FURTHER ASPECTS OF WEAK SHOCK THEORY APPLIED TO THE SOLAR CHROMOSPHERE

STUART D. JORDAN

Laboratory for Solar Physics, NASA-Goddard Space Flight Center, Greenbelt, Md. 20771, U.S.A.

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Abstract. The low chromosphere now seems definitely to require mechanical heating, and dissipation of initially acoustic waves by shocking is one of the most promising possibilities. Results of recent calculations indicate that the weak shock theory may be applicable here, but discrepancies exist among various applications of this theory, and the explanations offered to date are not completely satisfactory. It is shown here that the different approximations used by different authors to evaluate the mechanical flux integral play an important role in producing these discrepancies, in addition to the already well known effects of the density scale heights and the wave periods. Arguments are presented favoring Ulmschneider's method for evaluation of this flux integral.

There has been disagreement about weak shock theory applied to the solar chromosphere since Osterbrock (1961) claimed that it could predict the mechanical heating in both magnetic (network and plage) and non-magnetic regions. This conclusion has been challenged or, at least, modified many times. Pikel'ner and Livshits (1965) argued that weak shock propagation in magnetic regions \( B \geq 10 \, \text{G} \) in the chromosphere would be so strongly affected by non-linear processes that no reliable estimates of the heating could be obtained. Jordan (1970) then took the equations of Osterbrock and, retaining wave periods \( T \) of the order of 100 s or more in the non-magnetic regions, showed that, for more current atmospheric models unavailable to Osterbrock, a catastrophically rapid growth of the shock strength occurs.

On the other hand, Ulmschneider (1970, 1971a, b) has concluded that the shock strength grows slowly enough in the non-magnetic chromosphere to preserve the validity of the weak shock theory up to the base of the transition region. Furthermore, his calculations yield an almost constant value for the shock strength throughout this region, and the corresponding mechanical heating does not differ significantly from local net radiation loss estimates. Since these calculations are done for wave periods which are short compared to 100 s (cf. Stein, 1968), Ulmschneider concludes that this accounts for the smaller values of the computed shock strength.

While it is true that shorter periods yield smaller values for the shock strength, it is not obvious, and, in fact, proves to be untrue, that the discrepant results are due mainly to the different periods. The calculations reported here show that the major source of the discrepancy is due to differences in the way the mechanical flux integral is approximated.

To see this, consider the mechanical flux past a point in a fluid due to a periodic wave train of period \( T \):
\[ \pi F_+ = \frac{1}{T} \int_0^T [P(t) - P_0] U(t) \, dt, \]  
(1)

where \( P(t) \) and \( U(t) \) are the time dependent pressure and rest frame velocity, respectively, and \( P_0 \) is the equilibrium pressure at the point. Osterbrock evaluates this integral by assuming that the time dependence of \( P(t) \) and \( U(t) \) permits one to define a characteristic time \( t_0 \), so that the integral will reduce to

\[ \pi F_+ = U_2 (P_2 - P_0) t_0, \]  
(2)

where subscript 2 refers to the values of the parameters just behind the shock front. Application of the Rankine-Hugoniot conditions, dropping the subscripts from equilibrium values of the parameters, yields

\[ \pi F_+ = \left( \frac{t_0}{T} \right) \gamma P c \eta^2, \]  
(3)

where \( \gamma \) is the ratio of specific heats, \( c \) is the local sound speed, and \( \eta \) is the shock strength parameter, defined by densities \( \varrho_2 \) and \( \varrho_1 \) (subscript 1 for conditions just ahead of the shock) as

\[ \eta = \frac{\varrho_2 - \varrho_1}{\varrho_1}. \]  
(4)

An alternate method for evaluating the mechanical flux integral is to use an appropriate expression from fluid dynamics for relating \( P(t) \) and \( U(t) \), thus circumventing the ambiguities associated with the characteristic time parameter. If one assumes that all anisentropic effects associated with compressing and heating the gas occur in the very narrow shock discontinuity, the \( P(U) \) relation during the relaxation period can be approximated from the theory for a simple wave in the perfect gas. From Landau and Lifshitz (1959), this can be written as

\[ P = P_0 \left( 1 + \frac{1}{2} (\gamma - 1) \frac{U}{c} \right)^{2\gamma/(\gamma - 1)}. \]  
(5)

If one assumes further that the shock waves are weak enough to preserve \( U/c \ll 1 \), which is confirmed by the calculations for the shorter periods, expression (5) becomes

\[ P \approx P_0 \left( 1 + \gamma \frac{U}{c} \right). \]  
(6)

It is now possible to evaluate the mechanical flux integral with expression (6) for \( P(U) \), where \( U(t) \) is either a true sawtooth (relaxation time \( \tau_0 \) for \( U(t) \) equals the period \( T \)) or else a modified sawtooth for which \( \tau_0 < T \). The case \( \tau_0 = T \) was treated by Ulmschneider on the grounds that calculations by Kuperus (1969) strongly support
use of the sawtooth profile. Using Ulmschneider’s notation,

\[ U(t) = U_{00} - \frac{2U_{00}}{T} t, \quad \tau_0 = T \]  

(7)

where \( U_{00} \) is half the velocity jump across the shock front, and where the mean velocity \( U_0 = U(t) \) is assumed to be zero over a cycle, corresponding to negligible mass flow outward. For the more general case \( \tau_0 \ll T \), again using \( U_0 = 0 \),

\[ U(t) = U_{00} (2 - q) - \frac{2U_{00}}{\tau_0} t, \quad 0 < t < \tau_0 \]  

(8a)

\[ U(t) = -U_{00} q, \quad \tau_0 < t < T \]  

(8b)

where

\[ q = \frac{\tau_0}{T}. \]

Since expressions (8) are the more general, one needs only to evaluate the mechanical flux integral with them, using expression (6) for \( P(U) \), to obtain a very general expression for \( \pi F_+ \) due to a weak shock train. This becomes

\[ \pi F_+ = \frac{\gamma P}{c} U_{00}^2 \left( \frac{3}{4} - q \right) q, \]

(9)

which can be reduced further by recalling that \( U_{00} = \frac{1}{2} (U_2 - U_1) \) and invoking the Rankine-Hugoniot condition to give

\[ \pi F_+ = \frac{1}{2} \gamma P c \eta^2 \left( \frac{3}{4} - q \right) q, \quad q \leq 1. \]

(10)

This is the form for the mechanical flux integral used here. For \( q = 1 \), it reduces to the form used by Ulmscheider, whose linear \( P(U(t)) \) relation is consistent with the behavior of a simple wave subject to the weak shock approximation of expression (6). Note, in passing, that the numerical coefficient of the product \( \gamma P c \eta^2 \) changes very little in the range \( \frac{1}{2} < q < 1 \), and is, in fact, equal to \( \frac{1}{2} \) for both \( q = 1 \) and \( q = \frac{1}{2} \). Solutions of the shock differential equation for equal periods show that, only when \( q \) is much smaller than unity, corresponding to \( \tau_0 \ll T \), is the growth rate for the shock strength strongly dependent on \( q \). This corresponds to the relativity greater importance of using a correct \( P(U) \) correlation than in obtaining a correct picture, in detail, of the velocity field relaxation.

The basic differential equation usually used for the shock strength calculations is due, ultimately, to Brinkley and Kirkwood (1947). A convenient form for the calculations reported here is

\[ \frac{d}{dh} (\pi F_+) = -\frac{1}{12} \frac{\gamma (\gamma + 1) P}{T} \eta^3, \]

(11)

for one-dimensional upward propagation.
Osterbrock's Equation (74) follows directly from Equation (11), for the special case of zero magnetic field, one dimensional propagation, and $\pi F_+$ represented by expression (3). Ulmschneider's Equation (15) in the 1970 reference also follows from Equation (11), using expression (10) with $q=1$ for the mechanical flux. Multiplying the right-hand side of Equation (11) by $1/\cos \theta = \sqrt{3}$ to represent equal fluxes in all upward directions, and using expression (10) for the mechanical flux, we get

$$\frac{d}{dh} \left[ \ln \left( PT_e^{1/2} \eta^2 \right) \right] = -\frac{(\gamma + 1)}{\sqrt{3} \left( \frac{3}{4} - q \right) \epsilon \tau_0} \eta,$$

(12)

for the case of constant $\gamma$ and mean molecular weight, negligible refraction, and zero magnetic field. These assumptions are generally valid in the non-magnetic regions of the low chromosphere, below the zone of significant hydrogen ionization. The subscript on the kinetic temperature $T_e$ distinguishes it from the period $T$.

The calculations reported here are performed with Equation (12) and values of $(P, T_e)$ from both the Harvard Smithsonian Reference Atmosphere (Gingerich et al., 1971) and, for comparison with earlier results, with the PSC model due to Thomas and Athay (1961).

It is easy to integrate Equation (12) in one of several ways (cf. Jordan, 1970) to obtain $\eta(h)$ for a given atmospheric model. The calculations are done for the following ranges of values of the parameters $\tau_0$, $T$, and $\eta(0): 10 \leq \tau_0 \leq 100$ s in 10 s intervals subject to the condition $\tau_0 \leq T$; $T = 30, 90, 300$ s; and starting value $\eta(0) = 0.10, 0.21, 0.32, 0.43, 0.54$. These values span the ranges encountered in the literature, where they are justified by various theoretical and observational considerations. The starting point for the shock calculations is the height at which the temperature rise commences. The zero point in the models occurs at optical depth $\tau_{5000}$ (normal) $= 10^{-3}$.

Table I illustrates a few salient results of the calculations for the PSC model. The first two columns given the values of $\eta(h)$ reported in the Jordan reference, using the Osterbrock form for the mechanical flux $\pi F_+$ given by expression (3). One sees easily that, by substituting this expression for $\pi F_+$ into Equation (11), the resulting differential equation for $\eta(h)$ is independent of the period $T$. The first column shows that, even for $\eta(0) = 0.10$, a starting value which is probably too small, the shock strength $\eta$ grows very rapidly with height $h$, exceeding the limiting value for validity $\eta = 1$ below 750 km, so that, above this height, the theory would appear to have no validity whatsoever, even for $\tau_0 = 30$ s, a value much lower than the earlier studies used.

The results are quite different when the calculations are done using expression (10) for the flux $\pi F_+$. Two important effects show up immediately. First, for the period $T = 30$ s case, $\eta$ grows slowly with height, remains almost constant over a large part of the low chromosphere, and stays well within the range of validity of weak shock theory. In fact, for $T = 30$ s, the almost asymptotic values for $\eta$ above 750 km are only weakly dependent on $\eta(0)$. Second, for the longer period cases, the weak shock theory has become marginal where $T = 90$ s and has broken down completely for $T = 300$ s at some height in the low chromosphere.
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<tr>
<th>$h$ (km)</th>
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<td>$\eta(0) = 0.32$</td>
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<tr>
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Figure 1 shows how strongly the growth of the shock strength depends on the period $T$ for the HSRA model. Again, the striking difference between the short and long period cases is exhibited.

Figure 2 shows the dependence of the results on the parameter $q = \tau_0/T$, for the case $T = 90$ s, using the HSRA model. Note that, for $q = 1$ and $q = \frac{1}{3}$, the same $\eta(h)$ is obtained. There is, in fact, little variation in the solutions of Equation (12) in the range $\frac{1}{3} \leq q \leq 1$; and, as Figure 2 shows, only when $q$ becomes smaller than $\frac{1}{3}$ do we get a significant change in the values of $\eta(h)$.

These results demonstrate that, in several applications of weak shock theory in the solar chromosphere, the different ways of evaluating the mechanical flux integral have produced the major discrepancies in the results. Furthermore, it is shown that the method already employed by Ulmschneider is valid for low chromospheric conditions where the shock strength $\eta$ is initially small compared to one. One can show easily with such simple calculations the effects of the density scale height in the atmosphere, of the wave period, and of the approximations involved in evaluating the mechanical flux integral on the growth of the shock strength.

In addition, it is found that the behavior of $\eta(h)$ is very insensitive to the ratio $q$ of the relaxation time $\tau_0$ for the velocity field to the period of the disturbance $T$, if this ratio lies in the range $\frac{1}{3} \leq q \leq 1$. For smaller values of $q$, the $\eta(h)$ grows even more slowly with height than for $\frac{1}{3} \leq q \leq 1$. If the waves heating the chromosphere are initially sinusoidal disturbances forming a wave train, the case $q = 1$ is the only case of interest. However, the possibility that pulses occurring at irregular intervals might heat the
chromosphere is discussed and analyzed by Stein and Schwartz (1972). Their results for the behavior of $\eta(h)$ concur with the results given here, where $\tau_0 = T$. By noting that an average period for these pulses can be defined at a point, we see that Ulmschneider’s conclusion that the shock strength remains small in the chromosphere is correct, regardless of whether the wave is an initially sinusoidal short period wave train or a series of pulses, providing that, in the case of pulses, the passage time for the pulse is not too long (not larger than $\sim 100$ s).

Whether the weak shock theory is, in fact, appropriate for studying the chromospheric heating lies beyond the scope of this study. Reasons favoring it’s use as well as detailed applications appear in the Ulmschneider papers, and some reservations are found in Schwartz and Stein (1973). The remarks here serve only to elucidate the sources of discrepant results appearing in the literature and to point out what form of the theory is appropriate, given tangible evidence for the presence of relatively short period waves in the low chromosphere of the Sun.

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References