEL ATOM WITH STOCHASTIC VELOCITY FIELD

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SUMMARY

Following previous work on LTE stochastic transfer (Auvergne et al.; Frisch), transfer with incoherent scattering is considered for two-level atoms in the presence of turbulent velocity fields with finite eddy-size. Assuming that the velocity along each individual photon path may be represented by a Markov Process in time, we obtain a non-stochastic equation of transfer for the 'mean conditional intensity', from which the mean (observable) intensity may be recovered by integration over the velocity distribution. An integral equation of the Wiener-Hopf type is obtained for the 'mean conditional source function' from which it is shown by techniques introduced in Paper I (Frisch & Frisch) that the rms surface source function assumes the usual value $\sqrt{\epsilon B}$ for uniform thermal source $B$. An 'effective kernel and source function' approximation is introduced by which a standard transfer problem is recovered and worked out explicitly in a special case. It is shown that finite eddy-size effects can change the effective source function and the emergent profile by a factor 2 or more.

I. INTRODUCTION

The equation of transfer, for two-level atoms with complete frequency redistribution, in the presence of a velocity field may be written (Hummer & Rybicki 1968)

$$\Omega \cdot \nabla I = -k(r) \phi(x - \Omega \cdot v)(I - S) \quad (1.1)$$

$$S(r) = \epsilon B + (1 - \epsilon) \int \int \phi(x' - \Omega' \cdot v) I(r, x', \Omega') \, dx' \, \frac{d^2\Omega'}{4\pi} \quad (1.2)$$

$I$, the specific intensity, is a function of position $r$, frequency $x$ in thermal velocity (Doppler) units (thermal velocity uniform unless otherwise stated) and direction $\Omega'$; $S$ is the source function, $B$ is the thermal source function, $\phi$ the line profile normalized to unity and $\epsilon$ the collisional de-excitation probability. The absorption coefficient $k(r)$ has the form $N_1 B_{12} \nu / 4\pi$ ($N_1$ population of lower level, $B_{12}$ Einstein coefficient, $\nu$ frequency). The stimulated emission term $N_2 B_{21} \nu / 4\pi$ has been neglected. When this is not justified the problem becomes highly non-linear. We assume that the absorption coefficient $k(r)$ is given non-random. $v$ (in thermal velocity units) is a prescribed turbulent velocity field, assumed homogeneous and isotropic. A half-infinite space with no incident radiation and with thermal equilibrium at infinity ($I = B$) is assumed. We wish to compute the mean emergent intensity and are particularly interested in the following question: given the mean
emergent profile, what can we infer about the typical eddy-size of the stochastic velocity field?

Recent papers (Auvergne et al. 1973; Gail et al. 1974; Schmid-Burgk 1974; Gail & Sedlmayr 1974; Frisch 1975) have analysed the local thermodynamic equilibrium (LTE) case $S = B$ (corresponding to $\epsilon = 1$), assuming a space dependent stochastic velocity field with finite eddy-size. The usual macro- and microturbulent situations were recovered as limiting cases. The essential idea which allows the calculation of the mean intensity at LTE, is that a photon, once created, can only suffer absorption, but not scattering; therefore the intensity at $P$, for photons originating from $M$ is a functional only of the velocity field between $M$ and $P$ (see Fig. 1). If along each individual line of sight it is assumed that the velocity is a Markov Process, then the joint process $(v, I)$ will also be Markov, enabling one to compute the mean intensity by Fokker–Planck techniques (Frisch 1968; Gail et al. 1974).

![Fig. 1. Path of escaping photons in the LTE case.](image)

An attempt has been made recently by Traving (1975)* to extend this idea to non-local thermodynamic equilibrium (Non-LTE). What changes does Non-LTE introduce in the above picture? A photon, once created, may suffer an arbitrary number of scatterings in a three-dimensional (3-D) region (see Fig. 2). Since there is no 3-D analogue of a Markov process, it is hard to think of an exactly soluble model with 3-D velocity fields. Suppose, however, one considers the somewhat academic case, where photons can move only up and down ($\mu = \pm 1$) and the velocity depends only on the distance $z$ to the free surface; is it then possible to solve the problem when $v$ is a Markov Process? Because of back-scatterings, the intensity at a point $P$ is now a functional of the velocity everywhere, so the joint process $(v, I)$ is not Markov and the Fokker–Planck technique breaks down, contrary to the statement made in Traving (1975). There is, however, another method, namely the invariant imbedding or Riccati approach (Hummer & Rybicki 1967; Mihalas 1970; Engibaryan & Mnatsakanyan 1975). By this method the two-point boundary problem of transfer is transformed into a one-point boundary problem for the reflection and transmission operators; it becomes then possible again to use the

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* This paper contains a misleading statement at the end of Chapter III in connection with the paper by Auvergne et al. (1973): the method to handle $\tau$-dependent absorption coefficient, correlation length, thermal velocity, etc., is described in the Appendix of Auvergne et al. and worked out numerically in Frisch (1975); it is not at all related to the 'rather complicated' and 'bulky' expansion given by equations (3.30)–(3.32) of Traving.

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Markov property, but this procedure is exceedingly heavy if no approximation is made; a simplified version has been recently developed by Magnan (1975b). We wish to present now another approach based on the idea of a time-dependent velocity field, which allows again the use of Fokker–Planck techniques.

2. FROM SPACE TO TIME-DEPENDENT VELOCITY FIELD FOLLOWING PHOTON PATH

The equation of transfer being linear, there is no need to describe simultaneously the random velocities felt by each photon; it suffices to consider the fate of one particular photon. Indeed, the average over \( N \) photons will just be \( N \) times the average over one photon.

For a given photon created at \( r_0 \) at time \( t_0 \), with frequency \( \omega_0 \) and direction \( \Omega_0 \), we can define a time-dependent velocity \( \mathbf{v}(t) = \mathbf{v}(r(t)) \), where \( r(t) \) is the random\(^*\) position of the photon at time \( t \).\(^\dagger\) In principle it is very hard to obtain the statistical properties of \( \mathbf{v}(t) \), because the photon random walk is modified by the velocity field. To make the problem tractable, we assume that the statistical properties of \( \mathbf{v}(t) \) can be determined by neglecting the effect of the velocity on the photon path. This will be shown, \( a \ posteriori \), to be consistent with our results, particularly because we are not interested in a detailed statistical description of \( \mathbf{v}(t) \), but only in some overall statistical features. Specifically we ask for the probability density of \( \mathbf{v}(t) \) and for a typical correlation time, defined, e.g. as an integral time scale

\[
\tau_{\text{corr}} = \int_0^\infty \langle r(t) \cdot r(0) \rangle \, dt / \langle \mathbf{v}^2 \rangle, \tag{2.1}
\]

where angular brackets denote averaging over realizations. We must relate these quantities to the probability density \( P(\mathbf{v}) \) of \( \mathbf{v}(r) \) (also called the velocity distribution) and to a typical correlation length \( l_{\text{corr}} \) of \( \mathbf{v}(r) \), called here the 'eddy-size'.\(^\ddagger\)

\(^*\) Random refers here to the ordinary probabilistic interpretation of transfer as given, e.g. by Sobolev (1963), not to the random velocity field.

\(^\dagger\) The intrinsic time variation of the velocity field can be neglected whenever \( |\mathbf{v}| \ll c = \) speed of light.

\(^\ddagger\) We shall also use the mean number of eddies per unit optical depth \( n = 1/(kl_{\text{corr}}) \), where \( k \) is the absorption coefficient.
Since the turbulence is homogeneous, the probability density of \( v(t) \) is equal to the space-independent velocity distribution \( P(v) \), which is generally taken gaussian. To obtain the correlation time, one could argue that since photons move with the speed of light, the correlation time is given by

\[
t_{\text{corr}} = l_{\text{corr}}/c,
\]

which is just the time required to cross a typical eddy. This choice may appear questionable if \( l_{\text{corr}} \) is much larger than the photon mean free path at line centre, since photons will then undergo many diffusions before escaping (see Fig. 3);

![Diagram](image)

**Fig. 3.** Path of photons created at \( r_0 \), escaping from an eddy of size \( l_{\text{corr}} \) by a random walk with typical step \( a \).

should then not the correlation time be much larger? This would indeed be true if photon diffusion could be considered as an ordinary random walk with steps of finite variance; in that case \( t_{\text{corr}} \) would be given by

\[
t_{\text{corr}} = l_{\text{corr}}^2/4c,
\]

where \( a \) is a typical step-size. Actually, as shown for example by Field (1959), this picture is basically incorrect when there is complete frequency redistribution with a profile extending to infinite frequencies. Indeed, photons are emitted at \( r_0 \) with the distribution \( \phi(x) \), but as time elapses, this distribution broadens out with a half-width varying as \( \phi^{-1}(1/(ct)) \) where \( \phi^{-1} \) is the inverse function of \( \phi \) and \( c \) the absorption coefficient. Therefore the size of the photon-steps increases and the escape-path looks like in Fig. 4. A photon of frequency \( x \) has a mean free path before diffusion of the order of \( 1/(k\phi(x)) \). To estimate the time of escape, we notice that at \( t \approx l_{\text{corr}}/c \), the typical photon steps become now of the order of \( l_{\text{corr}} \). Therefore, at that time, photons have an appreciable probability to escape in one step. Since the time of escape can obviously not be smaller than \( l_{\text{corr}}/c \), we conclude that, within a numerical factor, the straight motion formula (2.2) is essentially correct. It is seen that this result is independent of the profile \( \phi(x) \), although the numerical factor may depend on \( \phi \); in particular it is not affected if \( \phi \) is replaced by a microturbulent profile. This justifies, \textit{a posteriori}, the assumption we made to neglect to effect of the turbulence on the overall statistical features of \( v(t) \).
3. REFORMULATION AS A TIME-DEPENDENT TRANSFER PROBLEM

Together with the time-dependent stochastic velocity field obtained in the previous section, we must of course use the time-dependent stochastic equation of transfer

\[
\frac{1}{c} \frac{\partial I(t, r, x, \Omega)}{\partial t} = -\nabla I - k(r) \phi(x - \Omega \cdot v(t))(I - S)
\]  
(3.1)

\[
S(t, r) = \varepsilon B(t) + (1 - \varepsilon) \int \phi(x - \Omega' \cdot v(t)) I(t, r, x', \Omega') \, dx' \frac{d^2 \Omega'}{4\pi}.
\]  
(3.2)

In addition, the initial intensity \(I_0 = I(0, r, x, \Omega)\) must be prescribed. Since we have taken the thermal source \(B\) time-independent, (3.1) will have a statistically stationary solution which is clearly independent of \(I_0\). We are interested in the mean of this stationary intensity, which is an observable quantity.

In order to obtain closed equations for mean quantities, without restricting ourselves to the macro- or microturbulent limits, we take \(v(t)\) to be a Markov Process (Uhlenbeck & Ornstein 1930; Wang & Uhlenbeck 1945; Bharucha-Reid 1960; Feller 1966). A stationary Markov Process can be characterized by its transition probability density \(P_{tr}(v, t|v')\) such that \(P_{tr} d^3v\) is the probability to find a velocity \(v\) (within \(d^3v\)) at time \(s + t\), given a velocity \(v'\) at time \(s\). The transition probability density satisfies the Fokker–Planck equation

\[
\frac{\partial P_{tr}(v, t|v')}{\partial t} = \int \Pi(v, v') P_{tr}(v', t|v') \, d^3v',
\]  
(3.3)

where \(\Pi\) is called the Fokker–Planck operator (Feller 1966). The stationary solution of this equation is the velocity distribution \(P(v)\). For diffusion processes (in the probabilistic sense) \(\Pi\) is a second-order differential operator; e.g. for the Uhlenbeck–Ornstein Process used by Gail et al. (1974) it reads

\[
\Pi_{\text{UO}} = \frac{1}{l_{\text{corr}}} \sum_{i=1,3} \left\{ \xi_i^2 \frac{\partial^2}{\partial v_i^2} + \frac{\partial}{\partial v_i} (\psi_i') \right\},
\]  
(3.4)

where \(\xi^2\) is the rms turbulent velocity measured in Doppler units. For the Kubo–Anderson Process used by Auvergne et al. (1973) which is a jump process, the
Fokker–Planck operator is

$$\Pi_{KAP}(v, v') = -\frac{1}{\tau_{corr}} \{\delta(v - v') - P(v)\}, \quad (3.5)$$

where $\delta$ is the Dirac distribution and where $P(v)$ will be taken gaussian. In both (3.4) and (3.5) $\tau_{corr}$ is the integral time scale defined by (2.1) and is related to the spatial correlation length by $\tau_{corr} = l_{corr}/c$. In the sequel the theory will be worked out for general Markov Processes; explicit examples will use the Kubo–Anderson Process.

4. AN EQUATION OF TRANSFER FOR THE MEAN CONDITIONAL INTENSITY

In order to exhibit the mathematical structure of the problem, it is convenient to eliminate the source function between (3.1) and (3.2) to obtain (putting $c = 1$

$$\frac{\partial I(t, r, x, \Omega)}{\partial t} = -\mathbf{\nabla} I - k(r) \phi(x - \Omega \cdot v(t)) \left\{ I(t, r, x, \Omega) - (1 - \epsilon) \right\}$$

$$\times \int \phi(x' - \Omega' \cdot v(t)) I(t, r, x', \Omega') \, dx' \frac{d\Omega'}{4\pi}$$

$$+ \epsilon k(r) \phi(x - \Omega \cdot v(t)) B. \quad (4.1)$$

Symbolically, this equation may be written

$$\frac{\partial I}{\partial t} = M(v(t)) I + L(v(t)) B, \quad (4.2)$$

where $M(v(t))$ and $L(v(t))$ are linear operators depending deterministically on the Markov Process $v(t)$. We are now ready to apply to (4.2) the standard technique for differential equations with Markov coefficients (Frisch 1968). The crucial observation is that the joint process $(\Omega(t), I(t))$ is also a Markov Process, of which the probability density $\mathcal{P}(\Omega, I; t)$ satisfies

$$\frac{\partial \mathcal{P}}{\partial t} = \Pi \mathcal{P} - \frac{\partial}{\partial I} \left\{ [M(v) I + L(v) B] \mathcal{P} \right\}. \quad (4.3)$$

The first operator on the rhs expresses the change between $t$ and $t + dt$ of the probability density of the velocity (cf. 3.3) and the second operator the change in the probability of the intensity which evolves according to (4.2), with a prescribed velocity at time $t$ (it is just a continuity operator).

The initial condition is

$$\mathcal{P}(0) = P(v) \delta(I - I_0) \quad (4.4)$$

where $P(v)$ is the velocity distribution and $I_0$ an arbitrary initial intensity (e.g. 0), which will not contribute to the stationary solution.

We now define the 'mean conditional intensity'

$$\mathcal{F}(t|v) = \frac{1}{P(v)} \int \mathcal{P}(v, I; t) I \, dI. \quad (4.5)$$

It is seen that $\mathcal{F}(t|v)$ is the mean intensity at time $t$, knowing that $v(t) = v$; indeed,
\( \mathcal{P}(v; t)/P(v) \) is the conditional probability density of \( I \), knowing \( v \). The mean observable intensity is obtained from \( \mathcal{P} \) by averaging over \( v \):

\[
\langle I \rangle = \int \mathcal{P}(t|v) P(v) \, dv.
\]  

(4.6)

From (4.3) a closed equation for the mean conditional intensity is obtained after multiplication by \( I \), integration over \( I \) and integration by parts of the last term

\[
\frac{\partial \mathcal{P}}{\partial t} = \Pi_1 \mathcal{P} + M(v) \mathcal{P} + L(v) B; \quad \mathcal{P}(v|t) = I_0,
\]

(4.7)

where

\[
\Pi_1(v, v') = \frac{1}{P(v)} \Pi(v, v') P(v').
\]

(4.8)

In explicit form (4.7) reads

\[
\frac{\partial \mathcal{P}(t, r, x, \Omega|v)}{\partial t} + \Omega \cdot \nabla \mathcal{P} = \int \Pi_1(v, v') \mathcal{P}(t, r, x, \Omega|v') \, dv' - k(r) \phi(x - \Omega \cdot v) \left( \mathcal{P} - (1 - \epsilon) \right)
\]

\[
\times \left[ \int \phi(x' - \Omega' \cdot v) \mathcal{S}(t, r, x', \Omega'|v) \, dx' \frac{d^{3} \Omega'}{4\pi} \right] + \epsilon k(r) \phi(x - \Omega \cdot v) B,
\]

(4.9)

with the initial condition

\[
\mathcal{P}(0, r, x, \Omega|v) = I_0(r, x, \Omega, v).
\]

(4.10)

We are only interested in the stationary solution, so we put \( \partial/\partial t = 0 \). We further assume that \( B \) and \( k \) are functions only of the distance \( z \) to the free surface and write as usual \( dr = -k(z) \, dz \). We finally obtain, making full use of the horizontal translational invariance,

\[
\mu \frac{\partial \mathcal{S}(\tau, x, \Omega|v)}{\partial \tau} = \phi(x - \Omega \cdot v) \mathcal{S} - \frac{1}{k(\tau)} \int \Pi_1(v, v') \mathcal{S}(\tau, x, \Omega|v') \, dv',
\]

(4.11)

where we define the mean conditional source function

\[
\mathcal{S}(\tau|v) = \epsilon B + (1 - \epsilon) \int \phi(x' - \Omega' \cdot v) \mathcal{S}(\tau, x', \Omega'|v) \, dx' \frac{d^{3} \Omega'}{4\pi}
\]

(4.12)

and where \( \mu \) is the \( z \)-component of \( \Omega \). The boundary conditions are as usual

\[
\mathcal{S}(0, x, \Omega|v) = 0 \quad \text{for} \quad \mu < 0,
\]

\[
\mathcal{S}(\infty, x, \Omega|v) = B(\infty).
\]

(4.13)

In addition, boundness for \( |v| \to \infty \) must be assumed to ensure uniqueness.

**Remark 1**

Equations (4.11) and (4.12) have the structure of an ordinary transfer equation but with an additional (three-dimensional) independent variable \( v \). Therefore

* Since \( I \) is a function of \( r, x \) and \( \Omega \), the integral in (4.5) is actually a functional integral (similarly in (4.3) \( \partial / \partial I \) is a functional derivative). Such notions will not appear in the sequel and the reader may therefore as well consider them as ordinary integrals and derivatives.
attempts to solve (4.11) and (4.12) numerically will have to face serious difficulties. Methods for ordinary transfer equations based on the differential equation make generally use of an equation for a suitable combination $I^+ + I^-$ of up and down going intensities. Because of the presence of the Fokker–Planck operator, they are hard to generalize. We shall see in Section 6 that an integral equation for $\mathcal{S}$ may be obtained, which generalizes the usual Wiener–Hopf integral equation; standard techniques may then be applied, but since $\mathcal{S}(\tau | v)$ is now a function of four variables, the memory requirements will be huge. Finally, we mention that an iterative method, where $\mathcal{S}$ is integrated upwards for $\mu > 0$ and downwards for $\mu < 0$ and which converges rather slowly has been derived for a somewhat simpler equation by Gail, Sedlmayr & Traving (1975).

**Remark 2**

In the derivation of the transfer equation for the mean conditional intensity and source function, we have assumed for simplicity homogeneous turbulence and uniform thermal velocity. Equations (4.11) and (4.12) remain valid when these conditions are not fulfilled, provided the following changes are made: $l_{\text{corr}}$ must be changed into $l_{\text{corr}}(\tau)$ and $\phi(x - \Omega \cdot v)$ into

$$\frac{c}{v_0 u_{\text{th}}(\tau)} \phi \left( \frac{c(v - v_0)/v_0 - \Omega \cdot v}{u_{\text{th}}(\tau)} \right),$$

where $v$ is the frequency and $u_{\text{th}}(\tau)$ the thermal velocity; if the Uhlenbeck–Ornstein Process is used, the dimensionless $\xi$ in (3.4) must be replaced by the $\tau$-dependent rms turbulent velocity, and if the Kubo–Anderson Process is used, $P(v)$ becomes a function of $\tau$ (velocities are now measured in dimensional units).

**Remark 3**

When $v$ is the Uhlenbeck–Ornstein Process, our transfer equation (4.11) and (4.12) is closely related to an equation first introduced by Traving (1975). The differences are the following:

(i) The velocity $v$ has only one component in the $z$-direction; this rules out isotropic turbulence, but appears reasonable in preliminary calculations when $\mu$ is for simplicity restricted to $\pm 1$.

(ii) In Traving's equation there is a $\mu$-factor, not only in $\partial \mathcal{S} / \partial \tau$, but also in front of the Fokker–Planck operator; this spurious factor comes from considering $v$ as a Markov Process in $\tau$ rather than in time (cf. Section 1). This factor does not appear anymore in the transfer equation for the mean conditional intensity given in Gail et al. (1975).

5. THE MICROTURBULENT LIMIT

The microturbulent limit is obtained by letting the eddy-size $l_{\text{corr}}$ go to zero without changing the amplitudes of the velocities. An equivalent assumption is to change $v(t)$ into $v(t/\eta)$ and let $\eta$ go to zero. In this case there is a more direct approach than the Fokker–Planck technique and the Markov property is superfluous. Indeed, in abstract form the transfer equation (4.2) becomes now

$$\frac{\partial I}{\partial t} = M \left( v \left( \frac{t}{\eta} \right) \right) I + L \left( v \left( \frac{t}{\eta} \right) \right) B .$$

(5.1)
It is a rather simple property of linear stochastic equations that as $\eta \to 0$ the mean intensity becomes the solution of the averaged equation (see Appendix)

$$\frac{\partial \langle I \rangle}{\partial t} = \langle M(v) \rangle \langle I \rangle + \langle I(v) \rangle B.$$  (5.2)

The explicit form of (5.2) (in the stationary case again), which was first derived by Magnan (1975a) from a different viewpoint, is

$$\mu \frac{\partial \langle I(\tau, x, \Omega) \rangle}{\partial \tau} = \langle \phi(x) \rangle \langle I(\tau, x, \Omega) \rangle - \langle \phi(x) \rangle B - (1 - \epsilon)
\times \int \int \langle \phi(x - \Omega \cdot v) \phi(x' - \Omega' \cdot v) \rangle \langle I(\tau, x', \Omega') \rangle dx \frac{d^2 \Omega'}{4\pi},$$  (5.3)

As noticed by Magnan the microturbulent limit becomes a redistribution problem: (5.3) is not the same as the original equation (1.1) with just $\phi$ chanted into $\langle \phi \rangle$, as is often implicitly assumed and which amounts in writing

$$\langle \phi(x - \Omega \cdot v) \phi(x' - \Omega' \cdot v) \rangle \approx \langle \phi(x) \rangle \langle \phi(x') \rangle.$$  (5.4)

It must however be noted (cf. Rybicki 1975) that the assumption of complete frequency redistribution in a fixed reference frame is already an approximation similar to (5.4). Thus approximation (5.4) is natural if one wants to avoid an unjustified discrimination between thermal and microturbulent broadening.

6. AN INTEGRAL EQUATION FOR THE MEAN CONDITIONAL SOURCE FUNCTION

In the absence of stochastic velocity fields, an integral equation may be obtained for the source function (Avrett & Hummer 1965; Mihalas 1970)

$$S(\tau) = eB(\tau) + (1 - \epsilon) \int_0^\infty K_1(\mid \tau - \tau' \mid) S(\tau') d\tau'$$  (6.1)

where

$$K_1(\mid \tau \mid) = \frac{1}{2} \int_0^1 \frac{d\mu}{\mu} \int_{-\infty}^{+\infty} \phi(x) \exp \left( - \frac{\mid \tau \mid}{\mu} \phi(x) \right).$$  (6.2)

With stochastic velocity fields, we can still derive an operator equation from the equation of transfer for the mean conditional intensity (4.11) and (4.12), by essentially the same technique. For simplicity we assume $\tau$-independence of the physical parameters (absorption coefficient, thermal velocity, correlation length, ...), except for $B(\tau)$ (this assumption is not essential; cf. Remark 2 of Section 4). Writing the Fokker-Planck operator $\Pi_1(\nu, \nu')$ as an abstract operator and incorporating into it the factor $1/k$, we can write the formal solution of (4.11)

$$\mathcal{S}(\tau, x, \Omega | v) = \int_\tau^\infty \exp \left\{ - \frac{\tau' - \tau}{\mu} (\phi - \Pi_1) \right\} S(\tau', \cdot) \frac{d\tau'}{\mu}; \mu > 0$$  (6.3)

$$\mathcal{S}(\tau, x, \Omega | v) = - \int_0^\tau \exp \left\{ - \frac{\tau' - \tau}{\mu} (\phi - \Pi_1) \right\} S(\tau', \cdot) \frac{d\tau'}{\mu}; \mu < 0.$$  (6.4)

Following standard mathematical notations, we write $\mathcal{S}(\tau, \cdot)$ to indicate that $\mathcal{S}$ is also function of the velocity variable, which cannot be written out explicitly because of the exponential operator.
Using this in (4.12) we obtain an operator integral equation
\[ \mathcal{S}(\tau | v) = eB(\tau) + (1 - e) \int_0^\infty d\tau' \mathcal{X}_1(|\tau - \tau'|) \mathcal{S}(\tau', \cdot) \] (6.5)
with
\[ \mathcal{X}_1(\tau) = \int_{\mu > 0} \frac{1}{4\pi} \int_{-\infty}^{+\infty} dx \phi \exp \left\{ - \frac{|\tau|}{\mu} (\phi - \Pi_1) \right\} . \] (6.6)
In (6.3)-(6.6), \( \phi \) is understood to be the operator of multiplication by \( \phi(x - v, \Omega) \).
The integral equation will now be used (i) to obtain an identity for the surface value of the conditional source function for uniform \( B \), and (ii) to construct a simple approximation for the source function itself.

7. Generalization of the \( \sqrt{\epsilon} \) Law for Stochastic Medium

In the absence of velocity field, it is well known that the following identity holds when the physical parameters are \( \tau \)-independent and \( B \) uniform (Avrett & Hummer 1965; see also Paper I)
\[ S(\tau = 0) = \sqrt{\epsilon}B. \] (7.1)
With a stochastic velocity field we wish to show that the root mean square surface value of the (conditional) source function is still equal to \( \sqrt{\epsilon}B \), in the following sense
\[ \int P(v) \mathcal{S}^2(\tau = 0 | v) d^3v = eB^2. \] (7.2)
This result (the proof of which is given below) implies that the surface value of the source function and therefore the emergent intensity at line centre are of the same order of magnitude (\( \sqrt{\epsilon}B \)) as in the absence of turbulence; this of course does not rule out sizeable differences.

Proof of (7.2)

It is easily checked that the operator \( \Pi_1 \) defined by (4.8), with \( \Pi \) given by either (3.4) or (3.5) is hermitian for the real scalar product
\[ (f, g) = \int P(v) f(v) g(v) d^3v. \] (7.3)
The operator \( \phi : f(v) \to \phi(x - v, \Omega) f(v) \), being diagonal, is trivially hermitian. Therefore the difference \( \phi - \Pi_1 \), the exponential operator \( \exp (- |\tau| (\phi - \Pi_1)/\mu) \) and the operator \( \mathcal{X}_1(\tau) \) itself are hermitian (for any fixed \( \tau \)). In Paper I we derived the \( \sqrt{\epsilon} \) result from the identity
\[ \int_0^\infty d\tau S(\tau) \frac{\partial}{\partial \tau} \int_0^\infty d\tau' K_1(|\tau - \tau'|) S(\tau') = \frac{1}{2} AS^2(\infty) \] (7.4)
where
\[ A = \int_{-\infty}^{+\infty} K_1(|\tau|) d\tau. \] (7.5)
This identity is easily generalized to real hermitian operators, yielding
\[ \int_0^\infty d\tau \mathcal{S}(\tau) \frac{\partial}{\partial \tau} \int_0^\infty d\tau' \mathcal{X}_1(|\tau - \tau'|) \mathcal{S}(\tau') = \frac{1}{2}(\mathcal{S}(\infty), \mathcal{S}(\infty)) \] (7.6)
where
\[ \mathcal{S} = \int_{-\infty}^{+\infty} \mathcal{X}_1(|\tau|) \, d\tau. \]  

At infinity the stochastic source function and therefore also the mean conditional source function are identically equal to \( B \):
\[ \mathcal{S}(\infty | v) = B. \]  

It remains to evaluate \( \frac{1}{2} B^2(1, \mathcal{X}) \), where \( 1 \) stands for the function identically one. From (6.6) we have
\[ (1, \mathcal{X}) = (1, \int_{-\infty}^{+\infty} dx \phi(\phi - \Pi_1)^{-1} \phi 1 \). \]  

We claim that
\[ (\phi - \Pi_1)^{-1} \phi = 1. \]  

Indeed, \( \Pi P(v) = 0 \) (a statement that \( P(v) \) is the stationary distribution of \( v \)), implies by (4.8) \( \Pi_1 1 = 0 \) and finally \( \phi = (\phi - \Pi_1) 1 \). Using (7.9) and (7.10) we have \( (1, \mathcal{X}) = 1 \). The derivation of (7.2) is then completed as in Paper I.

QED

8. THE EFFECTIVE SOURCE FUNCTION APPROXIMATION

In Non-LTE calculation the source function—but not the emergent intensity—is often found to be rather insensitive to velocity fields; it is therefore sometimes assumed that the source function may be calculated with no velocity field and then used to compute emergent profiles with velocities taken into account (Athay 1972, p. 55). This procedure is actually exact for macroturbulence. We wish to introduce a more refined approach in which the source function is replaced by an effective non-fluctuating source function which takes into account the turbulent velocity field.

Let us first consider the microturbulent case. The effective source function may be calculated by replacing in the basic transfer equation (1.1) \( \phi \) everywhere by \( \langle \phi \rangle \) (see Section 5 for a discussion of this procedure).

Clearly both in the macroturbulent limit and in the above microturbulent limit, the source function is non-random, so that the mean conditional source function does not depend on \( \tau \):
\[ \mathcal{S}(\tau | \tau') = S_{\text{eff}}(\tau). \]  

The idea is to use now (8.1) as an Ansatz in the operator equation (6.5) for the mean conditional source function. Of course this Ansatz is not strictly compatible with the equation since
\[ (1 - \epsilon) \int_0^\infty \mathcal{X}_1(|\tau - \tau'|) S_{\text{eff}}(\tau') \, d\tau' = f(\tau, \tau) \]  

is not independent of \( \tau \). We therefore replace \( f \) by its orthogonal projection on a \( \tau \)-independent function (with the scalar product given by (7.3)):
\[ f(\tau, \tau) \rightarrow \int f(\tau, \tau') P(\tau') \, d\tau'. \]
This procedure leads to a closed equation for the effective source function which is of the standard non-turbulent kind

\[ S_{\text{eff}}(\tau) = eB(\tau) + (1 - e) \int_0^{\infty} K_{1\text{eff}}(|\tau - \tau'|) S_{\text{eff}}(\tau') \, d\tau', \tag{8.4} \]

with the 'effective kernel', now a scalar function of \( \tau \).

\[ K_{1\text{eff}}(\tau) = \int_{\mu > 0} \frac{d^2\Omega}{4\pi} \int_{-\infty}^{\infty} dx \int d^3v P(v) \phi \exp \left\{ - \frac{|\tau|}{\mu} (\phi - \Pi_1) \right\} \phi, \tag{8.5} \]

where \( \phi \) stands as usual for \( \phi(x - \mathbf{v}, \Omega) \).

It is easily seen that

\[ \int_{-\infty}^{\infty} K_{1\text{eff}}(|\tau|) \, d\tau = 1. \tag{8.6} \]

Indeed,

\[ \int_{-\infty}^{\infty} K_{1\text{eff}}(|\tau|) \, d\tau = 2 \int_{\mu > 0} \frac{d^2\Omega}{4\pi} \int_{-\infty}^{\infty} dx \int d^3v P(v) \phi(\phi - \Pi_1)^{-1} \phi = 1 \tag{8.7} \]

(compare with (7.10)).

The normalization (8.8) of the effective kernel implies that, for uniform \( B \), the surface value of the effective source function is \( \sqrt{eB} \) (cf. Paper I).

Remark

The approximation introduced in this section by the Ansatz (8.1) followed by the projection (8.3) may be systematically refined by use of the Galerkin method (Lions 1969). In the \( N \)th Galerkin approximation one replaces the exact equation by its projection on a sub-space generated by the \( N \) first functions of a complete orthonormal system. One can take for example Hermite polynomials in \( v \) which are the eigenfunctions of \( \Pi_1 \) in the case of an Uhlenbeck–Ornstein Process, but can be used also for the Kubo–Anderson Process. With this choice, the lowest Galerkin approximation is identical to the effective source function approximation, because the lowest Hermite function is a constant.

9. THE EFFECTIVE KERNEL FOR A KUBO–ANDERSON PROCESS

9.1 Explicit form

We show that \( K_{1\text{eff}}(\tau) \) may be obtained by quadratures when \( \mathbf{v}(t) \) is a Kubo–Anderson Process with infinitesimal generator given by (3.5). Let us define the Laplace Transform of \( K_{1\text{eff}}(\tau) \)

\[ \hat{K}_{1\text{eff}}(p) = \int_0^{\infty} \exp (-p\tau) K_{1\text{eff}}(\tau) \, d\tau. \tag{9.1} \]

From (8.5) we get

\[ \hat{K}_{1\text{eff}}(p) = \int_{\mu > 0} \frac{d^2\Omega}{4\pi} \int_{-\infty}^{\infty} dx \int d^3v P(v) \phi \left\{ p + \frac{\phi - \Pi_1}{\mu} \right\}^{-1} \phi. \tag{9.2} \]
The calculation of the inverse of \( p + (\phi - \Pi_1)/\mu \) is equivalent to the solution of the integral equation

\[
pf(v) + \frac{\phi(x - v, \Omega)}{\mu} f(v) + \frac{n f(v)}{\mu} = \frac{n}{\mu} \int P(v') f(v') d^3v' = g(v), \quad (9.3)
\]

with \( n = 1/(kl_{\text{corr}}) \), \( l_{\text{corr}} = \text{correlation length} \) and \( k = \text{absorption coefficient} \). Simple algebra yields

\[
f(v) = g(v) \frac{n}{p + (\phi + n)/\mu} \int \frac{P(v') g(v') d^3v'}{p + (\phi + n)/\mu}
\]

where \( \phi \) and \( \phi' \) stand for \( \phi(x - v, \Omega) \) and \( \phi(x - v', \Omega) \). We use (9.4) in (9.2) with \( g(v) = \phi(x - v, \Omega) \); after suitable re-arranging of terms and use of \( \int P(v) d^3v = 1 \) and of \( \int \phi(x) dx = 1 \), we obtain finally

\[
K_{1e}^{\text{eff}}(\tau) = \frac{1}{2} \int_0^1 d\mu \left( 1 - \int_{-\infty}^{+\infty} dx \mu \frac{\phi(\phi + \phi + n)}{\phi(\mu + \phi + \phi + n)} \right), \quad (9.5)
\]

where \( \phi \) stands for \( \phi(x - v) \) with \( v = v, \Omega \) and \( < > \) means the average over the 'one-dimensional' velocity distribution \( P(v) \) of an arbitrary component of \( v \) (we have made use of isotropy).

The effective kernel may be recovered from (9.5) by inversion of the Laplace Transform. Actually (9.5) is useful only for mathematical analysis, to obtain, e.g. asymptotic expansions. A practical procedure to calculate \( K_{1e}^{\text{eff}}(\tau) \) is given in Section 9.4.

### 9.2 Macro- and microturbulent limits

If we let \( n = 1/(kl_{\text{corr}}) \) go to zero (macroturbulence), we get

\[
\lim_{n \to 0} K_{1e}^{\text{eff}}(\tau) = \frac{1}{2} \int_0^1 d\mu \left( 1 - \int_{-\infty}^{+\infty} dx \mu \phi \frac{\phi}{\phi + \phi} \right)
\]

which is the Laplace Transform of the ordinary, non-turbulent \( K_1(\tau) \) function (Avrett & Hummer 1965)

\[
K_1(\tau) = \frac{1}{2} \int_0^1 d\mu \int_{-\infty}^{+\infty} dx \phi^2(x) \exp \left( -\frac{1}{\mu} \phi(x) \right).
\]

Concerning the macroturbulent limit, it is important to stress, following Magnan (1975b), that in a finite slab this limit may not be achieved at all even if the slab-thickness \( T \) exceeds the thermalization length but is not large enough to ensure \( T\phi(x) \geq 1 \) for frequencies \( x \) of the order of a few turbulent Doppler widths.

If we let \( n \) go to infinity (microturbulence) we obtain

\[
\lim_{n \to \infty} K_{1e}^{\text{eff}}(\tau) = \frac{1}{2} \int_0^1 d\mu \left( 1 - \int_{-\infty}^{+\infty} dx \mu \phi \frac{\phi}{\phi + \phi} \right)
\]

which is the Laplace Transform of the ordinary, non-turbulent \( K_1(\tau) \) function (Avrett & Hummer 1965)

\[
K_1(\tau) = \frac{1}{2} \int_0^1 d\mu \int_{-\infty}^{+\infty} dx \phi^2(x) \exp \left( -\frac{1}{\mu} \phi(x) \right).
\]
which is the Laplace Transform of

$$K_1^{\text{micro}}(\tau) = \frac{1}{2} \int_0^1 \frac{d\mu}{\mu} \int_{-\infty}^{+\infty} \langle \phi(x) \rangle^2 \exp \left( -\frac{\tau}{\mu} \langle \phi(x) \rangle \right) dx. \quad (9.11)$$

We see that in this limit $K_1^{\text{eff}}$ and therefore $S_{\text{eff}}$ are exactly the kernel and source function obtained by the standard microturbulent procedure where $\langle \phi \rangle$ is used instead of $\phi$ (cf. Section 5).

For the special case of a Doppler profile and gaussian velocity distribution, it is easily checked that

$$\langle \phi(x) \rangle = \frac{1}{\sqrt{1+\xi^2}} \phi \left( \frac{x}{\sqrt{1+\xi^2}} \right) \quad (9.12)$$

where $\xi$ is the rms turbulent velocity in Doppler units. Using this in (9.11) we obtain

$$K_1^{\text{micro}}(\tau) = \frac{1}{\sqrt{1+\xi^2}} K_1 \left( \frac{\tau}{\sqrt{1+\xi^2}} \right). \quad (9.13)$$

### 9.3 Asymptotic expansion for $\tau \to \infty$ and Doppler profile

The asymptotic behaviour of $K_1^{\text{eff}}(\tau)$ for $\tau \to \infty$ may be obtained from the asymptotic behaviour of its Laplace Transform $\tilde{K}_1^{\text{eff}}(p)$ for $p \to 0$ (see for example Feller 1966, Chapter 13). Methods for asymptotic expansions of integrals of the form of (9.5) are described in Dingle (1973).

For $n \neq 0$ we have obtained the leading term

$$\tilde{K}_1^{\text{eff}}(p) \sim \frac{1}{2} - \frac{1}{2} p \sqrt{1+\xi^2} \sqrt{-\ln \left( p \sqrt{\pi(1+\xi^2)} \right)}, \quad p \to 0, \quad (9.14)$$

from which it follows that

$$K_1^{\text{eff}}(\tau) \sim \frac{\sqrt{1+\xi^2}}{4\tau^2 \sqrt{\ln (\tau/\sqrt{\pi(1+\xi^2)})}}, \quad \tau \to \infty. \quad (9.15)$$

Note that the result is independent of $n$ and holds also in the microturbulent limit.

For $n = 0$ we obtain, of course, the same result as in the absence of turbulence, i.e. (Avrett & Hummer 1965)

$$K_1^{\text{macro}}(\tau) \sim \frac{1}{4\tau^2 \sqrt{\ln (\tau/\sqrt{\pi})}}, \quad \tau \to \infty. \quad (9.16)$$

Actually, the function $K_1^{\text{eff}}(\tau)$ will behave essentially like the static (or macro) one for $0 \leq \tau \leq l_{\text{corr}}$ and like the microturbulent one for $\tau \gg l_{\text{corr}}$ (recall that $l_{\text{corr}} = n^{-1}$ is the eddy-size in dimensionless units).

### 9.4 A method for the practical calculation of the effective kernel

Let us introduce the effective escape probability $K_2^{\text{eff}}(\tau, x, \mu)$, which is the probability, averaged over all realizations, that a photon created at depth $\tau$ is emitted with frequency $x$ and direction $\mu$ and will escape directly to the surface. Using the Fokker–Planck technique or the same reasoning as in Auvergne et al. (1973), one can set up for the effective escape probability the convolution integral equation

$$K_2^{\text{eff}}(\tau, x, \mu) = \alpha_1(\tau, x, \mu) + \frac{n}{\mu} \int_0^\tau \alpha_0(\tau - \tau', x, \mu) K_2^{\text{eff}}(\tau', x, \mu) d\tau'. \quad (9.17)$$
where
\[ \alpha_i(\tau, x, \mu = \int_{-\infty}^{+\infty} P(v) \exp \left\{ -\tau \mu \left[ \phi(x-v) + n \right] \right\} \phi_i(x-v) \, dv, \quad i = 0, 1; \quad (9.18) \]

\( \phi_i \) are powers of \( \phi \); \( P(v) \) is the one-dimensional velocity distribution. A numerical procedure for this type of Volterra equation is described in Frisch (1975).

The effective kernel \( K_{1e}^{\text{eff}}(\tau) \) is simply related to the effective escape probability by
\[ K_{1e}^{\text{eff}}(\tau) = -\frac{1}{2} \frac{\partial}{\partial \tau} \int_{0}^{1} d\mu \int_{-\infty}^{+\infty} dx K_{2e}^{\text{eff}}(\tau, x, \mu). \quad (9.19) \]

The effective source function \( S_{e}^{\text{eff}}(\tau) \) is then given by the Wiener–Hopf equation (8.4). Finally the mean emergent intensity is recovered by integration
\[ \langle I_{\text{emergent}}(x, \mu) \rangle = \int_{0}^{K_{2e}^{\text{eff}}(\tau, x, \mu) S_{e}^{\text{eff}}(\tau) \, d\tau. \quad (9.20) \]

10. FINITE EDDY-SIZE INFLUENCE ON EFFECTIVE SOURCE FUNCTION AND EMERGENT INTENSITY

In order to investigate in detail the effect of turbulence with finite eddy-size on Non-LTE line profiles, one should in principle attempt to solve numerically the transfer equation (4.11) and (4.12) for the mean conditional intensity, using a realistic stellar atmosphere model. In view of the Remark 1 of Section 4 this is a rather difficult task if no approximations are used. Numerical solutions for Non-LTE stochastic transfer based on equations which are simpler but less justified than ours will be found in Magnan (1975b) and Gail et al. (1975).

In this section our purpose will be only to obtain order of magnitude of the effects of turbulence, without recourse to numerical procedures. For simplicity we consider only uniform \( B \). We have already seen in Section 8 that the surface value \( S_{e}^{\text{eff}}(\tau = 0) = \sqrt{\epsilon B} \) is not affected by turbulence. To determine \( S_{e}^{\text{eff}}(\tau) \) for \( \tau \neq 0 \), we use the approximation derived in Paper I
\[ S(\tau) \approx \frac{\sqrt{\epsilon B}}{\sqrt{P(\tau)}}, \quad (10.1) \]
\[ P(\tau) = \epsilon + (1 - \epsilon) K_{2e}^{\text{eff}}(\tau) \quad (10.2) \]
where
\[ K_{2e}^{\text{eff}}(\tau) = 2 \int_{\tau}^{+\infty} K_{1e}^{\text{eff}}(\tau') \, d\tau' \quad (10.3) \]
and \( K_{1e}^{\text{eff}} \) is defined by (8.5). In the region \( 1 \ll \tau \ll \epsilon^{-1} \) (\( \epsilon^{-1} \) is the thermalization length for a Doppler profile), we have
\[ S_{e}^{\text{eff}}(\tau) \approx \frac{\sqrt{\epsilon B}}{\sqrt{K_{2e}^{\text{eff}}(\tau)}}, \quad (10.4) \]

If \( l_{\text{corr}}k \) falls outside the interval \((1, \epsilon^{-1})\) the source function will be everywhere either macro if \( \epsilon^{-1} < l_{\text{corr}}k \) or micro if \( l_{\text{corr}}k < 1 \).

Suppose now that
\[ 1 \ll l_{\text{corr}}k \ll \epsilon^{-1}, \quad (10.5) \]
i.e. the eddy-size is larger than the unit optical depth at line centre and smaller than the thermalization length. From the asymptotic expansions of Section 9.3 we conclude that for $1 \ll \tau \ll l_{\text{corr}} k \ll \epsilon^{-1}$

$$S_{\text{eff}}(\tau) \approx S_{\text{macro}}(\tau) \approx \sqrt{\epsilon B} \left| 2\tau \sqrt{\ln (\tau/\sqrt{\pi})} \right|^{1/2}$$

(10.6)

whereas for $l_{\text{corr}} k \ll \tau \ll \epsilon^{-1}$

$$S_{\text{eff}}(\tau) \approx S_{\text{micro}}(\tau) \approx \sqrt{\epsilon B} \left\{ \frac{2\tau}{\sqrt{1 + \xi^2}} \sqrt{\ln \frac{\tau}{\sqrt{\pi (1 + \xi^2)}}} \right\}^{1/2}.$$  \hspace{1cm} (10.7)

We see that, due to finite eddy-size, the source function is essentially unaffected for $\tau \ll l_{\text{corr}} k$ and reduced by a factor $(1 + \xi^2)^{1/4}$ for $\tau \gg l_{\text{corr}} k$. For heavy elements, this factor can become appreciable, even for subsonic turbulence. In Fig. 5 we represent schematically the effect of a finite eddy-size on the effective source function, assuming that inequations (10.5) hold.

The emergent profile is affected by the turbulence in two ways. First by the change in $S_{\text{eff}}$, second by the fluctuations of the absorption coefficient seen by escaping photons. To get a feeling for the combined effect, let us consider the two limiting cases of macro- and mitroturbulence; the general case will lie somewhere in between. The following very simple argument is independent of the mathematical formalism introduced in the paper.

Upon use of (9.13) in the integral equation (8.4) for the effective source function, we obtain, for uniform $B$ and without recourse to the approximation (10.1), that

$$S_{\text{micro}}(\tau) = S \left( \frac{\tau}{\sqrt{1 + \xi^2}} \right).$$  \hspace{1cm} (10.8)

FIG. 5. Sketch of the effective source function; dots: non-turbulent (static or macro); dashes: microturbulent calculated with $\langle \phi \rangle$ and $\xi = 2$; continuous: effective source function for $l_{\text{corr}} k \approx 10^3$. 

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where $S$ is solution of the non-turbulent problem. The emergent intensity will be given by

$$I_{\text{micro}}(\alpha, x, \mu) = \int_0^\infty \exp \left( -\frac{\tau}{\mu} \phi_{\text{micro}}(\alpha) \right) S_{\text{micro}}(\tau) \phi_{\text{micro}}(x) \frac{d\tau}{\mu}. \quad (10.9)$$

Since

$$\phi_{\text{micro}}(x) = \langle \phi(x) \rangle = \frac{1}{\sqrt{1 + s^2}} \phi \left( \frac{x}{\sqrt{1 + s^2}} \right) \quad (10.10)$$

we obtain

$$I_{\text{micro}}(\alpha, x, \mu) = I \left( \alpha, \frac{x}{\sqrt{1 + s^2}}, \mu \right), \quad (10.11)$$

where $I(\alpha, x, \mu)$ is the non-turbulent emergent intensity.

For the macroturbulent case we obtain

$$I_{\text{macro}}(\alpha, x, \mu) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left( -\frac{\tau}{\mu} \phi(x-v) \right) S(\tau) P(v) \phi(x-v) \frac{d\tau}{\mu} dv, \quad (10.12)$$

$$= \int_{-\infty}^{+\infty} I(\alpha, x-v, \mu) P(v) dv.$$

We see that microturbulence just broadens out the profile without affecting the central intensity, whereas macroturbulence both broadens and changes the central intensity (it is increased because the line is in absorption).

In conclusion it must be stressed that finite eddy-size effects can easily change the effective source function and the emergent profiles by a factor 2 or more. This suggests that it may be possible to deduce the scale of turbulence in a stellar atmosphere from a careful analysis of spectral lines, although it may be difficult to disentangle finite eddy-size from all the other effects.

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REFERENCES


NOTE ADDED IN PROOF


APPENDIX

THE MICROTURBULENT LIMIT

We wish to show that for $\eta \to 0$, the mean solution of

$$\frac{\partial I}{\partial t} = M \left( v \left( \frac{t}{\eta} \right) \right) I + L \left( v \left( \frac{t}{\eta} \right) \right) B; \quad I(0) = I_0 = \text{non-random}, \quad (A1)$$

where $v$ is a given stationary random function of $t$ with finite correlation time, satisfies the averaged equation

$$\frac{\partial \langle I \rangle}{\partial t} = \langle M(v) \rangle \langle I \rangle + \langle L(v) \rangle B; \quad \langle I(0) \rangle = I_0. \quad (A2)$$
Formally the solution of (A1) may be written as a series

\[ I(t) = \left\{ 1 + \int_0^t dt_1 M \left( \nu \left( \frac{t_1}{\eta} \right) \right) + \int_0^t dt_1 \int_0^{t_1} dt_2 M \left( \nu \left( \frac{t_1}{\eta} \right) \right) M \left( \nu \left( \frac{t_2}{\eta} \right) \right) + \ldots \right\} I_0 \]

\[ + \left\{ \int_0^t dt_1 L \left( \nu \left( \frac{t_1}{\eta} \right) \right) + \int_0^t dt_1 \int_0^{t_1} dt_2 M \left( \nu \left( \frac{t_1}{\eta} \right) \right) L \left( \nu \left( \frac{t_2}{\eta} \right) \right) \right\} \]

\[ + \int_0^t \int_0^{t_1} dt_1 \int_0^{t_1} dt_2 M \left( \nu \left( \frac{t_1}{\eta} \right) \right) M \left( \nu \left( \frac{t_2}{\eta} \right) \right) L \left( \nu \left( \frac{t_3}{\eta} \right) \right) + \ldots \} B. \quad (A3) \]

The crucial remark is that for \( \eta \to 0 \) and for given \( t > t_1 > t_2 > t_3 > \ldots \)

\[ \langle M \left( \nu \left( \frac{t_1}{\eta} \right) \right) M \left( \nu \left( \frac{t_2}{\eta} \right) \right) \ldots M \left( \nu \left( \frac{t_n}{\eta} \right) \right) \rangle \to \langle M(\nu) \rangle^n \]

\[ \langle M \left( \nu \left( \frac{t_1}{\eta} \right) \right) M \left( \nu \left( \frac{t_2}{\eta} \right) \right) \ldots M \left( \nu \left( \frac{t_n}{\eta} \right) \right) L \left( \nu \left( \frac{t_{n+1}}{\eta} \right) \right) \rangle \to \langle M(\nu) \rangle^n \langle L(\nu) \rangle. \quad (A5) \]

Indeed, as \( \eta \to 0 \), the various time differences \( (t_i - t_j)/\eta \) become much larger than the correlation time, so that the moments may be factorized. Averaging (A3) and using (A4) and (A5), we obtain a series equivalent to (A2).