THE STABILITY OF A SOLAR MODEL TO NON-RADIAL OSCILLATIONS

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SUMMARY

The stability of a solar model to low-order gravity modes has been investigated in quasiadiabatic approximation, neglecting any direct effects of convection on the oscillations. The zero age main sequence model was found to be stable, but after about $2 \times 10^8$ yr the $g_1 (l = 1)$ mode became unstable. The dominant destabilization comes from the dependence of the pp chain reaction rate on the abundances of $^3$He and H when nuclear equilibrium is disturbed, and is felt once a sufficient $^3$He inhomogeneity is established in the energy generating core.

1. INTRODUCTION

It has been suggested (Dilke & Gough 1972) that at certain epochs of its main sequence evolution the Sun has been unstable to low-order gravity modes. The instability caused transient mixing of the solar core which temporarily depressed both the neutrino generation rate and the surface luminosity. If, as many climatologists believe, reduction of the solar constant induces terrestrial glaciation, instability would last have been initiated at the beginning of the Pleistocene Epoch, too recent for the Sun to have yet regained its thermal balance. This would imply that the surface luminosity of the Sun now is not exactly equal to the thermonuclear energy generation rate, and that the solar neutrino flux is at present anomalously low.

The gravity waves thought to be responsible for triggering this process are driven principally in the convectively stable, thermonuclear energy generating core. Once the Sun has had time to generate a spatial composition variation in its unmixed core the downward displaced fluid is richer than its surroundings in hydrogen and $^3$He. Consequently it generates energy faster. This tends to enhance the restoring buoyancy force which would drive the fluid back past its equilibrium position at a speed greater than it had previously, thus increasing the amplitude of the wave (cf. Cowling 1957). On the other hand radiative heat transfer generally acts oppositely, since on the whole downward displaced fluid has a positive temperature fluctuation and so loses heat to its surroundings.

Because radiative heat transfer occurs throughout the whole of the star, and thermonuclear reactions only at its centre, the most unstable modes are likely to be those that are most confined to the core. It is the standing gravity waves ($g$ modes) which have this property. Furthermore, since the solar interior is optically dense, radiative damping decreases with increasing length scale; nuclear driving does not depend explicitly on length scale so it is to be expected that the lowest order $g$ modes, whose characteristic scale is comparable with the radius of the Sun, are the most unstable.
The ratio of the driving to the damping is essentially the ratio of the characteristic thermal energy generation time to the radiative diffusion time. For the lowest-order modes of oscillation of a main sequence star this ratio must be of order unity, provided the variation of the amplitude of the oscillations from centre to surface is not very great. A detailed perturbation analysis is necessary, therefore, before one can assess whether the Sun is likely to have ever been unstable. Dilke & Gough estimated the result of such a calculation using the adiabatic eigenfunction $g_1 (l = 2)$ of the polytrope of index 3 published by Cowling (1941). (This was the lowest order mode they could find in print.) Their tentative conclusion was that the Sun was stable at the beginning of its main sequence history, but that, as a result of the composition gradients subsequently generated in its convectively stable core, it became unstable after a time of order $10^8$ yr. The resulting mixing of the core material destroyed the composition gradients which had initiated the instability, the Sun relaxed back to its normal quiescent state in the thermal diffusion (Kelvin-Helmholtz) time of about $10^7$ yr, and the whole process started again. Thus the interval between major terrestrial ice ages was identified as a solar thermonuclear time.

It is the purpose of this paper to present results of a more refined stability analysis. In this preliminary investigation linearized non-radial adiabatic oscillations of what is hoped is a realistic model of the Sun in the early stages of its main sequence history were considered. The neglected non-adiabatic terms were then estimated, as is usual, by substituting the adiabatic eigenfunctions in the work integral. The zero-age model was found to be stable to all the modes considered, the lowest order $g$ mode being least stable. The $g_1 (l = 2)$ mode became unstable after about $10^8$ yr, and the $g_1 (l = 1)$ mode became unstable sooner, as expected, after about $2 \times 10^8$ yr.

The sensitivity of the instability to the parameters determining the solar model (e.g. chemical composition) has not been investigated. We plan to do this later with a non-adiabatic computation.

It must be pointed out that Dziembowski & Sienkiewicz (1973) have performed a similar stability analysis to that reported here, and find no instability. This may be due in part to their choice of a solar model which probably does not have the correct solar luminosity and effective temperature at the present age of the Sun. But more important is their neglect of the deviations from nuclear equilibrium which we find provides the dominant destabilizing mechanism. We return to this point in the penultimate section of this paper.

### 2. THE SOLAR MODEL

A solar model evolving on the main sequence was computed using the programme described by Eggleton (1971). A spatial mesh of 201 points was used and time steps were typically $10 \times 10^8$ yr, though shorter steps were used for the first few hundred million years of evolution. Hydrogen and heavy elements abundances $X = 0.735$, $Z = 0.020$ and a mixing length of 1.1 pressure scale heights in Vitense's formulation of the theory were chosen to produce a model which had the present solar luminosity and effective temperature at an age of $4.7 \times 10^9$ yr.

In the stellar evolution programme it is assumed that the abundances of the elements generated by the pp chain are such that the nuclear reactions are always in equilibrium, in the sense of, for example, Clayton (1968). That is to say, that
the rates of all the reactions in the chain balance at any instant so that the abundances of the intermediate products of reaction depend only on $X$ and $Z$ and the thermodynamic state and not explicitly on time. For the purposes of calculating the energy generation in main sequence solar models this is a good approximation, apart possibly for the first few hundred thousand years. But the stability calculation requires explicit knowledge of $X_3$, the $^3\text{He}$ abundance. To assume the equilibrium abundance throughout the model would lead to absurdly high values in the outer envelope, so the $^3\text{He}$ distribution was estimated by the procedure described below.

In hydrostatic equilibrium the $^3\text{He}$ abundance satisfies the equation

$$\frac{N}{A_3} \frac{\partial X_3}{\partial t} = \frac{1}{2} \lambda_{11} X^2 - \lambda_{33} X_3^2 - \lambda_{34} Y X_3,$$

(2.1)

where $\lambda_{11} X^2$, $\lambda_{33} X_3^2$ and $\lambda_{34} Y X_3$ are the rates per unit mass of the $p(p, \beta^+\nu) D(p, \gamma) ^3\text{He}$, $^3\text{He} (^3\text{He}, 2p) ^4\text{He}$ and $^3\text{He}(^4\text{He}, \gamma) ^7\text{Be}$ reactions, $Y$ is the abundance of $^4\text{He}$, $A_3$ is the atomic weight of the $^3\text{He}$ and $N$ is Avogadro’s number. Since at any particular point in the region where energy generation is significant the time scale for $X_3$ to attain equilibrium is much shorter than the time scale associated with

![Graph](image-url)
the evolution of $X$, $Y$ and the $\lambda$'s, equation (2.1) can be adequately solved by regarding the latter to be constant. The resulting solution satisfying $X_3 = 0$ at $t = 0$ is thus approximately

$$X_3 = \frac{(1 + 2\alpha) \tanh \frac{1}{2} \nu t}{1 + \alpha (1 + \tanh \frac{1}{2} \nu t)} X_{3e},$$

(2.2)

where $t$ is time, $\nu = (A_3/\nu) (2\lambda_{11}\lambda_{33}X^2 + \lambda_{34}X^2)^{1/2}$, $\alpha = \lambda_{34}Y/2\lambda_{33}X_{3e}$, and $X_{3e} = \{(N/A_3) \nu - \lambda_{34}Y\}/2\lambda_{33}$ is the equilibrium $^3$He abundance. Equation (2.2) was used to determine $X_3$ for the stability analysis, using values of $\nu$, $\alpha$ and $X_{3e}$ obtained from the appropriate solar model computed with Eggleton's programme.

The result is illustrated in Fig. 1. On a time scale of $10^8$ yr the region in which $X_3 \approx X_{3e}$ grows to include the bulk of the energy generating core; 95 per cent of the energy is generated in the inner 25 per cent by radius of the model. In addition, on a time scale of $10^9$ yr, the gradient in the equilibrium abundance steepens due to the preferential depletion of hydrogen near the centre.

Nuclear reaction rates were computed from the cross-sections quoted by Bahcall & Sears (1972) and the $^7$Be electron capture rate derived by Bahcall & Moeller (1969), and the opacities used were those provided by Cox & Stewart (1970).

### 3. Perturbation Equations

Since we are seeking modes which grow in a time much shorter than the evolution time, for the purposes of the linear stability analysis the evolving solar model can be regarded as a steady 'equilibrium' model. The linear equations governing a small displacement $\xi(r, t)$ from such a model can then be written (e.g. Ledoux & Walraven 1958)

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\nabla p' + \rho' \nabla \Phi_0 + \rho_0 \nabla \Phi',$$  

(3.1)

$$\rho' + \text{div} \rho_0 \xi = 0,$$  

(3.2)

$$\frac{\partial p'}{\partial t} - c^2 \frac{\partial \rho'}{\partial t} + \left( \frac{dp_0}{dr} - c^2 \frac{dp_0}{dr} \right) \frac{\partial \xi}{\partial t} = (\Gamma_{30} - 1)(\rho \epsilon - \text{div } F'),$$  

(3.3)

$$\nabla^2 \Phi' = -4\pi G \rho',$$  

(3.4)

where a subscript zero denotes the equilibrium value of a quantity and a prime denotes the Eulerian perturbation from it. Here $r$ is the distance from the centre of the Sun, $\rho$ denotes density, $p$ pressure, $\Phi$ the gravitational potential, $\epsilon$ the thermonuclear energy generation rate per unit mass and $F$ the heat flux; $\xi_t$ is the radial component of $\xi$, and $\Gamma_{30} - 1 = (\partial \ln T/\partial \ln \rho)_s$ where $T$ is temperature, the derivative being taken at constant specific entropy $s$; also $c$ is the local adiabatic sound speed $(\Gamma_0 \rho_0/\rho_0)^{1/2}$ where $\Gamma_1 = (\partial \ln p/\partial \ln \rho)_s$ and $G$ is the constant of gravitation. The equilibrium model satisfies the equations

$$\frac{dp_0}{dr} = \rho_0 \frac{d\Phi_0}{dr},$$  

(3.5)

$$\frac{d\Phi_0}{dr} = -\frac{4\pi G}{r^2} \int_0^r \rho_0 \rho \, dr,$$  

(3.6)

$$\text{div } F_0 = \rho_0 \epsilon_0.$$  

(3.7)

These equations must be supplemented by equations for $\Gamma_1$, $\Gamma_3$, $F$ and $\epsilon$.  

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At the photosphere \( r = r_s \) the perturbation in the gravitational field was matched to a vacuum field by insisting that \( \Phi' \) and \( \nabla \Phi' \) be continuous on the perturbed surface \( r = r_s + \xi(r_s) \). The dynamical boundary condition was approximated by \( \rho' = 0 \) on the perturbed surface, which is equivalent to the vanishing of the Lagrangian pressure perturbation on the unperturbed surface. At the singular point \( r = 0 \), the boundary conditions that isolate the regular solution were chosen.

### 4. ADIABATIC OSCILLATIONS

The equations governing adiabatic oscillations are simply (3.1) to (3.4) with the right-hand side of equation (3.3) set to zero. These equations, when expressed with respect to spherical polar coordinates \((r, \theta, \phi)\), have coefficients which are independent of \( \theta \) and \( \phi \) and, for the purposes of the stability analysis, can be regarded as independent of \( t \) too. They therefore admit separated solutions of the form

\[
\xi_r = r_s \operatorname{Re} \{ \Xi(x) S_{\text{Im}}(\theta, \phi) e^{-i\omega t} \} \\
\xi_\theta = r_s \operatorname{Re} \left\{ H(x) \frac{\partial S_{\text{Im}}}{\partial \theta} e^{-i\omega t} \right\} \\
\xi_\phi = r_s \operatorname{Re} \left\{ H(x) \frac{1}{\sin \theta} \frac{\partial S_{\text{Im}}}{\partial \phi} e^{-i\omega t} \right\} \\
\rho' = \rho_c \operatorname{Re} \{ \Psi(x) S_{\text{Im}}(\theta, \phi) e^{-i\omega t} \} \\
\rho' = \rho_c \operatorname{Re} \{ \Phi(x) S_{\text{Im}}(\theta, \phi) e^{-i\omega t} \} \\
\Phi' = \frac{\rho_c}{\rho_0} \operatorname{Re} \{ \Phi(x) S_{\text{Im}}(\theta, \phi) e^{-i\omega t} \}
\]  

(4.1)

where \( \xi_\theta \) and \( \xi_\phi \) are the \( \theta \) and \( \phi \) components of \( \xi \), \( \rho_c \) and \( \rho_0 \) are the values of \( \rho_0 \) and \( \rho_0 \) at \( r = 0 \), \( S_{\text{Im}} \) is a surface harmonic \( P^m_l(\cos \theta)^m \sin m\phi \), \( \omega \) is the (real) frequency of oscillation and \( x = r/r_s \) is now the independent variable. It is then straightforward to derive from equations (3.1) to (3.4), with the help of equation (3.5), the following equations for the adiabatic dimensionless perturbation amplitudes:

\[
\frac{d\Psi}{dx} - \rho \frac{d\Phi}{dx} = A\Psi + B\Xi
\]  

(4.2)

\[
\frac{d\Xi}{dx} = \frac{C}{x^2} \Psi - \frac{l(l+1)}{\sigma^2 x^2} \Phi \left( A + \frac{2}{x} \right) \Xi
\]  

(4.3)

\[
\frac{d^2\Phi}{dx^2} + \frac{2}{x} \frac{d\Phi}{dx} - \frac{l(l+1)}{x^2} \Phi = -D\Psi - \sigma \Xi
\]  

(4.4)

where

\[
A = \frac{1}{\Gamma_{10}} \frac{d \ln \rho}{dx}, \quad B = \sigma^2 \rho + \frac{1}{\Lambda^2} \frac{d \rho}{dx}, \quad C = \frac{l(l+1)}{\sigma^2 \rho} - \frac{x^2}{\Gamma_{10} \rho'}, \quad D = \frac{\Lambda \rho}{\Gamma_{10} \rho'}, \quad \sigma = \Lambda \left( \frac{1}{\Gamma_{10}} - 1 \right) \frac{d \rho}{dx},
\]

(4.5)

\( \rho = \rho_0 / \rho_c, \rho = \rho_0 / \rho_c \) are now dimensionless pressure and density of the equilibrium model,

\[
\sigma = r_s \left( \frac{\rho_c}{\rho_0} \right)^{1/2} \omega
\]  

(4.6)
is a dimensionless frequency, \( \Gamma = \frac{d \ln \rho}{d \ln \rho} \), and \( \Lambda = 4 \pi G \rho c^2 r^2 / \rho_0 \) is a dimensionless quantity of order unity.

At the photosphere the boundary conditions are

\[
\begin{align*}
\Psi' + \frac{d \rho}{dx} = 0 \\
\frac{d \Phi}{dx} + \frac{l+1}{x} \Phi = \Lambda \rho \Xi
\end{align*}
\]

at \( x = 1 \).

The centre is a singular point, and boundary conditions must be applied at a finite distance \( x = x_0 \) from it. These were obtained by expanding in powers of \( x \) the regular solutions of equations (4.2)-(4.4) about \( x = 0 \). Only the first two non-zero terms in the expansions of each of the variables were retained for this purpose. From these can be obtained

\[
\begin{align*}
\Xi & \approx \Xi_0 x^{l-1} + \Xi_2 x^{l+1} \\
\Psi & \approx \Psi_0 x^l + \Psi_2 x^{l+2} \\
\Phi & \approx \Phi_0 x^l + \Phi_2 x^{l+2}.
\end{align*}
\]

The coefficients \( \Xi_2, \Psi_2 \) and \( \Phi_2 \) can be expressed in terms of the coefficients of the leading powers of \( x \) which themselves are related by \( \Psi_0 = (\sigma^2 / l) \Xi_0 + \Phi_0 \) (see Appendix). In the vicinity of \( x = 0 \) the differential equations imply no further relation between the coefficients: the leading term of the expansion of the gravitational potential is a solution of Laplace's equation, and its amplitude is determined by the structure of the density perturbation throughout the star. The boundary condition on the perturbed potential can be written without reference to \( \Phi_0 \), however, by eliminating \( \Phi_0 \) between the last of equations (4.8) and

\[
\frac{d \Phi}{dx} \approx l \Phi_0 x^{l-1} + (l+2) \Phi_2 x^{l+1},
\]

with the help of equation (4.9) for \( \Phi_2 \). The problem is finally closed by arbitrarily specifying the amplitude of the solution. We chose \( \Xi_0 = 1 \), which yields solutions for \( \Xi, \Psi \) and \( \Phi \) which are real.

Equations (4.2)-(4.4) were differenced by representing derivatives by centred second-order accuracy differences. The resulting equations, together with the boundary conditions (4.7) and those implied by (4.8) and (4.9) at \( x = x_0 \), were solved numerically by a Newton–Raphson iterative procedure on the eigenfunctions and the eigenfrequency in a way similar to that described by Baker, Moore & Spiegel (1971). To save computer time only every other mesh point at which the equilibrium solution had been computed was used; the value of \( x_0 \) specifying the position of the innermost point was typically about \( 10^{-2} \).

Some solutions were recomputed using all the mesh points. The eigenfunctions differed from the less accurate results typically by about 1 or 2 per cent, the periods by less than 0.1 per cent. The work integral was about 5 per cent different up to the base of the convection zone but, presumably because of poor resolution in the outer regions, diverged more in the convection zone.

5. ESTIMATION OF THE GROWTH RATE OF NON-ADIABATIC OSCILLATIONS

If the right-hand side of equation (3.3) is not neglected eigenfunctions of the form (4.1) are still possible, but now \( \omega \) and the amplitude functions are complex.
The imaginary part of each eigenfrequency can be estimated by considering the energy equation for the oscillations. This is obtained by taking the scalar product of equation (3.1) with $\xi^*$, the complex conjugate of $\xi$, and integrating over a volume $V$ within the star. With appropriate use of equations (3.2)-(3.4) and the usual use of the divergence theorem (cf Ledoux & Walraven 1958; Chandrasekhar 1964) one can easily show that

$$\omega^2 \int_V \rho_0 \xi^* \cdot \xi \, d\tau = \int_V \left[ \Gamma_{10} \rho_0 \text{div} \xi^* \, \text{div} \xi + \frac{d\rho_0}{dr} \left( \xi^* \text{div} \xi + \xi \text{div} \xi^* \right) \frac{1}{\rho_0} \frac{d\rho_0}{dr} \xi^* \xi \right] \, d\tau$$

$$- G \int \int_V \frac{\text{div} \left[ \rho_0(\mathbf{r}) \xi^*(\mathbf{r}) \right]}{|\mathbf{r} - \mathbf{r}'|} \text{div}' \left[ \rho_0(\mathbf{r}') \xi(\mathbf{r}) \right] \, d\tau \, d\tau'$$

$$- \int \Sigma \xi^* \left( \frac{d\rho_0}{dr} \xi + \Gamma_{10} \rho_0 \text{div} \xi + \rho_0 \Phi' \right) \, dS$$

$$+ \frac{i}{\omega} \left[ - \int_V (\Gamma_{30} - 1)(\rho \epsilon - \text{div} \mathbf{F}') \text{div} \xi^* \, d\tau \right.$$

$$+ \int \Sigma (\Gamma_{30} - 1)(\rho \epsilon - \text{div} \mathbf{F}') \xi^* \, dS \right]$$  \hspace{1cm} (5.1)

where $\mathbf{r}$ and $\mathbf{r}'$ are position vectors, $\Sigma$ is the surface enclosing $V$ and the eigenfrequency $\omega$ is now complex.

Equation (5.1) can be formally written

$$\omega^2 I = K + \frac{i}{\omega} J$$  \hspace{1cm} (5.2)

where $I$ and $K$ are integrals not explicitly dependent upon $\omega$; $J$ depends on $\omega$ through the perturbation in the $^7$Li and $^7$Be abundances (see equation 5.7). Since throughout almost the entire star the motion is very nearly adiabatic, it is usually assumed that the low order eigenfunctions and their corresponding eigenfrequencies differ little from those computed using the adiabatic approximation described in the previous section. The small correction to the frequency can therefore be estimated from equation (5.2) by substituting the adiabatic eigenfunctions into the integrals. This has been called the quasiaadiabatic approximation. In this approximation the integrals $I$ and $K$ are real, and since $|J| \ll |\omega K|$ the growth rate of the oscillations is given approximately by

$$\eta = \text{Im}(\omega) \simeq \frac{\text{Re}(J)}{2\omega_a I}$$  \hspace{1cm} (5.3)

where $\omega_a$ is the frequency of the adiabatic mode.

In terms of the adiabatic amplitude functions introduced in (4.1) the quasiaadiabatic approximations to the integrals $I$ and $J$ are

$$I = 8^3 \rho_c \int_0^{r_s} \rho [x^2 \Xi^2 - \sigma - l(l+1)(\rho^{-1} \Phi - \Phi)^2] \, dx,$$  \hspace{1cm} (5.4)

$$J = -8^3 \rho_c \epsilon_c \left\{ \int_0^{r_s} (\Gamma_{30} - 1)(\epsilon P + \rho E - \Delta) \left[ \frac{d}{dx} (x^2 \Xi) - \sigma - l(l+1)(\rho^{-1} \Phi - \Phi)^2 \right] \, dx \right.$$

$$+ (\Gamma_{30} - 1) x^2 \Delta \Xi \big|_{x=x_a} \right\},$$  \hspace{1cm} (5.5)
where $E$ and $\rho e$ are the dimensionless amplitudes of $e'$ and $\text{div} F'$, measured in units of $e_c^2$ and $\rho e e r^2/T e$. The subscript $c$ denotes central value of the appropriate equilibrium quantity, and now $e = e_0/e_c$. It has been assumed that the volume of integration $V$ is the sphere $r = r_a = r_{ac}$.

When evaluating the integral $J$ only perturbations of the nuclear energy generation rate arising from the ppI and ppII chains were considered. Thus $\rho e$ was considered to depend explicitly only on the density and temperature and the abundances of $H$, $^3\text{He}$, $^7\text{Li}$ and $^7\text{Be}$. The Lagrangian perturbation in $X$ (which is required also for computing the opacity perturbation) and the perturbation in $X_3$ were taken to be zero:

$$X' + \Xi \frac{dX_0}{dx} = 0, \quad X_3' + \Xi \frac{dX_3}{dx} = 0, \quad (5.6)$$

since the time scales on which they are modified by nuclear reactions or diffusion are very much greater than the oscillation periods. The $^7\text{Li}$ and $^7\text{Be}$ abundance perturbations, however, were computed from the appropriate reaction rate equations. For example, the equation used to calculate the amplitude of the perturbation in the $^7\text{Be}$ abundance $X_7$ was

$$[-\frac{i}{2} \omega_b N + \lambda_7(1 + X)] X_7' = \frac{i}{2} \omega_b N \frac{dX_7}{dx} \Xi + \lambda_{34} X_3 Y$$

$$\times \left[ (\frac{d \ln \lambda_3}{d \ln T} - \frac{d \ln \lambda_7}{d \ln T}) \Theta - \frac{2X'}{1 - X^2} \frac{X_7}{X_3} \right], \quad (5.7)$$

where $X_7 = \lambda_{34} X_3 Y/[\lambda_7 (1 + X)]$ and the subscript $\circ$ has been omitted from equilibrium quantities. Here, $\lambda_{34} X_3 Y$ and $\lambda_7 (1 + X) X_7$ are the rates of the $^3\text{He}(^4\text{He}, \gamma) ^7\text{Be}$ and the $^7\text{Be}(e^-, \nu) ^7\text{Li}$ reactions. In deriving this equation it was assumed that $Y = 1 - X$ and that the density of free electrons is $\frac{2}{3}(1 + X)$ per nucleon, the latter being assumed also in obtaining a similar equation for the perturbation in the abundance of $^7\text{Li}$. Only the real parts of these abundance perturbations contribute in leading order to the growth rate. The characteristic adjustment time scale for $^7\text{Li}$ is about 0.4 hr at the centre and about 4 hr at the radius within which half the energy is generated, and is therefore comparable with the oscillation period which, of the $g_1(l = 1)$ mode, is about 2 hr. The adjustment time for $^7\text{Be}$ is typically about 3000 hr, which is admittedly considerably greater than the periods of the modes discussed in this paper. The contribution to the work integral from the $^7\text{Li}$ and $^7\text{Be}$ abundance perturbations was in any case quite small.

In the convection zone only the contribution to $\text{div} F'$ from the radiative flux was considered. Reynolds stresses and perturbations of the convective heat flux were ignored entirely because there is no adequate theory with which to evaluate them. Thus the expression for $\Delta$ is simply

$$\Delta = -\left[ \frac{K}{x^2} \frac{d^2}{dx^2} (x\Theta) - l(l+1) \frac{K}{x^2} \Theta + \frac{dK}{dx} \frac{d\Theta}{dx} + \frac{dT}{dx} \frac{dK'}{dx} + \frac{1}{x} \frac{d^2(xT)}{dx^2} K' \right], \quad (5.8)$$

where

$$\Theta = (\Gamma_{30} - 1) \frac{T}{\rho} P - \left[ \frac{dT}{dx} + (\Gamma_{30} - 1) \frac{T}{\rho} \frac{d\rho}{dx} \right] \Xi \quad (5.9)$$
is the temperature fluctuation amplitude, measured in units of $T_c$, and $K$ and $K'$ are the dimensionless equilibrium radiative conductivity and the amplitude of its perturbation, measured in units of $r_s^2 p_c e_c / T_c$.

If the integrals in (5.2) are evaluated over the whole star, the surface terms are negligible compared with the volume integrals. If they are truncated at a radius $r_a$, the surface integrals then provide the contribution to the frequency and growth rate arising from the material outside $r = r_a$, provided the correct nonadiabatic eigenfunctions are used. But when adiabatic eigenfunctions are substituted some care must be taken. Near the surface of the star the thermal diffusion time over a length scale characteristic of the perturbation is shorter than the oscillation period, and the adiabatic approximation is invalid. The volume integrals must therefore be truncated at a radius $r_a$ small enough to exclude this region. Furthermore, since to obtain the adiabatic eigenfunctions it was necessary to integrate the differential equations (4.2)–(4.4) through the non-adiabatic region $r > r_a$ in order to apply the boundary conditions at the surface, the contribution to the growth rate from material in this region cannot be estimated from the surface integrals even though they are evaluated where locally the adiabatic approximation may be valid. Indeed, since the structure of the eigenfunctions depends on the boundary conditions, even the truncated volume integrals are not accurately estimated when the adiabatic eigenfunctions are used. In common with, for example, Robe, Ledoux & Noels (1972) and Dziembowski & Sienkiewicz (1973), growth rates were estimated by truncating the volume integrals at a radius $r_a$ and ignoring the surface integrals. Such a procedure is not unreasonable provided the amplitude of the mode in the surface regions is not high. The $f$ and $p$ modes do not have this property. The low order $g$ modes have small surface amplitudes in comparison, however, and the estimated growth rates are not very sensitive to minor changes in the surface boundary conditions. This provides some comfort, but does not guarantee the reliability of the estimate. The sensitivity of the predicted growth rate to the truncation radius $r_a$ can be seen in Fig. 3.

6. RESULTS

The $l = 1$ $g$ modes comprise the most unstable sequence of perturbations we considered; they are the only modes whose displacement does not vanish at $x = 0$ and thus they have relatively large amplitude in the energy generating core. Stability increases with $l$ and with the order of the modes, as can be seen in Table I.

The growth rates $\eta$, in units of $10^{-7}$ yr$^{-1}$, are plotted in Fig. 2 as a function of solar age on the main sequence. They all increase with time and eventually become positive. The growth rates here, and in Tables I and II, were evaluated using a truncation radius $x_a = r_a / r_s = 0.99$.

To illustrate how destabilization comes about we have plotted in Figs 3 and 4 various components of the volume work integral $J_v$ for the $g_1(l = 1)$ mode at $t = 5 \times 10^8$ yr. Fig. 3 shows $J_v/I$ plotted against the dimensionless truncation radius $x_a$ together with its components $J_e/I$ and $J_r/I$ which arise from fluctuations in the energy generation rate and radiative heat transfer. As one would expect, the contribution from $J_e$ is destabilizing; but though the total contribution from the integral $J_r$ is stabilizing, radiation does not damp everywhere. In the region between about $x = 0.3$ and $x = 0.5$ radiative transfer acts to excite the oscillations, partly because $\text{div} \, \xi$ is not everywhere positive for upward moving fluid elements,
Table I

Periods $\Pi$ (in hours) and growth rates $\eta$ (in units of $10^{-7}$ yr$^{-1}$) of some low order $g$ modes at various ages of the equilibrium model. Only the interesting lowest order modes were computed at $10^8$ and $2 \times 10^8$ yr. The $l = 2$ modes are not included beyond $1.5 \times 10^9$ yr because the results are unreliable.

<table>
<thead>
<tr>
<th>Age (yr)</th>
<th>$L/L_0$</th>
<th>$T_e$ (K)</th>
<th>$g_1(l = 1)$</th>
<th>$g_2(l = 1)$</th>
<th>$g_3(l = 1)$</th>
<th>$g_4(l = 1)$</th>
<th>$g_1(l = 2)$</th>
<th>$g_2(l = 2)$</th>
</tr>
</thead>
<tbody>
<tr>
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and near the centre it is exciting because upward displaced elements have positive Eulerian temperature fluctuations. The latter phenomenon, which is contrary to the expectations of Dilke & Gough (1972), appears to result from the perturbation in the gravitational potential (see Fig. 5) which was neglected in their analysis.

The oscillatory behaviour of $J_F$ near the base of the convection zone arises from the rapidly varying temperature gradient in the equilibrium model. This region was not well resolved, so the details are not to be trusted. For this reason the integral here has been drawn with a broken line. The magnitude of the variation of $J_F$ in this region increases with solar age for all modes, and for the $l = 1 g$ modes

![Graph](image_url)

**Fig. 2.** Growth rates $\eta$ of three low order $g$ modes in units of $10^{-7}$ yr$^{-1}$ as a function of the age $t$ (measured in units of $10^8$ yr) of the equilibrium model.
is comparable with the total integral after about $4 \times 10^9$ yr. For the $l=2$ $g$ modes it is appreciable by about $1.5 \times 10^9$ yr. We cannot be sure, therefore, that the $g_2(l=2)$ mode really is unstable as indicated in Table I. The oscillation in $J_F$ is so pronounced in the $f$ and $p$ modes that the quasiadiabatic stability calculation is hardly meaningful. However, if the work integral is truncated at the base of the convection zone, the $f$ and low order $p$ modes are predicted to be stable.

In Fig. 4 the major components of $J_\varepsilon$ are plotted separately. These are $J_{\varepsilon_3}$, $J_{\varepsilon_T}$ and $J_{\varepsilon_\rho}$ which measure the contributions from the fluctuations in $\rho e$ arising directly from the Eulerian perturbations in $^3$He concentration, temperature and density. $J_{\varepsilon_\rho}$ is damping, as one would expect, since upward displaced fluid elements are generally denser than their surroundings in a convectively stable region and so there is a tendency for them to generate energy faster. Outside $x = 0.16$ Eulerian temperature perturbations excite the oscillations, but $J_{\varepsilon_T}$ is dominated by the stronger stabilization near the centre arising from the unexpected sign of $\Theta$ mentioned above (see Fig. 5).

The main contribution to the growth rate comes from the perturbation in the $^3$He abundance. In regions where $^3$He is in nuclear equilibrium, $dX_{^3\text{He}}/dx > 0$ and upward displaced fluid elements have a lower $^3$He abundance than their surroundings and hence generate energy more slowly. Only where $X_{^3\text{He}}$ is still far from its equilibrium value is $dX_{^3\text{He}}/dx$ negative and stabilizing, but once this region has moved to the edge of the energy generating core, as in the case illustrated in
Figs 3–5, its influence is barely discernible. The equilibrium hydrogen abundance $X_0$ also increases outwards, and the contribution to $\eta$ from the fluctuations is also positive. However, in the relatively young model illustrated here a steep gradient in $X_0$ has not yet developed and perturbations in $X$ contribute less than 1 part in 1000 to $J_\eta$. As the model evolves, however, the relative contribution from $X'$ increases, and by $t = 4 \times 10^9$ yr it is about 25 per cent of $J_{\epsilon 3}$.

![Graph showing contributions $J_{\epsilon 3}$, $J_\epsilon$, $J_{\epsilon T}$, and $J_{\epsilon \rho}$ to the volume work integral $J_\nu$ from perturbations in the energy generation rate arising directly from perturbations in the $^3$He abundance, temperature and density, and their sum $J_\epsilon$. The integrals are measured in units of $r_s^3 \rho_c \epsilon_c$, where $r_s$ is the radius of the equilibrium model and $\rho_c$ and $\epsilon_c$ are the values of $\rho$ and $\epsilon$ at the centre.]

The reason for the shape of the growth rate curves depicted in Fig. 2 is now apparent. The initial rapid rise, on a time scale of $10^8$ yr, occurs as the proportion of the energy generating core of the static model in which $^3$He has reached nuclear equilibrium increases to near unity. After that there is a slow rise, on a time scale of $10^9$ yr, as a gradient in $X_0$ and an associated augmented gradient in $X_{30}$ (see Fig. 1) is established. Strictly speaking, this argument assumes that the shapes of the perturbation eigenfunctions do not change with time. Actually the increasing central concentration of the equilibrium model with time also concentrates the shapes of the eigenfunctions (e.g. the nodes become closer to the centre) which reduces the efficiency of the $^3$He driving and is presumably responsible for the positive curvature in the growth rate curves in Fig. 2 in the vicinity of $t = 2 \times 10^8$ yr. It also causes a decrease in the period (see Table I).

These effects are evident also in the higher modes, but because the eigenfunctions
have a more intricate structure the growth rates depend on time in a rather more complicated way.

Several authors (e.g. Cowling 1941; Kopal 1949; Owen 1957) have computed non-radial oscillations using the simplified set of perturbation equations obtained by neglecting the perturbations in the gravitational potential. Cowling (1941) argued that this would make little difference to the eigenfrequencies and Robe (1968) demonstrated by direct computation that for polytropes the error is less than 10 per cent for $g$ modes. This is true also of the solar model we use here. For

![Amplitude of the radial component of displacement $\Xi$ (continuous lines) and Eulerian density and temperature perturbation amplitudes $P$, $\Theta$ (broken and dotted lines) as functions of $x = r/r_s$. The thick lines represent the mode calculated including the perturbation in the gravitational potential; the thin lines are modes computed ignoring it. Both eigenfunctions are $g_1(l = 1)$ modes at an age of $5 \times 10^8$ yr. Their periods and growth rates are included in Table II.](image)

illustration we compare in Table II the periods and growth rates of the $g_1(l = 1)$ modes computed with and without considering $\Phi'$.

It is also of interest to compare the eigenfunctions. In Fig. 5 is plotted the amplitudes $\Xi$ of the vertical components of displacement for the $g_1(l = 1)$ mode, with and without $\Phi'$, at $t = 5 \times 10^8$ yr, together with the amplitudes $P$ and $\Theta$ of the Eulerian density and temperature fluctuations. As in many other situations, the most obvious consequence of including the disturbance in the gravitational field is to accentuate the density perturbation. In the central regions the gravitational compression is so great that the sign of $\Theta$ is reversed so that dense upward displaced fluid is actually hotter than its surroundings. Consequently fluctuations in nuclear energy generation arising directly from the temperature dependence of the reaction rates damp the oscillations, in contrast to what calculations that ignore $\Phi'$ suggest. However, the growth rate of the oscillations is not dramatically affected
because it is controlled principally by the abundance perturbations. These are determined by only the vertical component of the displacement which does not depend sensitively on perturbations in the gravitational field.

Our results do not agree with those of Dziembowski & Sienkiewicz (1973). Partly, this is because we were not able to reproduce their equilibrium models exactly. We obtained a model with \( L = 0.972 L_\odot \), \( R = 1.051 R_\odot \) and a central hydrogen content \( X_c = 0.40 \), using their initial composition and mixing length. This approximates their Model 2 but is not an acceptable solar model because its age was only \( 3.6 \times 10^9 \) yr. The period of the \( g_1 (l = 1) \) oscillation was \( 0.0402 \) d compared with \( 0.0474 \) d found by Dziembowski & Sienkiewicz, and as far as we could judge from their Fig. 1 our relative Lagrangian pressure perturbation \( \delta p/p \) was in good agreement with theirs. But whereas Model 2 was found to be stable with an e-folding time of \( 1.85 \times 10^6 \) yr ours was unstable with an e-folding time of \( 1.25 \times 10^7 \) yr. Much of this discrepancy has arisen because Dziembowski & Sienkiewicz neglected to perturb the balance between the reactions in the pp chain (Dziembowski, private communication): our \( g_1 \) mode was stable if nuclear equilibrium was artificially imposed, but with a damping rate of only about one-third of that found by Dziembowski & Sienkiewicz.

7. DISCUSSION

The quasadiabatic calculation presented in this paper supports the suggestion that at some epochs the Sun has been unstable to non-radial oscillations. The principal factor responsible for the driving of the modes is the dependence of the nuclear energy generation rate on the \(^3\text{He} \) abundance, and becomes sufficiently important to destabilize the oscillations once the \(^3\text{He} \) abundance has reached its equilibrium value throughout most of the energy generating core.

The calculations need improving, however. To obtain reliable estimates of the growth rates of the linear modes it is necessary both to perform a consistent non-adiabatic calculation and to take into account the interaction between the oscillations and convection. The first is straightforward, and we hope to report on this in the near future. The second is more difficult, however, because the reliability of recipes for treating even equilibrium stellar convection zones is not known. There is reason to believe that in some regions of a star convection acts to stabilize the large-scale oscillations whereas in others it is destabilizing, so we know not even the sign of the integrated effect. Rough estimates of its magnitude, however, indicate that a more detailed investigation is crucial. Central to any such investigation is a treatment of the Reynolds stresses, which includes incorporating them into the equilibrium model. This has never been done in a consistent way, even within the framework of mixing-length theory.

Proving instability of the Sun to infinitesimal gravity wave perturbations is not sufficient to establish the occurrence of the mixing process discussed by Dilke & Gough. This is a non-linear instability. And even if mixing does occur it may not be complete, and hence intermittent in the way Dilke & Gough predict. It may, for example, take place continuously at just the rate to maintain almost neutral stability of the oscillations, in much the same way as many astrophysicists believe to be the case in semiconvection zones. If that is so, the correspondence between the time for \(^3\text{He} \) to come to equilibrium throughout the energy generating core and the intervals between the major terrestrial ice ages is a fortuitous coincidence.
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REFERENCES


APPENDIX

The behaviour of the solutions of equations (4.2)-(4.4) in the neighbourhood of $x = 0$ can readily be found in the usual way by substituting into the equations

$$\Xi = x^a \sum_{n=0}^{\infty} \Xi_n x^n, \quad \Xi_0 \neq 0, \quad (A1)$$

and similar expansions for $\Psi$ and $\Phi$. It is necessary also to expand the coefficients in the equations about the origin. The indicial equation for $\Xi$ has roots (Hurley, Roberts & Wright 1966; Smeyers 1967):

$$a = -(l+2), \quad -(l-1), \quad l+1 \quad (A2)$$

the first two of which must be rejected to keep $\Xi$ bounded at $x = 0$.

Because $\partial \rho / \partial x$ and $\partial T / \partial x$ are zero at the origin and $\partial X / \partial x = 0$ at $t = 0$, it follows that at $t = 0$, $\partial \rho / \partial x = 0$ and therefore $\partial \rho / \partial x = 0$ at $x = 0$. Consequently $\partial X / \partial x$ and $\partial \rho / \partial x$ remain zero at the origin for all $t$, since $\partial X / \partial t \propto \epsilon$. As a result...
the second terms of expansion (A1) and the corresponding expansions of $\Phi$ vanish, so that

$$\Xi = \Xi_0 x^{-l-1} + \Xi_2 x^{l+1} + \ldots \quad (A3)$$

and

$$\Psi = \Psi_0 x^l + \Psi_2 x^{l+2} + \ldots, \quad (A4)$$

$$\Phi = \Phi_0 x^l + \Phi_2 x^{l+2} + \ldots$$

The leading coefficients are related by

$$\Psi_0 = \frac{\sigma^2}{l} \Xi_0 + \Phi_0, \quad (A5)$$

where $\sigma$ is defined by equation (4.6), and $\Phi_0$ is obtained in terms of $\Phi$ and $d\Phi/dx$ at $x = x_0$ by eliminating $\Phi_2$ between equations (4.9) and the last of equations (4.8):

$$\Phi_0 = \frac{1}{2} x_0^{-l-1} \left[ (l+2) \Phi - x \frac{d\Phi}{dx} \right]_{x = x_0}. \quad (A6)$$

The second-order coefficients are given by

$$\Xi_2 = \frac{1}{2(2l+3)} \left\{ - \left[ (l+1) \frac{d^2 \rho}{dx^2} + \frac{1}{\Gamma_{10}} \frac{d^2 \rho}{dx^2} + \frac{l(l+1)}{\sigma^2} \frac{d^2 \rho}{dx^2} \frac{d^2 \rho}{dx^2} - \frac{1}{\Gamma_{10}} \frac{d^2 \rho}{dx^2} \right] \Xi_0 + \left( \frac{l+2}{\Gamma_{10}} \right)^2 \Xi_0 \right\},$$

$$\Psi_2 = \frac{1}{2(2l+3)} \left\{ \left[ \frac{\sigma^2}{\Gamma_{10}} \frac{d^2 \rho}{dx^2} + 3\frac{\sigma^2}{\Gamma_{10}} \frac{d^2 \rho}{dx^2} - (l+3) \frac{d^2 \rho}{dx^2} \frac{d^2 \rho}{dx^2} - \frac{1}{\Gamma_{10}} \frac{d^2 \rho}{dx^2} \right] \Xi_0 \right\}$$

$$+ \left\{ \frac{l^2}{\Gamma_{10}} \frac{d^2 \rho}{dx^2} + l + 3 \frac{d^2 \rho}{dx^2} - \frac{\sigma^2}{\Gamma_{10}} \right\} \Phi_0 + \Phi_2, \quad (A7)$$

$$\Phi_2 = \frac{-\Lambda}{2(2l+3)} \left\{ \left( \frac{\sigma^2}{\Gamma_{10}} + \frac{1}{\Gamma_{10}} \frac{d^2 \rho}{dx^2} - \frac{d^2 \rho}{dx^2} \right) \Xi_0 + \frac{1}{\Gamma_{10}} \Phi_0 \right\}, \quad (A8)$$

which are evaluated at $x = 0$.  

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