APPROXIMATE SOLUTION TO THE EQUATION OF
RADIATIVE TRANSFER IN LINE FREQUENCIES

V. V. Ivanov

A. A. Zhdanov Leningrad State University
Translated from Astronomicheskii Zhurnal, Vol. 49, No. 1,
pp. 115-120, January-February, 1972
Original article submitted December 28, 1970

A simple approximation is given for the resolvent function of the integral equation of
transfer for the line radiation in a plane layer. The approximate value of the source
function appears to deviate from the exact value by no more than a factor of 2 for
any optical thickness $\tau_0$ of the layer ($0 < \tau_0 \leq \infty$), any particle albedo $\lambda$ ($0 < \lambda \leq 1$),
any plausible line absorption coefficient, and an arbitrary source distribution. The
approximation rests on certain asymptotic properties of the solutions to the equation of
transfer.

1. INTRODUCTION

Various authors have obtained solutions to a
large number of model problems concerning the
transfer of resonance radiation, and have investi-
gated them in detail (see, in particular, the sur-
veys by Jeffries [1] and the present author [2]).
Several efficient methods have been developed for
numerically solving the equation of transfer of line
radiation on an electronic computer [3]. However,
no one has yet found an approximate solution to the
problem, a solution that would at once be simple,
accurate, and universal enough to satisfy the re-
quirements of those who merely wish rapidly to
obtain a reliable estimate. The reason is that in
principle the diffusion approximation is not appli-
cable to these problems.

For a long time it remained unclear what ought to
replace the diffusion approximation. This ques-
tion has now been answered in recent studies by
Abramov, Dykhne, and Napartovich [4, 5]. But the
generalized diffusion approximation (Abramov et al.
call it the large-scale approximation), as applied
to problems of resonance-radiation transfer, turns
out to be too complicated to be useful for obtaining
rapid estimates. Biberman's simple approxima-
tion [6], which has been applied rather widely, also
cannot be considered entirely satisfactory, for in
the case of nearly conservative scattering it may
yield results near the boundaries that are in error
by an order of magnitude (in this connection, see
also [7, 8]).

In this paper we shall describe a simple ap-
proximate solution to the integral equation of trans-
fer for resonance radiation in a plane layer; the
solution will yield a result that is correct to a fac-
tor hardly anywhere exceeding 2. The approxi-
mation rests on certain asymptotic properties of
the solutions to the equation of transfer [2].

2. BASIC EQUATIONS

In the standard approximation of complete
redistribution over frequencies in the presence of
scattering, the equation of transfer for line radia-
tion in a plane layer, namely,

$$\frac{dI(\tau, \mu, x)}{d\tau} = -a(x)I(\tau, \mu, x)$$

$$+ a(x) \frac{\lambda}{2} A \int_{-\pi}^{\pi} a(\mu') \int_{-1}^{1} I(\tau, \mu', x') d\mu'$$

$$+ a(x) S(\tau)$$

(1)

with the boundary conditions

$$I(0, \mu, x) = 0, \quad \mu > 0,$$

$$I(\tau_n, \mu, x) = 0, \quad \mu < 0,$$

(2)

expressing the absence of incident radiation at the
boundaries, is equivalent to an integral equation for the source function
\[ S(\tau) = \frac{\lambda}{2} A \int_{-\infty}^{\infty} a(x') dx' \int_{-1}^{1} I(\tau, \mu', x') d\mu' + S^*(\tau). \tag{3} \]

This equation is of the form
\[ S(\tau) = \frac{\lambda}{2} \int_{-\infty}^{\infty} K(|\tau - \tau'|) S(\tau') d\tau' + S^*(\tau), \tag{4} \]
where
\[ K(\tau) = A \int_{-\infty}^{\infty} a(x) E_1(a(x) \tau) dx, \tag{5} \]

\[ E_1(\tau) = \int_{0}^{\tau} e^{-\mu} d\mu / \mu, \]
and \( A \) is a normalizing constant such that
\[ A \int_{-\infty}^{\infty} a(x) dx = 1, \tag{6} \]
whereby
\[ \int_{-\infty}^{\infty} K(\tau) d\tau = 1. \tag{7} \]

Here and subsequently we will use the notation adopted in our book [2]: \( I \) is the radiant intensity; \( a(x) \) represents the profile of the absorption coefficient, normalized according to Eq. (6); the function \( S^* \) describes the distribution of primary sources; \( x \) is a dimensionless frequency, measured from the center of the line \( (x = 0) \) in some suitable units, say, in Doppler half-widths; \( \tau \) is the optical depth at the central frequency of the line; \( \tau_0 \) is the optical thickness of the layer \( (0 < \tau_0 < \infty) \), \( \mu \) is the cosine of the angle between the direction of propagation of the radiation and the inner normal to the boundary \( \tau = 0 \); and \( \lambda \) is the particle albedo or the probability for survival of a photon upon being scattered \( (0 < \lambda < 1) \). We shall assume that the absorption coefficient is symmetrical about the line center \( [a(-x) = a(x)] \), decreases monotonically with increasing \( |x| \), and does not vanish for \( |x| < \infty \), so that the line will have infinitely extended wings. Our treatment is intended to provide an approximate solution to Eq. (4) under these assumptions.

It is well understood (see, for example, Sec. 8.1 of [2]) that the solution of Eq. (4) for arbitrary \( S^*(\tau) \) can be expressed in terms of a resolvent function \( \Phi(\tau; \tau_0) \), representing the solution of the equation
\[ \Phi(\tau; \tau_0) = \frac{\lambda}{2} \int_{-\infty}^{\infty} K(|\tau - \tau'|) \Phi(\tau'; \tau_0) d\tau' + \frac{\lambda}{2} K(\tau). \]

The problem thereby consists in obtaining an approximate expression for \( \Phi(\tau; \tau_0) \).

3. SEMIINFINITE MEDIUM

In the case of a semiinfinite medium \( (\tau_0 = \infty) \), the resolvent function \( \Phi(\tau) = \Phi(\tau, \infty) \) may be approximated, with the assumptions formulated above, by the expression
\[ \Phi(\tau) = \frac{\lambda}{2} K(\tau) \frac{1}{[1 - \lambda + \lambda L(\tau)]^{1/2}}, \tag{8} \]
where
\[ L(\tau) = \int_{\tau}^{\infty} K(\tau') d\tau'. \tag{9} \]

In Eq. (8) and henceforth, the superscript \( \alpha \) means that the expression is approximate rather than exact.

The function \( \Phi(\tau) \) satisfies the integral relations
\[ \int_{\tau}^{\infty} \Phi(\tau) d\tau = (1 - \lambda)^{-\alpha} - 1, \tag{10} \]
\[ \int_{\tau}^{\infty} L(\tau) \Phi(\tau) d\tau = \frac{2}{\lambda} (1 - \sqrt{1 - \lambda}) - 1. \tag{11} \]

The next three equations represent expansions of \( \Phi(\tau) \) with only the leading term retained:
\[ \Phi(\tau) = \frac{\lambda}{2} K(\tau) + \ldots, \quad \tau \to 0, \tag{12} \]
\[ \Phi(\tau) = \frac{\lambda}{2} K(\tau) + \ldots, \quad \tau \to \infty, \quad \lambda < 1, \tag{13} \]
\[ \Phi(\tau) = \frac{C}{2} \frac{K(\tau)}{L^{1/2}(\tau)} + \ldots, \quad \tau \to \infty, \quad \lambda = 1, \tag{14} \]
where the coefficient \( C \) is given by
\[ C = \frac{1}{\Gamma(\gamma)} \left( \frac{2}{\pi} \Gamma(2\gamma) \sin \pi \gamma \right)^{1/2}; \tag{15} \]
\( \Gamma(\gamma) \) is Euler's gamma function and \( \gamma \) is a characteristic index determined by the profile of the ab-
sorption coefficient (see Sec. 2.6 of [2]). Under the assumptions we have made, \( 0 < \gamma \leq 1/2 \). The more slowly the absorption coefficient falls off in the wings, the smaller \( \gamma \) will be. In particular, for a Doppler profile \( \gamma = 1/2 \); for Lorentz and Voigt profiles \( \gamma = 1/4 \). It is important to note that in all cases the quantity \( C \) is close to unity; as \( \gamma \) increases, \( C \) decreases from \( C = 1 \) at \( \gamma = 0 \) to \( C = 3^{3/2}/\pi = 0.990 \) at \( \gamma = 1/2 \). In practice, the range of applicability of Eq. (13) will be limited to the values \( \tau \gg 1 \) for which \( L(\tau) \ll 1 \). Equation (14), however, can be used not only for \( \lambda = 1 \), but also for \( 1 - \lambda \ll 1 \). In the latter case it will be applicable for values \( \tau \gg 1 \) such that \( L(\tau) \gg 1 - \lambda \). For a proof of Eqs. (10)-(14), see Sec. 5.5 of [2].

The approximation (8) is so devised that it will strictly satisfy the exact relations (10) and (11) in the same regions as \( \Phi(\tau) \); the expansions (12)-(14) will be applicable for if it is replaced by unity. Presumably, then, the values \( \Phi(\tau) \) should be very close to \( \Phi(\tau) \) for all \( \tau \) (0 \( \leq \tau < \infty \)) and \( \lambda \) (0 \( \leq \lambda \leq 1 \)) and for an arbitrary profile with infinitely extended wings.

For the resolvent function of an infinite medium, that is, for the solution of the equation

\[
\Phi_\infty(\tau) = \frac{\lambda}{2} \int_{-\infty}^{\infty} K(\tau - \tau') \Phi_\infty(\tau') d\tau' + \frac{\lambda}{2} K(\tau),
\]

(16)

the expression

\[
\Phi_\infty^*(\tau) = \frac{\lambda}{2} K(\tau) \left[ 1 \right. \left. - \lambda + \lambda L(\tau) \right]^{-1/2}
\]

(17)

introduced by Rybicki and Hummer [9] is analogous to the approximation (8). The expression (17) has a simple probabilistic interpretation [9]. The quantity \( 2(\lambda - 1) \Phi_\infty(\tau) d\sigma \) represents the probability that, in an infinite homogeneous medium with a plane isotropic source, the longest straight section in a photon trajectory selected at random will have a length between \( \tau \) and \( \tau + d\tau \). We may accordingly expect that Eq. (8) would also admit of a probabilistic interpretation.

Let us illustrate the application of the approximation (8). The standard problem of a half-space with uniformly distributed sources (see, for example, Secs. 6.1-6.3 of [2]) reduces to the equation

\[
S(\tau) = \frac{\lambda}{2} \int_{-\infty}^{\infty} K(\tau - \tau') S(\tau') d\tau' + 1,
\]

(18)

whose solution has the following expression in terms of the resolvent function:

\[
S(\tau) = (1 - \lambda)^{-\frac{1}{2}} \left( 1 + \int_{\frac{1}{2}}^{\infty} \Phi(\tau') d\tau' \right) = (1 - \lambda)^{-\frac{1}{2}} \Psi(\tau).
\]

(19)

The quantity \( S(\tau) \) is equal to the average number of scattering events experienced by a photon starting at depth \( \tau \) in a seminfinite medium (here the original radiation is regarded as the first scattering event). Substituting Eq. (8) into Eq. (19), we obtain a simple approximation for \( S(\tau) \):

\[
S^*(\tau) = (1 - \lambda)^{\frac{1}{2}} (1 - \lambda + \lambda L(\tau))^{-\frac{1}{2}} = (1 - \lambda)^{-\frac{1}{2}} \Psi^*(\tau).
\]

(20)

4. LAYER OF FINITE OPTICAL THICKNESS

The approximation (8) may be generalized to the case of a layer of finite optical thickness \( \tau_0 \) as follows:

\[
\Phi(\tau; \tau_0) = \frac{\lambda}{2} \frac{K(\tau)}{[1 - \lambda + \lambda L(\tau)]^{1/2}} \left( 1 - \lambda + \lambda L(\tau) \right)^{1/2}.
\]

(21)

This is an interpolation equation. It has been derived from these considerations; As Abramov et al., [4, 5] have shown, in the limiting cases of weakly dissipative \( 1 - \lambda \ll L(\tau_0) \) and strongly dissipative \( 1 - \lambda \gg L(\tau_0) \) media, the asymptotic relation

\[
\Phi(\tau; \tau_0) = \Phi(\tau) \frac{\Psi(\tau - \tau)}{\Psi(\tau)}
\]

(22)

holds for \( \tau_0 \gg 1 \). Equation (22) would be expected to be satisfied to fairly good accuracy for a layer of any optical thickness with arbitrary dissipation. Equation (21) can now be obtained simply by replacing the \( \Phi \) and \( \Psi \) in Eq. (22) by \( \Phi^* \) and \( \Psi^* \).

We next consider some implications of Eq. (21). For the quantity

\[
X(\infty; \tau_0) = 1 + \int_{\tau_0}^{\infty} \Phi(\tau; \tau_0) d\tau,
\]

(23)

which is often encountered in scattering problems for a plane layer and which satisfies the equation (see Sec. 8.2 of [2])

\[
\frac{dX(\infty; \tau_0)}{d\tau_0} = X(\infty; \tau_0) \Phi(\tau_0; \tau_0)
\]

(24)

with the boundary condition \( X(\infty; 0) = 1 \), we obtain from Eqs. (24) and (21) the approximation

\[
X^*(\infty; \tau_0) = [1 - \lambda + \lambda L(\tau_0)]^{-1/2}.
\]

(25)
Using Eqs. (21) and (25), we can readily find an explicit approximate expression for the source function in a layer with uniformly distributed sources, that is, for the solution of the equation

$$S(\tau) = \frac{\lambda}{2} \int_0^\infty K(|\tau - \tau'|)S(\tau') d\tau' + 1.$$  (26)

The quantity $S(\tau)$ is numerically equal (see, for example, Sec. 8.9 of [2]) to the average number of scattering events experienced by a photon "starting" at depth $\tau$, and is expressed in terms of $\Phi(\tau; \tau_0)$ and $X(\infty; \tau_0)$ by

$$S(\tau) = X(\infty; \tau_0) \left( 1 + \int_0^\infty \Phi(t; \tau_0) dt - \int_0^\infty \Phi(\tau_0 - t; \tau_0) dt \right).$$  (27)

If we substitute Eq. (21) into Eq. (27) and integrate by parts in the first integral on the right-hand side, we will obtain

$$S(\tau) = \frac{1 - \lambda + \lambda L(\tau)}{1 - \lambda + \lambda L(\tau_0 - \tau)}.$$  (28)

This simple expression yields an accuracy that is sufficient for the great majority of applications of the theory. The accuracy evidently is no lower than that generally afforded by the approximation of complete redistribution over frequencies. In particular, for $\lambda = 1$ and $\tau_0 \to \infty$, we find

$$S \left( \frac{\tau_0}{2} \right) \frac{\sin \pi \gamma}{\pi \gamma}, \quad \frac{S(0)}{S(0)} = \left( 1 + \frac{n}{2} \right) \Gamma(\gamma + 1) \frac{\sin \pi \gamma}{2\pi \gamma}.$$  (29)

where $\gamma$ is the characteristic index. The question of the accuracy provided by Eqs. (28) and (21) warrants a more careful examination.

We should point out in conclusion that Shcherbakov and the author [10] have tabulated the functions $K(\tau)$ and $L(\tau)$ for a Doppler profile. Furthermore, for Doppler and Voigt profiles (with a Voigt parameter $a = 10^{-4}$ and $10^{-5}$) tabulations have been prepared [11, 12] for the coefficients $A_1$ and $c_1$ in approximate representations of the form

$$K(\tau) = \sum_{n=1}^N A_n e^{-t_n}, \quad L(\tau) = \sum_{n=1}^N c_n e^{-t_n}$$

for $K$ and $L$, enabling these functions to be computed rapidly for a wide range of values of $\tau$ (see also [3]).

LITERATURE CITED

10. V. V. Ivanov and V. T. Shcherbakov, Astrofizika, 1, 31 (1965).