FREQUENCY-QUADRATURE PERTURBATIONS IN
RADIATIVE-TRANSFER THEORY

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Received 1973 January 19; revised 1973 April 30

ABSTRACT
A new numerical method is presented for solving the equation of radiative transfer for spectral-line formation. This method enables the integral over frequency to be represented by a quadrature sum of lower order than that previously used to obtain solutions of the required accuracy. In particular, it is found that the method is best suited to slab geometry where, using only one point quadrature over frequency, results are obtained at least as accurate as those obtained from other existing numerical methods. Although more frequency-quadrature points are necessary in semi-infinite situations, the method gives sufficiently accurate solutions for $\alpha \sim C_{ul}/A_{ul}$ of order $10^{-4}$, for example, with just three quadrature points. Thus, since at least seven quadrature points have been required in the past for numerical solution of even the most simple problem in spectral-line formation theory, the computer time and storage required for this new method is significantly reduced. Further, the technique is quite general and may be used in any radiative-transfer problem involving integration over frequency. It may also be used, if so desired, in conjunction with other methods and may therefore be easily incorporated into existing computer programs.

Subject headings: line formation — radiative transfer

I. INTRODUCTION

The problems of astrophysical interest occurring in the theory of spectral-line formation have, apart from some simple model situations, necessitated the numerical solution of the equation of radiative transfer. This has almost always resulted in the integral over frequency being approximated by a quadrature sum with the quadrature used and, more particularly, the spacing of the frequency-quadrature points, conforming to definite specifications for the computation of solutions to satisfactory accuracy. For example, the spacing of adjacent frequency points should be no larger than approximately one-half a Doppler width (see Hummer and Rybicki 1967). Even so, the number of quadrature points required in the past was no larger than about 10 and thus, limitations in available computer storage and time were of no significant concern. In recent years, however, such complex problems as Hα and Lα line formation involving continuum absorption, and the formation of spectral lines in atmospheres exhibiting large velocity gradients (for example, Wolf-Rayet stars and Cepheid variables), have resulted in the numerical solution of the transfer equation using more than 100 frequency-quadrature points. Further, optically thin atmospheres are quite often considered in these situations so that the well-known Eddington approximation breaks down. This then necessitates the use of more than one-point quadrature over angle. The corresponding computer time and storage required to solve such problems increases very significantly, and therefore the computer limitations become an ever increasing restriction on the solutions obtained. In a recent paper (Cannon 1973, hereinafter referred to as Paper I), the author developed a method, called the angular-quadrature perturbation technique (AQPT), which overcomes the difficulty associated with integration with respect to angle. This method enables the replacement of the integral by a one-point quadrature sum (i.e., the Eddington approximation) such that, with the use of a perturbation series, solutions to the transfer equation may be obtained at least as accurate as those using
higher-order quadrature. The basic advantage of the method is that the time required for the computation of the solution is only fractionally more than that using the Eddington approximation directly. The method can also be used together with any of the existing numerical methods of solution, and may be generalized to any transfer problem in which integration over angle is required.

Here we wish to generalize the results presented in paper I, with their subsequent advantages, to frequency quadrature in an attempt to overcome the computing difficulties mentioned above. We test this new frequency-quadrature perturbation technique (FQPT) on two rather difficult (numerically) radiative-transfer problems of spectral-line formation involving large velocity gradients in slab geometry where it is found that satisfactory results may be obtained using just one frequency-quadrature point and one angle-quadrature point. Corresponding calculations using the FQPT in semi-infinite atmospheres indicate that one-point quadrature over frequency is insufficient for adequate convergence. Satisfactory results can, however, be obtained using higher-order quadrature, and although the computer time required is therefore increased, the total time taken for the FQPT is still significantly smaller than that required using other methods.

II. QUADRATURE PERTURBATION ANALYSIS

For clarity in exposition, we will illustrate the development of the method on the one-dimensional equation of transfer, specifying the interaction of spectral-line radiation with a differentially moving atmosphere. We take the atmosphere to consist of two-level atoms and electrons in complete redistribution. It should be emphasized, however, that the method can be generalized to any problem occurring in radiative-transfer theory.

The equation to be solved then has the form (see, e.g., Hummer and Rybicki 1967)

\[ \frac{\partial I_v}{\partial \tau} (\tau, \mu) = \phi(\nu - V\mu)[I_v(\tau, \mu) - S(\tau)] , \quad \mu = \cos \theta , \quad (1) \]

\[ S(\tau) = \frac{1}{2}(1 - \epsilon) \int_{-1}^{1} d\mu \int_{-\infty}^{\infty} d\nu \phi(\nu - V\mu)I_v(\tau, \mu) + \epsilon B_v(T_e) \quad , \quad (2) \]

where \( I_v(\tau, \mu) \) is the specific intensity of radiation at mean optical depth \( \tau \) in a direction \( \theta \) from the normal to the surface at frequency \( \nu \) from line center. The probability of absorption \( \phi(\nu - V\mu) \) is also dependent upon the macroscopic velocity \( V(\tau) \) of the atmosphere, where \( V(\tau) \) is measured in units of Doppler widths. The quantity \( B_v(T_e) \) is the Planck function at the local electron temperature \( T_e \) and \( \epsilon \) is essentially the ratio of collisional to radiative de-excitation of the atoms.

To solve equations (1) and (2) we approximate the integrals occurring in equation (2) by quadrature sums. This is standard procedure in numerical solutions, but here we also include an error term \( E(\tau) \) which thus renders the approximation exact. We therefore have

\[ (1 - \epsilon) \int_{-1}^{1} d\mu \int_{-\infty}^{\infty} d\nu \phi(\nu - V\mu)I_v(\tau, \mu) \]

\[ = (1 - \epsilon) \sum_{i=1}^{2N_p} \sum_{j=1}^{2N_p} a_{ij}\phi(\nu_j - V\mu_i)I_v(\tau, \mu_i) + E(\tau) \quad , \quad (3) \]

where \( a_{ij}, \mu_i, \) and \( \nu_j \) are the quadrature weights and points corresponding to quadrature.
of order \(N_A\) and \(N_F\) over angle and frequency. Clearly, the larger \(N_A\) and \(N_F\), the smaller will be the error \(E\).

Here we wish to develop a perturbation series in the radiation field with \(N_A\) and \(N_F\) as small as possible. Although in practice we solve equations (1) and (2) using the Feautrier (1964; Auer 1967) technique, we formally illustrate the frequency-quadrature perturbation technique by combining equations (1) and (2) in integral form such that

\[
S(\tau) = \Lambda S + \epsilon B ,
\]

where the operator \(\Lambda\) has the form

\[
\Lambda = (1 - \epsilon) \int_0^T d\tau' K_1(\tau, \tau') ,
\]

where \(T\) is the optical thickness of the medium, and where

\[
K(\tau, \tau') = \frac{1}{2} \int_{-\infty}^\infty dv \int_0^1 \frac{d\mu}{\mu} \phi[v - \mu\nu(t)]\phi[v - \mu\nu(\tau')]
\times \exp \left\{ - \int_{\tau}^{\tau'} \phi [v - \mu\nu(t)] \frac{dt}{\mu} \right\} .
\]

One may think of \(\Lambda\) as the operation corresponding to a large number of quadrature points denoted by \(N_A^{(2)}\) and \(N_F^{(2)}\). We then define \(\Lambda^*\) as the more approximate operator corresponding to a smaller number of quadrature points which we denote by \(N_A^{(1)}\) and \(N_F^{(1)}\).

Equation (4) may then be rewritten as

\[
S(\tau) = (1 - \Lambda)^{-1} \epsilon B ,
\]

\[
= [1 - \Lambda^* - (\Lambda - \Lambda^*])^{-1} \epsilon B ,
\]

\[
= [1 - (1 - \Lambda^*)^{-1} (\Lambda - \Lambda^*)]^{-1} (1 - \Lambda^*)^{-1} \epsilon B ,
\]

\[
= [1 + (1 - \Lambda^*)^{-1} (\Lambda - \Lambda^*)
+ [(1 - \Lambda^*)^{-1} (\Lambda - \Lambda^*)]^2 + \cdots ] (1 - \Lambda^*)^{-1} \epsilon B ,
\]

\[
= \sum_{k=1}^{\infty} S_k(\tau) ,
\]

where \(S_1 = (1 - \Lambda^*)^{-1} \epsilon B\), and \(S_k = (1 - \Lambda^*)^{-1} (\Lambda - \Lambda^*) S_{k-1}\) for all \(k \geq 2\), i.e.,

\[
S_1 = \Lambda^* S_1 + \epsilon B
\]

and

\[
S_k = \Lambda^* S_k + E_{k-1} ,
\]

where

\[
E_{k-1} = (\Lambda - \Lambda^*) S_{k-1} \quad \text{for all} \quad k \geq 2 .
\]

It is not difficult to show that the above series converges to the required source function.
Clearly, equation (9) is the usual equation of transfer for spectral-line radiation in which integrals over angle and frequency are approximated by quadrature sums of order \(N_A\) and \(N_F\), respectively. Equation (10), however, specifies the corrections to be made to the radiation field determined from equation (9).

In performing the computations, one solves equation (9) in its integro-differential form using \(N_A^{(1)}\) and \(N_F^{(1)}\) order quadrature. One then obtains \(\Delta S_1\) using \(N_A^{(2)}\) and \(N_F^{(2)}\) quadrature by solving equation (1) with \(S = S_1\). Since \(\Delta S_1\) is already known from equation (9), one may obtain the first-order error term \(E_1(\tau)\) immediately. One then solves equation (10) for \(S_2\) and the process is repeated. The computed \(E_2(\tau)\), however, are accurate only to order \(N_A^{(2)}\) and \(N_F^{(2)}\), and thus the choice of \(N_A^{(2)}\) and \(N_F^{(2)}\) will directly affect the accuracy of the resulting solution. Any further errors will be a result of the depth discretization used to solve equations (9) and (10).

It is important to realize that equation (10) is identical to equation (9) except for the nonhomogeneous source terms \(E_k\). Thus, equation (10) may be solved in exactly the same manner as equation (9). More particularly, if one uses the Feautrier method of solution (Feautrier 1964; Auer 1967), the required matrices and their inverses need only be calculated once and thence stored for future use in equation (10). Most important, however, since the determination of these contributes overwhelmingly to the total computer time needed, the extra time required to solve for the correction terms is insignificant. Further, no extra reformulation of the transfer equation is required to obtain equation (9). Thus, the technique presented here may be used in conjunction with any of the existing numerical methods of solution, and may be applied to any radiative-transfer problem involving both angle and frequency integration.

### III. Calculations

We apply the frequency-quadrature perturbation technique to spectral-line transfer problems in an infinite slab of thickness \(\tau = T\) (case I) and to a semi-infinite atmosphere (case II). We take \(\epsilon\) and \(B_\epsilon(T_s)\) to be constant and assume a purely Doppler broadened line such that

\[
\phi(\nu) = \frac{1}{\Delta v_D \sqrt{\pi}} \exp\left[-\left(\frac{\Delta \nu}{\Delta v_D}\right)^2\right],
\]

where the Doppler width \(\Delta v_D\) is also taken to be constant. In case I, we test the method on the problem of a differentially expanding atmosphere. The macroscopic velocity field is assumed to have the form

\[
V(\tau) = \frac{2V_0}{T} \left(\frac{T}{2} - \tau\right),
\]

where \(|V_0|\) is the maximum of the velocity field, occurring at the surface of the slab, in units of Doppler widths.

The depth grid used in performing the calculations consisted of a first step of \(\tau = 0.01\), each subsequent step being twice the former, with a depth point for case I at the center of the slab. We also incorporate the improved boundary condition for Feautrier's technique developed by Auer (1967). Gaussian quadrature over heliocentric angle was used such that \(N_A^{(1)} = 1\) (i.e., the Eddington approximation with \(\mu = \pm 1/\sqrt{3}\)) and \(N_F^{(2)} = 3\). The trapezoidal rule was employed for quadrature over frequency with \(N_F^{(2)} = 12\).

Before actually discussing the results of the computations in detail, one should mention the problem associated with the frequency bandwidth and the corresponding
frequency-quadrature points $\nu_j^{(1)}$. In effect, when performing the numerical integration over frequency, we have

$$\int_{-\infty}^{\infty} \phi(v - V\mu)I_s(\tau, \mu)dv \simeq \int_{-\beta}^{\beta} \phi(v - V\mu)I_s(\tau, \mu)dv,$$

(12)

so that equation (3) is only an approximate relationship, and thus another source of error is introduced. Of course, one chooses the bandwidth $\beta$, sufficiently large for the desired accuracy (see, e.g., Hummer and Rybicki 1967). For the velocity fields considered in this paper, it was found that a bandwidth of $\beta = 6.2\Delta\nu_D$ gave a solution accurate to four significant figures. Thus, when solving equation (1) to obtain $\Delta S_{k}$, one uses this $\beta$ together with the appropriate $N_{F}^{(2)}$ to determine the higher-order frequency-quadrature points $\nu_j^{(2)}$. However, the choice of $\nu_j^{(1)}$ is quite arbitrary. For example, if one chooses $N_{F}^{(1)} = 1$, i.e., one-point quadrature over frequency, then one has a choice of $\nu_j^{(1)}$ such that $0 \leq \nu_j^{(1)} \leq \beta$. Clearly, then, the value of $\nu_j^{(1)}$ giving the best convergence will depend upon the problem being considered. We will return to this point later.

The accuracy of the results obtained is best illustrated using the ratio $\chi_{k}(\tau)$ given by

$$\chi_{k}(\tau) = \frac{S_{k}(\tau) - S_{(k)}(\tau)}{S_{k}(\tau)},$$

(13)

where $S_{(k)}(\tau)$ is the source function obtained after $k$ perturbations, i.e.,

$$S_{(k)}(\tau) = \sum_{k'=1}^{k} S_{k'}(\tau),$$

(14)

$S_{k}(\tau)$ is the "exact" source function obtained by solving equation (9) directly using $N_{A}^{(1)} = 3$ and $N_{F}^{(1)} = 12$. It should be emphasized here that differences between $S_{k}(\tau)$ and the true exact source function are a result of the depth discretization used, and the approximation introduced by a finite bandwidth (i.e., eq. [12]). In fact, with the chosen numerical depth grid, $S_{k}(\tau)$ calculated in this paper is accurate to at least four significant figures. Most important, however, since the same depth discretization is used to determine both $S_{k}(\tau)$ and $S_{(k)}(\tau)$, the ratio $\chi_{k}(\tau)$ offers a highly satisfactory means of studying the convergence of the series given by equation (8).

IV. RESULTS

Calculations were made for two slabs of thickness $T = 2$ and $T = 10$, and for a semi-infinite atmosphere. In all three situations we take zero incident radiative flux at the surfaces of the media, with $\varepsilon = 10^{-4}$ and $V_D = 3\Delta\nu_D$. For case I, the source functions, and thus $\chi_{k}(\tau)$, are symmetric about the center of the slabs.

Table 1 shows the results of the computation of $\chi_{k}(\tau)$ for the slab of thickness $T = 2$ using both $N_{F}^{(1)} = 1$ and $N_{F}^{(1)} = 3$. We arbitrarily put $\nu_1^{(1)} = 3.1\Delta\nu_D$ for the one-point quadrature over frequency, and choose a trapezoidal rule for $N_{F}^{(1)} = 3$ such that $\nu_1^{(1)} = 0$, $\nu_2^{(1)} = 3.1\Delta\nu_D$ and $\nu_3^{(1)} = 6.2\Delta\nu_D$. Clearly, the convergence is excellent in both quadrature situations. For example, with just one-point quadrature over angle and frequency, the solution converges from an accuracy of order 18–32 percent after the zeroth-order perturbation to an accuracy of order 0.01–0.02 percent after only seven terms in the perturbation series. The convergence is even faster for $N_{F}^{(1)} = 3$.

The results of $\chi_{k}(\tau)$ for the slab of thickness $T = 10$ are shown in table 2. First, we consider one-point quadrature over frequency (i.e., $N_{F}^{(1)} = 1$) with the three different frequency points $\nu_1^{(1)} = 0$, $\nu_1^{(1)} = 1\Delta\nu_D$, and $\nu_1^{(1)} = 3.1\Delta\nu_D$. Clearly, $\nu_1^{(1)} = 1\Delta\nu_D$ offers the superior convergence of the three $\nu_1^{(1)}$. In this case, $S_{(1)}(\tau)$ is accurate
TABLE 1
VALUES OF $\chi_k(\tau)$ FOR $T = 2$ USING THE FQPT

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\chi_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.18 0.048 0.014 0.0040 0.0012 0.00034 0.00010</td>
</tr>
<tr>
<td>1</td>
<td>0.32 0.097 0.029 0.0084 0.0025 0.00073 0.00021</td>
</tr>
</tbody>
</table>

B. $N_f^{(1)} = 3; \nu_1^{(1)} = 0, \nu_2^{(1)} = 3.1\Delta\nu_D, \nu_3^{(1)} = 6.2\Delta\nu_D$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\chi_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.083 0.0032 0.00044 0.0000080</td>
</tr>
<tr>
<td>1</td>
<td>0.064 0.0080 0.00065 0.000073</td>
</tr>
</tbody>
</table>

Also shown in table 2 are the results of $\Lambda$-iteration for $T = 10$. In performing the $\Lambda$ operation, we start from an initial solution obtained from equation (9) with $N_A^{(1)} = 1, N_f^{(1)} = 1$ and recompute the source function from equation (1) with $N_A^{(2)} = 3$ and $N_f^{(2)} = 12$. We have only listed the values of $\chi_k(\tau)$ using $\nu_1^{(1)} = 1\Delta\nu_D$. Clearly, the convergence is far inferior to the corresponding convergence obtained using the FQPT. In fact, the error is a factor of at least 30 greater than that using the FQPT after the same number of perturbations and iterations. Furthermore, the convergence for $\nu_1^{(1)} = 0$ and $\nu_1^{(1)} = 3.1\Delta\nu_D$ using the FQPT is oscillatory in nature, and this compares favorably with that using $\Lambda$-iteration.

The results of $\chi_k(\tau)$ for $T = 10$ using third-order quadrature over frequency (i.e., $N_f^{(1)} = 3$) with $\nu_1^{(1)} = 0, \nu_2^{(1)} = 3.1\Delta\nu_D, \nu_3^{(1)} = 6.2\Delta\nu_D$ are also shown in table 2. Here we have convergence to within approximately 0.3 percent after four terms in the perturbation series, and an accuracy of order 0.02 percent after eight terms. It was found that in all calculations made using the FQPT, the convergence increased for increasing $N_f^{(1)}$. This is not surprising since one has, for larger $N_f^{(1)}$, a more representative knowledge of photon loss from the atmosphere than otherwise.

Difficulties did arise, however, when using the FQPT in semi-infinite atmospheres with $N_f^{(1)} = 1$. First, the source functions $S_k$, and thus $\Lambda S_k$ and $\Lambda^* S_k$, are at least a factor of $\epsilon^{-1/2}$ larger than the error source terms $E_k(\tau)$. For example, in a stationary constant property atmosphere, $\sqrt{\epsilon B} \leq S_k \leq B$, whereas the source term is only $\epsilon B$. Thus, since the $k$th error source term $E_k = \Lambda S_k - \Lambda^* S_k$ involves the difference of two relatively large quantities, the $\Lambda S_k$ and $\Lambda^* S_k$ must be determined accurate to a sufficient number of significant figures to ensure $|E_{k+1}(\tau)| < |E_k(\tau)|$. Otherwise, convergence will not be reached. This difficulty can be overcome, however, by using a larger $N_f^{(1)}$ or, in some cases, a finer depth-discretization grid. It should be emphasized that the requirement $|E_{k+1}(\tau)| < |E_k(\tau)|$ applies only for depths at and below a
TABLE 2
VALUES OF $x_{rc}(\tau)$ FOR $T = 10$ USING THE FQPT AND $\Lambda$-ITERATION

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>0</th>
<th>5</th>
<th>$\tau$</th>
<th>0</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) FQPT: $N_f^{(1)} = 1, \nu_{1}^{(1)} = 0$</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>1.</td>
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<td>-0.45</td>
<td>7.</td>
<td>-0.026</td>
<td>-0.035</td>
</tr>
<tr>
<td>2.</td>
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<td>+0.36</td>
<td>8.</td>
<td>+0.012</td>
<td>+0.016</td>
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<tr>
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</tr>
<tr>
<td>4.</td>
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<td>+0.13</td>
<td>10.</td>
<td>+0.0043</td>
<td>+0.0057</td>
</tr>
<tr>
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<td>-0.0044</td>
</tr>
<tr>
<td>6.</td>
<td>+0.034</td>
<td>+0.045</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) FQPT: $N_f^{(1)} = 1, \nu_{1}^{(1)} = 1\Delta\nu_D$</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>c) FQPT: $N_f^{(1)} = 1, \nu_{1}^{(1)} = 3\Delta\nu_D$</td>
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<td>d) FQPT: $N_f^{(1)} = 3, \nu_{1}^{(1)} = 0, \nu_{2}^{(1)} = 3.1\Delta\nu_D, \nu_{3}^{(1)} = 6.2\Delta\nu_D$</td>
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<td>e) A-Iteration: $\nu_{1}^{(1)} = 1\Delta\nu_D$</td>
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<tr>
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</tbody>
</table>

thermalization path length from the surface since it is the source terms at these depths (and below) which nonlocally control the radiation field. Clearly, since $S_1 \approx T e B$ for $T \ll 1/e$, this difficulty does not arise for slabs of optical thickness significantly smaller than a thermalization path length. In fact, it was found that the computations were particularly straightforward and easy to program for all such slabs.

The second, and as it happens, associated difficulty involves the choice of $\nu_{1}^{(1)}$ when $N_f^{(1)} = 1$. To illustrate this, we write each term in the series given by equation (8) as

$$S_k = \Lambda^* S_k + (\Lambda - \Lambda^*) S_{k-1},$$

where the kth term $S_k^{(A)}$ using $\Lambda$-iteration has the form

$$S_k^{(A)} = \Lambda S_{k-1}^{(A)}.$$
Thus, for example, if we consider an infinite (constant property) atmosphere such that

\[ \Lambda = (1 - \varepsilon) \int_{-\infty}^{\infty} \, d\tau' K_1(|\tau - \tau'|), \quad (17) \]

with a corresponding relationship for \( \Lambda^* \), the Fourier transform of equations (15) and (16) yields the following "convergence rates":

\[ \frac{\mathcal{S}_k}{\mathcal{S}_{k-1}} = \frac{(1 - \varepsilon)(\bar{K}_1 - \bar{K}_1^*)}{1 - (1 - \varepsilon)\bar{K}_1^*}, \]

\[ = 1 - \left( 1 - (1 - \varepsilon)\bar{K}_1^* \right), \quad (18) \]

and

\[ \frac{\mathcal{S}_{k-1}^{(\Lambda)}}{\mathcal{S}_{k+1}^{(\Lambda)}} = (1 - \varepsilon)\bar{K}_1, \quad (19) \]

where, for example, \( \mathcal{S}_k \) is the Fourier transform of \( S_k \). Clearly, the FQPT will converge if the right-hand side of equation (18) lies between +1 and −1. This implies that \( (1 - \varepsilon)\bar{K}_1 < 1 \) (which is also the requirement for the convergence of \( \Lambda \)-iteration), and \( \frac{1}{2}[1 - (1 - \varepsilon)\bar{K}_1] < 1 - (1 - \varepsilon)\bar{K}_1^* \). The first of these inequalities is, of course, satisfied for \( 0 < \varepsilon \leq 1 \), and therefore presents no difficulty, whereas the second inequality may be violated if an injudicious choice of \( \bar{K}_1^* \) is made. The choice of \( \bar{K}_1^* \), however, is

**Fig. 1.—The ratio \( \chi_k(\tau) \) using \( N_x^{(1)} = 3 \) for a semi-infinite stationary isothermal atmosphere with \( \varepsilon = 10^{-4} \) is plotted against log \( \tau \) for \( k = 1, 2, 3, \) and 4. Curves for \( \chi_k(\tau) \) with \( k > 4 \) are not shown since these \( \chi_k(\tau) \) are less than 0.01 for all \( \tau \). Note that the errors tend to zero as \( \tau \to \infty \). In the Feautrier technique one actually puts \( S_1 = B_1 \) at the deepest point in the grid so that all errors at this point must be zero.**

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directly dependent upon the choice of the $\nu_1^{(1)}$, $i = 1$, $N_F^{(1)}$. If $1 - \tilde{K}_1^* \simeq 1 - \tilde{K}_1$, i.e., the operation $\Lambda^*$ in some way approximates the $\Lambda$ operation, the rate of convergence given by equation (18) is clearly far superior to that for $\Lambda$-iteration. If, however, $\tilde{K}_1^* \ll \tilde{K}_1$, the right-hand side of equation (18) approaches $(1 - \epsilon)\tilde{K}_1$ and thus the convergence of the FQPT and $\Lambda$-iteration will be essentially identical. This is obviously an unsatisfactory situation.

In slab geometry, one may choose a $\nu_1^{(1)}$ for $N_F^{(1)} = 1$ such that the operation $\Lambda^* \simeq \Lambda$. This reflects the dependence of the physics on the local conditions in such an atmosphere. For a semi-infinite medium, this is more difficult since nonlocal considerations come into effect. Thus, although the FQPT may be made to work for a semi-infinite atmosphere with $N_F^{(1)} = 1$ (by a suitable choice of $\nu_1^{(1)} > \nu_0$ where $\nu_0$ is some critical frequency away from line center), it is more satisfactory to use larger $N_F^{(1)}$.

Figure 1 shows the results of several $\chi_\delta(\tau)$ for a stationary semi-infinite atmosphere with $\epsilon = 10^{-4}$ and $N_F^{(1)} = 3$. We use a trapezoidal rule for the frequency quadrature with a bandwidth $\beta$ of $5.0\Delta\nu_0$. Clearly, good convergence is attained. Near the surface of a semi-infinite isothermal atmosphere with constant $\epsilon$, one has $S_1 \simeq \sqrt{\epsilon} B_1(T_e)$ independent of the quadrature order taken, whereas deep within the atmosphere one puts, from physical considerations, $S_1 = B_1(T_e)$ which thus stipulates $E_\delta(\tau \gg \epsilon^{-1}) = 0$ for all $k \geq 1$. These points are manifested in figure 1 where the errors are obviously smaller both near the surface and for $\tau > \epsilon^{-1}$.

As $\epsilon$ decreases the thermalization depth increases and thus the bandwidth must increase. The problem then becomes more nonlocal and thus still larger $N_F^{(1)}$ are required. It should be emphasized, however, that even though $N_F^{(1)}$ must be increased in such situations, the size of the matrices to be inverted, and thus the computer time required for solution, is far smaller for the FQPT than for other methods.

V. CONCLUSION

We have presented here a new numerical method, the frequency-quadrature perturbation technique, which significantly reduces the computer time and storage required to obtain solutions of the equation of radiative transfer specifying spectral-line formation. The method enables the approximation of the integral over frequency by a quadrature sum of lower order than that used in other existing techniques. In particular, it was found that solutions could be obtained in slab geometry, equal in accuracy to those obtained using these other methods, with only one-point quadrature over both angle and frequency, and thus the computer time saved in such a situation is very considerable. Further, the technique may be applied to any radiative-transfer problem involving integration over frequency, and may be used in conjunction with other methods of solution.

Difficulties were encountered when computing solutions with $N_F^{(1)} = 1$ for semi-infinite atmospheres, however, and it was found that both the choice of the frequency quadrature and the accuracy to which the $\Lambda S_k$ terms were calculated were critical in determining convergence. This difficulty was entirely overcome by increasing $N_F^{(1)}$. No such problems were encountered in slab geometry.

The frequency-quadrature perturbation technique has, therefore, obvious advantages when attempting problems which have, in the past, involved a large number of frequency-quadrature points. More particularly, calculations in multidimensional atmospheres (see Cannon 1970; Cannon and Rees 1971) can involve even larger matrices than those used in one-dimensional situations, and thus the FQPT may be again used to reduce the time taken to compute solutions.

The author is grateful to the referee for suggestions regarding the presentation of the method and the discussion of convergence.
This work was supported by a grant from the Australian Research Grants Committee.

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